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Application of Sub-Predictor Control for Linear Delayed Stochastic Systems

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Abstract: The problem of robust predictor-based H_∞ state-feedback control for uncertain continuous-time stochastic state-multiplicative retarded systems is extended to include a set of sub-predictors, thus considerably improving the control performance. The multiplicative noise appears in the system dynamic matrix and in the measurement matrix of the observed signal, while the delay resided in the input to the system. In this problem, a cost function is defined which is the expected value of the standard H_∞ performance index with respect to the uncertain parameters. In the robust case, the solution is obtained via a simple set of linear matrix inequalities. We bring a detailed numerical example that demonstrates the superiority of designing state-feedback control via a set of sub predictor compared to a single predictor design.

Keywords: Stochastic H_∞ control, robust control, sub-predictor control, delayed systems.

1. INTRODUCTION

We consider continuous-time, state-multiplicative noisy linear systems with time-delayed input and either norm-bounded or polytopic-type uncertainties and we address the problem of robust H_∞ state-feedback control by applying a set of Luenberger-type sub-predictors. We thus obtain non predictive systems which can be efficiently treated via the input-output approach which was shown to yield simple and tractable LMI conditions in the stochastic case.

The field of stochastic control has greatly advanced in the last four decades since the emergence of the H_∞ control theory. Starting with the stochastic H_2 counterpart in the early 60's, this field has accumulated a host of solution methods which were aimed to tackle of problems stochastic stability, control and estimation of both: continuous- and discrete-time systems. Focusing on the special structure of stochastic state-multiplicative noisy linear systems, the stability and control of stochastic retarded systems of various types (i.e constant time-delay, slow and fast varying delay) have been central issues in the theory of stochastic state-multiplicative systems for more than three decades (Verriest and Florchinger [1995], Mao [1996], Chen et al. [2005], Verriest [2004], Gao and Chen [2007], Yue et al. [2009]). We note that many of the results, including solution methods and mathematical techniques, that have been obtained for the stability of deterministic retarded systems, since the 90's, have been applied to the stochastic case, mainly for continuous-time systems (Kolmanovskii and Richard [1999], Kao and Lincoln [2004], Li et al. [2011], Mazenc and Normand-Cyrot [2013], see also Fridman [2014] for a comprehensive review).

The control and estimation theory of systems with stochastic uncertainties has been mainly developed in the last three decades (see Gershon et al. [2005] and Gershon and Shaked [2013] and the references therein). Numerous solutions to various stochastic control and filtering problems, including those that ensure a worst case performance bound in the H_∞ sense, have been derived and solved for both: delay-free (see, for example, Dragan and Morozan [1997], Hinriechsen and Pritchard [1998]) and retarded linear stochastic systems (see Verriest and Florchinger [1995], Verriest [2004] for the continuous-time case and Xu et al. [2004], Yue et al. [2009] for the discrete-time counterpart).

In the continuous-time stochastic setting, the predominant tool for the solution of the traditional control and estimation stochastic problem is the Lyapunov-Krasovskii (LK) approach. For example, the H_∞ state-feedback control for systems with time-varying delay is treated in S.Xu and Chen [2002] for restricted LK functions that provide delay-independent, rate dependent results. Also Boukas and Liu [2002] considers H_∞ control (both state and output feedback) and estimation of time delay systems.

In the discrete-time setting, the mean square exponential stability and the control and filtering problems of these systems were treated by several groups (Xu et al. [2004]-Yue et al. [2009]). In Xu et al. [2004], the state-feedback control problem solution is solved for norm-bounded uncertain systems, for the restrictive case where the same multiplicative noise sequence multiplies both the states and the input of the system. The solution there is delay-dependent.

In the last two decades, the input-output approach has been applied to both continuous- and discrete-time stochastic settings where solutions for the stability and Bounded Real Lemma (BRL) problems were obtained mainly via the use of LMI conditions Gershon and Shaked [2013]. Based on the later approach, the solution to the state-feedback control, filtering and measurement problems were obtained for the state-delayed case (see Gershon and Shaked [2013] and the references therein). We note that the *input-output* approach is based on the representation of the system's delay action by linear operators, without delay, which allows one to replace the underlying system with an equivalent one that possesses a norm-bounded uncertainty, and therefore may be treated by the well developed theory of norm bounded uncertain, non-retarded systems with state-multiplicative noise Gershon and Shaked [2013]. The major advantage of the *input-output* method is its ability to yield simple LMI conditions which are amenable to various techniques especially in the treatment of uncertain systems.

Recently, the problem of state-feedback control of stochastic state-multiplicative systems with delayed input has been solved where a predictor-type state-feedback controller is applied for the solution Gershon et al. [2017],Gershon and Shaked [2019] . In Gershon et al. [2017], a new state vector is defined there which, given the feedback gain matrix, predicts the value of the true state of the resulting closed-loop. A condition for closed-loop stability is derived there which is used to find a stabilizing state-feedback control. Also, similar conditions have been obtained in Gershon et al. [2017] that guarantee prescribed bounds on the L_2-gain of the resulting closed-loop systems. The major drawback of the predictor based solution method of Gershon et al. [2017] is that it can not be extended either to the uncertain polytopic case or to the uncertain norm-bounded case. This handicap has been tackled via the new approach of Gershon and Shaked [2019] where a Luenberger-type predictor is applied. However in the uncertain case, especialy in case of a large input delay, the use of a single predictor may produce

In this work we bring the solution of the robust state-feedback control based on a host of sub-predictors which are of a Luenberger type. These predictors are applied to the measurement signal, thus transforming the system into a non-predictive one. We first introduce, as a preliminary result, the solution of the robust H_∞ Luenberger-type filter for norm-bounded uncertain systems. This is followed by the solution of a single predictor based state-feedback control for nominal systems which is then extended to include two types of uncertain systems: norm bounded and polytopic-type uncertainties. In the latter case the solution is obtained by applying a single Lyapunov function over all the uncertainty polytope. Similarly to the single predictor case, we then apply two sub-preditors [chosen for simplicity without harming the general nature of the solution] based on the structure of the Luenberger-type single predictor. In the example section we bring a numerical example where we bring the various solution methods used to solve the state-feedback control problem for both the nominal retarded systems [i.e with no uncertainties] and uncertain retarded systems

The paper is organized as follows: Starting with the problem formulation of Section II, The solution of the robust Luenberger H_∞ filter which serves as a preliminary result is given in Section III. This latter result is followed by the solution of the robust norm-bounded and polytopic single predictor-based state-feedback control in Section IV. In Section V, the results of Section IV are extended to to the case where two sub-predictors [for simplicity] are applied for the design of the state-feedback controller. In Section VI, we bring a numerical example that demonstrates the various solution methods used to design a robust H_∞ controller.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space and $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices. For a symmetric $P\in\mathcal{R}^{n\times n}$, $P>0$ means that it is positive definite. We denote expectation by $\mathcal{E}\{\cdot\}$ and we provide all spaces $\mathcal{R}^k,\ k\geq 1$ with the usual inner product $<\cdot,\cdot>$ and with the standard Euclidean norm $||\cdot||$. The space of vector functions that are square integrable over $[0\quad\infty)$ is denoted by $\mathcal{L}_2$ and $col\{a,\ b\}$ implies $[a^T\ b^T]^T$. We denote by $L^2(\Omega,\mathcal{R}^k)$ the space of square-integrable $\mathcal{R}^k-$ valued functions on the probability space $(\Omega,\mathcal{F},\mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_t)_{t>0}$ we denote an increasing family of σ-algebras $\mathcal{F}_t\subset\mathcal{F}$. We also denote by $\tilde{L}^2([0,T);\mathcal{R}^k)$ the space of nonanticipative stochastic process $f(\cdot)=(f(t))_{t\in[0,T]}$ in $\mathcal{R}^k$ with respect to $(\mathcal{F}_t)_{t\in[0,T)}$ satisfying

$$||f(\cdot)||^2_{\tilde{L}_2}=\mathcal{E}\{\int_0^T ||f(t)||^2dt\}=\int_0^T\mathcal{E}\{||f(t)||^2\}dt<\infty.$$

Stochastic differential equations will be interpreted to be of *Itô* type Klebaner [2012].

2. PROBLEM FORMULATION

We consider the following linear system:

$$\begin{aligned}dx(t)&=Ax(t)dt+Dx(t)d\nu(t)+B_1w(t)dt+\\ &B_2u(t-h)dt,\quad x(\tau)=0,\ \tau\leq 0\end{aligned}\tag{1}$$

with the objective vector

$$z(t)=C_1x(t)+D_{12}u(t-h),\quad C_1^TD_{12}=0,\tag{2}$$

where $x(t)\in\mathcal{R}^n$ is the system state vector, $w(t)\in\mathcal{R}^p$ is the exogenous disturbance signal, and $z(t)\in\mathcal{R}^r$ is the state combination (objective function signal) to be estimated. The variable $\nu(t)$ is a zero-mean real scalar Wiener processes that satisfy:

$$\mathcal{E}\{d\nu(t)\}=0,\ \mathcal{E}\{d\nu(t)^2\}=dt.$$

In the case where we measure the entire state vector $x(t)$ we build the following single predictor $\hat{x}(t)$ for $x(t+h)$

$$d\hat{x}(t)=\bar{A}\hat{x}(t)dt+\bar{B}_2u(t)dt+L[x(t)-\hat{x}(t-h)]dt.\tag{3}$$

Assuming that $A=\bar{A}+\Delta A$ and $B_2=\bar{B}_2+\Delta B_2$ we denote,

$$e(t)=x(t)-\hat{x}(t-h)\tag{4}$$

and obtain the following:

$$\begin{aligned} &dx(t) = (\bar{A} + \Delta A)x(t)dt + Dx(t)d\nu(t) + B_1 w(t)dt + \\ &B_2 u(t-h)dt, \\ &de(t) = [\bar{A}e(t) - Le(t-h)]dt + \Delta A x(t)dt + \\ &Dx(t)d\nu(t) + B_1 w(t)dt + \Delta B_2 u(t-h)dt. \end{aligned} \tag{5}$$

Applying the 'state' feedback control

$$u(t) = K\hat{x}(t) \tag{6}$$

and considering the following index of performance,

$$J_E \stackrel{\Delta}{=} ||z(t)||^2_{\tilde{L}_2} - \gamma^2 [||w(t)||^2_{\tilde{L}_2}, \tag{7}$$

we seek a state-feedback control law in the form of (6) such that J_E of (7) is negative for all nonzero $w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$. In the sequel, we first bring a preliminary result, which will be used latter, concerning the robust Luenberger filtering problem. We then bring the solution of the single predictor state-feedback control followed by a solution of the sub-predictor based counterpart problem.

3. PRELIMINARY RESULT

In this section we bring the solution of the Luenberger-type filter for norm-bounded uncertain systems. We will use this solution latter for the derivation of the predictor based state-feedback control. We note that the solution of the Luenberger filter for the systems under study has already been solved in Gershon and Shaked [2019] only for the nominal case. We consider the following delay-free linear system:

$$\begin{aligned} dx(t) &= Ax(t)dt + Dx(t)d\nu(t) + B_1 w(t)dt, \\ dy(t) &= C_2 x(t)dt + Fx(t)d\zeta(t) + D_{21} w(t)dt \end{aligned} \tag{8}$$

with the objective vector

$$z(t) = C_1 x(t), \tag{9}$$

where $y(t) \in \mathcal{R}^m$ is the measured output and where the variables $\nu(t)$ and $\zeta(t)$ are zero-mean real scalar Wiener processes that satisfy:

$$\mathcal{E}\{d\nu(t)\} = 0,\ \mathcal{E}\{d\zeta(t)\} = 0,\ \mathcal{E}\{d\nu(t)^2\} = dt,$$
$$\mathcal{E}\{d\zeta(t)^2\} = dt,\ \mathcal{E}\{d\nu(t)d\zeta(t)\} = 0.$$

Denoting

$$\Delta A = A - \bar{A} \quad \text{and} \quad \Delta C_2 = C_2 - \bar{C}_2$$

where $\bar{A}$ and $\bar{C}_2$ are the nominal values for A and C_2, respectively, we consider,

$$\begin{aligned} &d\hat{x}(t) = \bar{A}\hat{x}(t)dt + L(dy(t) - \bar{C}_2\hat{x}(t)dt) = \\ &\bar{A}\hat{x}(t)dt + L\bar{C}_2 e(t)dt + L\Delta C_2 x(t)dt + LF\xi(t)x(t)dt \\ &+ LD_{21}w(t)dt, \\ &\hat{z}(t) = \bar{C}_1\hat{x}(t), \end{aligned} \tag{10}$$

where

$$e(t) \stackrel{\Delta}{=} x(t) - \hat{x}(t).$$

Denoting $\bar{z}(t) = z(t) - \hat{z}(t)$, we consider the following cost function:

$$J_F \stackrel{\Delta}{=} ||\bar{z}(t)||^2_{\tilde{L}_2} - \gamma^2 [||w(t)||^2_{\tilde{L}_2}. \tag{11}$$

Given $\gamma > 0$, we seek an estimate $\bar{C}_1\hat{x}(t)$ of $C_1 x(t)$ over the infinite time horizon $[0,\infty)$ such that J_F is negative for all nonzero $w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$.

It is readily found that

$$\begin{aligned} de(t) &= Ax(t)dt + Dx(t)d\nu(t) + B_1 w(t)dt - \bar{A}\hat{x}(t)dt \\ &\quad - LC_2 e(t)dt - LFx(t)d\xi(t) - LD_{21}w(t)dt \end{aligned}$$

or

$$\begin{aligned} de(t) &= (\bar{A} - L\bar{C}_2)e(t)dt + \Delta A x(t)dt - L\Delta C_2 x(t)dt \\ &+ Dx(t)d\nu(t) - LFx(t)d\xi(t) + (B_1 - LD_{21})w(t)dt. \end{aligned} \tag{12}$$

Denoting $\eta(t) = col\{x(t),\ e(t)\}$ we obtain:

$$d\eta(t) = \tilde{A}\eta(t)dt + \tilde{D}d\nu(t)\eta(t) + \tilde{F}d\xi(t)\eta(t) + \tilde{B}_1 w(t)dt$$

where

$$\tilde{A} = \begin{bmatrix} \bar{A} + \Delta A & 0 \\ \Delta A - L\Delta C_2 & \bar{A} - L\bar{C}_2 \end{bmatrix},\ \tilde{D} = \begin{bmatrix} D & 0 \\ D & 0 \end{bmatrix}, \tag{13}$$

$$\tilde{F} = \begin{bmatrix} 0 & 0 \\ -LF & 0 \end{bmatrix},\ \tilde{B}_1 = \begin{bmatrix} B_1 \\ B_1 - LD_{21} \end{bmatrix}$$

and

$$\bar{z}(t) = \tilde{C}_1\eta(t),$$

where $\Delta C_1 = C_1 - \bar{C}_1$ and $\tilde{C}_1 = [\Delta C_1 \ \ \bar{C}_1]$. We thus arrive at the following result:

Lemma 1: Consider the system of (8a,b) and (9). For a prescribed scalar $\gamma > 0$, a necessary and sufficient condition for J_F of (11) to be negative for all nonzero $w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$, is that there exist $0 < P \in \mathcal{R}^{n\times n}$, $L \in \mathcal{R}^{n\times n}$ such that the following LMI holds:

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} + \tilde{D}^T P\tilde{D} & P\tilde{B}_1 & \tilde{C}_1^T & \tilde{F}^T P \\ * & -\gamma^2 I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -P \end{bmatrix} < 0. \tag{14}$$

Remark 1: In the case where there is no uncertainty, $A = \bar{A}$, $C_2 = \bar{C}_2$ and $C_1 = \bar{C}_1$, the standard result of (Gershon et al. [2005], see p. 31) is retrieved. In case of uncertainty one may consider two types. Norm-bounded and polytopic uncertainties. In the norm-bounded case we assume that:

$$A = \bar{A} + \Delta A, \quad C_2 = \bar{C}_2 + \Delta C_2$$
$$\text{and } [\Delta A,\ \Delta C_2] = H\delta(x,t)[E_1 \ \ E_2],\ \ ||\delta(x,t)|| < 1. \tag{15}$$

We then have:

$$\tilde{A} = \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} - L\bar{C}_2 \end{bmatrix} + \begin{bmatrix} H \\ H \end{bmatrix} \delta(x,t)\,[E_1 \ \ 0\,] + \begin{bmatrix} 0 \\ -LH \end{bmatrix} \delta(x,t)\,[E_2 \ \ 0\,].$$

Using Young's inequality (Cvetkovski [2012], Huijsmans et al. [1995]) and (15d) the LMI solution readily follows. Note that for simplicity we took $\Delta C_1 = 0$. The polytopic type uncertainty can be readily handled in the case where nominal $\bar{A}$ and $\bar{C}_2$ are given and where the matrices A, C_2 and C_1 lie in the polytope (21). Assigning a single Lyapunov function over all the uncertainty polytope, the latter solution is achieved by solving (14) for each vertex. The so-called 'quadratic' solution is thus obtained.

4. ROBUST PREDICTOR-BASED STATE-FEEDBACK CONTROL

The above derivation of the observer was partially motivated by the need for a robust predictor in the control of state multiplicative systems with an input delay. We start by augmenting the system of (1) to include the control input $u(t-h)$ where h is the time delay.

Denoting $\eta(t) = col\{\bar{x}(t),\ e(t)\}$, where $\bar{x}(t) \triangleq x(t) - e(t)$, it readily follows that

$$d\eta(t) = \hat{A}\eta(t)dt + \hat{D}\eta(t)d\nu(t) + \hat{E}\eta(t-h)dt + \hat{B}_1 w(t)dt$$

and

$$z(t) = [C_1\ C_1]\eta(t) + D_{12}[K\ 0]\eta(t)$$

where

$$\hat{A} = \begin{bmatrix} \bar{A}+\bar{B}_2K & 0 \\ \Delta A + \Delta B_2 K & \bar{A}+\Delta A \end{bmatrix},\ \hat{D} = \begin{bmatrix} 0 & 0 \\ D & D \end{bmatrix}, \quad (16)$$

$$\hat{E} = \begin{bmatrix} 0 & L \\ 0 & -L \end{bmatrix}, \hat{B}_1 = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}.$$

Assuming that

$$\begin{bmatrix} \Delta A \\ \Delta B_2 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \delta(x,t)E_1, \text{ where } ||\delta(x,t)|| \leq 1, \quad (17)$$

we write the matrix $\hat{A}$ of (16c) as:

$$\hat{A} = \hat{A}_0 + \hat{B}_2[K\ \ 0] + \begin{bmatrix} 0 \\ H_1 \end{bmatrix} \delta(x,t)\,[E_1\ \ E_1] + \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \delta(x,t)E_1\,[K\ \ 0]\,. \quad (18)$$

where $\hat{A}_0 = \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{bmatrix}$ and $\hat{B}_2 = \begin{bmatrix} \bar{B}_2 \\ 0 \end{bmatrix}$.

The system of (16) is a simple state time-delayed system with multiplicative noise acting only on the state vector $\eta(t)$ and with norm bounded uncertainty. Applying Theorem 2.7 of (Gershon and Shaked [2013]) we seek the matrices: $P = diag\{P_1,\ P_2\} > 0$, m_p, $Y_K = KP_1$, $Y_L = LP_2$, R_p and a tuning parameter $\epsilon > 0$ that, for $E_1 = 0$, satisfy the following LMI:

$$\begin{bmatrix} \Upsilon_1 & \hat{E}P - m_p & m_p & \hat{B}_1 & \Upsilon_2 & \Upsilon_3 & P\hat{D}^T \\ * & -R_p & 0 & 0 & \Upsilon_4 & 0 & 0 \\ * & * & -\epsilon P & 0 & -h\epsilon m_p^T & 0 & 0 \\ * & * & * & -\gamma^2 I & h\epsilon \hat{B}_1^T & 0 & 0 \\ * & * & * & * & -\epsilon P & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -P \end{bmatrix} < 0 \quad (19)$$

where

$$\Upsilon_1 = P\hat{A}_0^T + \hat{A}_0 P + \hat{B}_2\,[Y_K\ \ 0] + \begin{bmatrix} Y_K^T \\ 0 \end{bmatrix} \hat{B}_2^T + m_p + m_p^T + \frac{1}{1-d}R_p,\ \Upsilon_2 = \epsilon h[P\hat{A}_0^T + m_p^T + \begin{bmatrix} Y_K^T \\ 0 \end{bmatrix} \hat{B}_2^T],$$

$$\Upsilon_3 = P[C_1\ C_1]^T + \begin{bmatrix} Y_K^T \\ 0 \end{bmatrix} D_{12}^T,$$

and where $\Upsilon_4 = \epsilon h[\begin{bmatrix} 0 \\ Y_L^T \end{bmatrix}[I\ \ -I] - m_p^T]$.

The decision matrix variables Y_K and Y_L are defined by $Y_K = KP_1$ and $Y_L = LP_2$.

Considering the index of performance of (7) where we seek the state-feedback control law of (6), we obtain the following result:

Theorem 1: Consider the system of (1a,b) and (2). For a prescribed scalar $\gamma > 0$ and a given tuning scalar parameter $\epsilon > 0$, J_E of (7) is negative for all nonzero $w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$, if there exist: $0 < P = diag\{P_1,\ P_2\} \in \mathcal{R}^{2n\times 2n}$, $0 < R_p \in \mathcal{R}^{2n\times 2n}$, $m_p \in \mathcal{R}^{2n\times 2n}$, $Y_K \in \mathcal{R}^{l\times n}$ and $Y_L \in \mathcal{R}^{n\times n}$ such that the LMI of (19) holds.

The latter result can be readily extended to the norm bounded uncertain case (where $E_1 \neq 0$) using Young's inequality. We thus obtain the following result:

Corollary 1: Consider the uncertain system of (1a,b), (2) and (17). For a prescribed scalar $\gamma > 0$ and a given tuning scalar parameter $\epsilon > 0$, J_E of (7) is negative for all nonzero $w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$, if there exist: $0 < P = diag\{P_1,\ P_2\} \in \mathcal{R}^{2n\times 2n}$, $0 < R_p \in \mathcal{R}^{2n\times 2n}$, $m_p \in \mathcal{R}^{2n\times 2n}$, $Y_K \in \mathcal{R}^{l\times n}$, $Y_L \in \mathcal{R}^{n\times n}$ and scalars $\bar{\epsilon}_1 > 0$, $\bar{\epsilon}_2 > 0$ such that the following LMI holds:

$$\begin{bmatrix} \bar{\Upsilon}_1 & \bar{\epsilon}_2 & m_p & \hat{B}_1 & \bar{\Upsilon}_2 & \bar{\Upsilon}_3 & P\hat{D}^T & P\begin{bmatrix} E_1^T \\ E_1^T \end{bmatrix} & P\begin{bmatrix} E_1^T \\ 0 \end{bmatrix} \\ * & -R_p & 0 & 0 & \bar{\Upsilon}_4 & 0 & 0 & 0 & 0 \\ * & * & -\epsilon P & 0 & -h\epsilon m_p^T & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & h\epsilon\hat{B}_1^T & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon P & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & -P & 0 & 0 \\ * & * & * & * & * & * & * & -\bar{\epsilon} I & 0 \\ * & * & * & * & * & * & * & * & -\bar{\epsilon} I \end{bmatrix} < 0, \quad (20)$$

where:

$$\bar{\Upsilon}_1 = P\hat{A}_0^T + \hat{A}_0 P + \hat{B}_2\,[Y_K\ \ 0] + \begin{bmatrix} Y_K^T \\ 0 \end{bmatrix} \hat{B}_2^T + m_p + m_p^T + \frac{1}{1-d}R_p + \begin{bmatrix} 0 \\ H_1 \end{bmatrix} \bar{\epsilon}_1[0\ H_1^T] + \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \bar{\epsilon}_2[0\ H_2^T],$$

$$\bar{\Upsilon}_2 = \epsilon h[P\hat{A}_0^T + m_p^T + \begin{bmatrix} Y_K^T \\ 0 \end{bmatrix} \hat{B}_2^T] + \epsilon h[\begin{bmatrix} 0 \\ H_1 \end{bmatrix} \bar{\epsilon}_1[0\ H_1^T] + \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \bar{\epsilon}_2[0\ H_2^T]],\ \ \bar{\epsilon} = \begin{bmatrix} \bar{\epsilon}_1 I & 0 \\ 0 & \bar{\epsilon}_2 I \end{bmatrix}$$

and where Υ_3 and Υ_4 are given above.

In the above, uncertainty was assumed to be of the norm-bounded type. Uncertainty of the polytopic type can also be solved for, around nominal values of A and B_2. We thus assume that the system matrices in (8a,b), (9) lie within the following polytope:

$$\bar{\Omega} = \mathcal{C}o\{\bar{\Omega}_1, \bar{\Omega}_2, ..., \bar{\Omega}_N\}, \quad (21)$$

where

$$\bar{\Omega}_i \triangleq \begin{bmatrix} A^{(i)} & B_2^{(i)} \end{bmatrix} \quad (22)$$

and where N is the number of vertices. In other words:

$$\bar{\Omega} = \sum_{i=1}^{N} \bar{\Omega}_i f_i, \quad \sum_{i=1}^{N} f_i = 1 \quad , f_i \geq 0. \quad (23)$$

Denoting, for the uncertain poytopic case,

$$\hat{A}^{(i)} = \begin{bmatrix} \bar{A} & 0 \\ A^{(i)} - \bar{A} & A^{(i)} \end{bmatrix} \text{ and } \hat{B}_2^{(i)} = \begin{bmatrix} \bar{B}_2 \\ B_2^{(i)} - \bar{B}_2 \end{bmatrix},$$

$i = 1, 2, ..., N$, we obtain the following result by applying a single Lyapunov function over the whole uncertainty polytope.

Corollary 2: Consider the uncertain system of (1a,b), (2) where the matrices A and B_2 reside in the polytope of (21). For a prescribed scalar $\gamma > 0$, J_E of (7) is negative for all nonzero $w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$, if there exist: $0 < P = diag\{P_1,\ P_2\} \in \mathcal{R}^{2n\times 2n}$, $0 < R_p \in \mathcal{R}^{2n\times 2n}$, $m_p \in \mathcal{R}^{2n\times 2n}$, $Y_K \in \mathcal{R}^{l\times n}$ and $Y_L \in \mathcal{R}^{n\times n}$ such that the following set of LMIs holds $\forall i,\ i = 1, 2, ..N$:

$$\begin{bmatrix} \Upsilon_1^{(i)} & \hat{E}P - m_p & m_p & \hat{B}_1^{(i)} & \Upsilon_2^{(i)} & \Upsilon_3^{(i)} & P\hat{D}^T \\ * & -R_p & 0 & 0 & \Upsilon_4^{(i)} & 0 & 0 \\ * & * & -\epsilon P & 0 & -h\epsilon m_p^T & 0 & 0 \\ * & * & * & -\gamma^2 I & h\epsilon \hat{B}_1^T & 0 & 0 \\ * & * & * & * & -\epsilon P & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -P \end{bmatrix} < 0, \qquad (24)$$

where

$$\Upsilon_1^{(i)} = P\hat{A}^{(i),T} + \hat{A}^{(i)}P + \hat{B}_2^{(i)}[Y_K\ 0] + \begin{bmatrix} Y_K^T \\ 0 \end{bmatrix}\hat{B}_2^{(i),T}$$
$$+ m_p + m_p^T + \frac{1}{1-d}R_p,$$
$$\Upsilon_2^{(i)} = \epsilon h[P\hat{A}^{(i),T} + m_p^T + \begin{bmatrix} Y_K^T \\ 0 \end{bmatrix}\hat{B}_2^{(i),T}],$$
$$\Upsilon_3^{(i)} = P[C_1\ C_1]^T + \begin{bmatrix} Y_K^T \\ 0 \end{bmatrix} D_{12}^T,$$
$$\Upsilon_4^{(i)} = \epsilon h[\begin{bmatrix} 0 \\ Y_L^T \end{bmatrix}[I\ -I] - m_p^T].$$

5. APPLICATION OF PREDICTORS IN CASCADE

The predictors of the above section applied a single predictor that predicts, h seconds ahead, the state vector $x(t)$. Since this prediction is not based on measurements for h seconds, the prediction error may be too large for achieving a stable closed-loop design. In order to circumvent this difficulty one can apply the method of Najafi et al. [2013]. For simplicity we demonstrate the method for two sub-predictors. It can be readily extended to the case of many sub predictors in cascade. Dividing the delay h into two equal parts, we build the following sequential predictors:

$$\begin{aligned} d\hat{x}_1(t) &= \bar{A}\hat{x}_1(t)dt + \bar{B}_2u(t)dt + L_1[\hat{x}_2(t) - \hat{x}_1(t-\frac{h}{2})]dt \\ d\hat{x}_2(t) &= \bar{A}\hat{x}_2(t)dt + \bar{B}_2u(t-\frac{h}{2})dt + L_2[\hat{x}(t) - \hat{x}_2(t-\frac{h}{2})]dt \end{aligned} \qquad (25)$$

and consider the application of

$$u(t) = K\hat{x}_1(t). \qquad (26)$$

We denote:

$$e_1(t) = \hat{x}_2(t-\frac{h}{2}) - \hat{x}_1(t-h),\ e_2(t) = x(t) - \hat{x}_2(t-\frac{h}{2}) \qquad (27)$$

and obtain the following:

$$\begin{aligned} &dx(t) = (\bar{A} + \Delta A)x(t)dt + Dx(t)d\nu(t) + B_1w(t)dt \\ &+ B_2u(t-h)dt, \\ &de_1(t) = [\bar{A}e_1(t) - L_1e(t-\frac{h}{2})]dt + L_2e_2(t-\frac{h}{2})dt, \\ &de_2(t) = [\bar{A}e_2(t) - L_2e(t-\frac{h}{2})]dt + \Delta Ax(t)dt + Dx(t)d\nu(t) \\ &+ B_1w(t)dt + \Delta B_2u(t-h)dt \end{aligned} \qquad (28)$$

Applying the 'state' feedback control of (26) and denoting $\eta(t) = col\{x(t) - e_1(t) - e_2(t),\ \ e_1(t),\ \ e_2(t)\}$, it readily follows that

$$d\eta(t) = \hat{A}\eta(t)dt + \hat{D}\eta(t)d\nu(t) + \hat{E}\eta(t-\frac{1}{2}h)dt + \hat{B}_1w(t)dt$$

and

$$z(t) = [C_1\ C_1\ C_1]\eta(t) + D_{12}[K\ 0\ 0]\eta(t)$$

where

$$\hat{A} = \begin{bmatrix} \bar{A} + \bar{B}_2K & 0 & 0 \\ 0 & \bar{A} & 0 \\ \Delta A + \Delta B_2K & \Delta A & \bar{A} + \Delta A \end{bmatrix},$$
$$\hat{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D & D & D \end{bmatrix}, \hat{E} = \begin{bmatrix} 0 & L_1 & 0 \\ 0 & -L_1 & L_2 \\ 0 & 0 & -L_2 \end{bmatrix}, \hat{B}_1 = \begin{bmatrix} 0 \\ 0 \\ B_1 \end{bmatrix}. \qquad (29)$$

Assuming that (17) holds, we denote:

$$\begin{aligned} \hat{A} &= \hat{A}_0 + \hat{B}_2[K\ \ 0\ \ 0] + \begin{bmatrix} 0 \\ 0 \\ H_1 \end{bmatrix}\delta(x,t)\,[\,E_1\ \ E_1\ \ E_1\,] \\ &+ \begin{bmatrix} 0 \\ 0 \\ H_2 \end{bmatrix}\delta(x,t)E_1\,[K\ \ 0\ \ 0\,]. \end{aligned} \qquad (30)$$

where $\hat{A}_0 = \begin{bmatrix} \bar{A} & 0 & 0 \\ 0 & \bar{A} & 0 \\ 0 & 0 & \bar{A} \end{bmatrix}$ and $\hat{B}_2 = \begin{bmatrix} \bar{B}_2 \\ 0 \\ 0 \end{bmatrix}$. The system that is described in (29) is a state-delayed system with a state-multiplicative noise that acts on the state $\eta(t)$. Defining $P = diag\{P_1,\ P_2,\ P_3\}$, $Y_K = KP_1$, $Y_{L_1} = L_1P_2$ and $Y_{L_2} = L_2P_3$ the following result is obtained:

Theorem 2: Consider the system of (1a,b) and (2). For a prescribed scalar $\gamma > 0$ where $E_1 = 0$ and a given tuning scalar parameter $\epsilon > 0$, a sufficient condition for J_E of (7) to be negative for all $w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$, is that there exist $0 < P = diag\{P_1,\ P_2\ \ P_3\} \in \mathcal{R}^{3n\times 3n}$, $0 < R_p \in \mathcal{R}^{3n\times 3n}$, $m_p \in \mathcal{R}^{3n\times 3n}$, $Y_K \in \mathcal{R}^{l\times n}$ and $Y_{L_1} \in \mathcal{R}^{n\times n}$, $Y_{L_2} \in \mathcal{R}^{n\times n}$ such that the LMI of (19) holds with a delay that is half of the delay in (25).

6. EXAMPLE

We consider the example that is given in Gershon and Shaked [2019] in which the theory is limited to the application of a single predictor. We consider the system of (1a,b) and (2) with

$$A = \begin{bmatrix} 0.1 & 0.6 \pm a \\ -1 & 0.2 \end{bmatrix},\ \ D = \begin{bmatrix} 0 & 0.189 \\ 0 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.225 \\ 0.45 \end{bmatrix},\ \ B_2 = \begin{bmatrix} 0.04 \\ 0.05 \end{bmatrix} \text{ and } C_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$

where $D_{12} = [0\ \ \ 0.1]^T$ and $a \in [-0.2\ \ \ 0.2]$. We seek a state-feedback controller that stabilizes the system and achieves a negative J_E for a minimum γ. Considering the nominal case (i.e $a = 0$) and applying Theorem 1 in Gershon and Shaked [2019], where a single predictor is applied, it was found that the system can be stabilized for $h \le 0.63$ secs. For $h = 0.63$ secs., a stabilizing feedback gain $K = [-91.86\ \ -1.72]$ is obtained for $\epsilon = 1.1$, where $\gamma = 37.65$.

Applying the new approach where a set of two sub-predictors is applied, an upper-bound of $h = 0.8$ secs. is obtained by applying the result of Theorem 2. The latter solution was obtained for $\epsilon = 1.6$ where the minimum attenuation level is $\gamma = 32.21$. The controller gain is $K = [-183.34 \;\; 32.37]$ and the two predictors gains are:

$$L_1 = \begin{bmatrix} 0.727 & 0.185 \\ -0.308 & 0.888 \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 0.448 & 0.028 \\ -0.309 & 0.622 \end{bmatrix}.$$

Solving for the uncertain polytopic case, a maximal delay of $h = 0.32$ secs. is obtained for $\epsilon = 0.23$ by the state-feedback control solution in Gershon and Shaked [2013] which is based on augmenting the system dynamics to include the delayed input as a delayed state vector. Applying the result of Theorem 1 in Gershon and Shaked [2019], which is based on a single predictor, an upper bound of $h = 0.49$ secs. is obtained taking $\epsilon = 2.0$. Using the method of of Corollary 2 in this work, where two sub predictors are applied, an upper bound of $h = 0.62$ secs. is obtained taking $\epsilon = 2.1$.

7. CONCLUSIONS

In this work we bring the solution of the sub-predictors based state-feedback control of linear retarded continuous-time stochastic systems. The solution is obtained by utilizing the theory that is adopted for the solution of a single predictor control. The later solution is based on a preliminary result concerning the robust H_∞ Luenberger filter. In our systems, the multiplicative noise appears in the system dynamic matrix where the delay appears in the input of the system. The delay is assumed to be unknown and time-varying where only the bound on its size is given. The robust solution is obtained for both norm-bounded and polytopic-type uncertainties resulting in the latter case in a simple set of LMIs condition. The numerical example clearly demonstrates the improvement in the control performance achieved by applying a set of two sub-predictors compared to a single one. Obviously, one may expect to improve the results by a applying a multiple set of sub-predictors. We note that an inherent overdesign is admitted to our solution due to the use of the bounded operators which enable us to transform the retarded predictor-based system to a norm-bounded one in the final stage of the solution.

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