Distributed Model Predictive Control with Asymmetric Adaptive Terminal Sets for the Regulation of Large-scale Systems ⋆

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Abstract: In this paper, a novel distributed model predictive control (MPC) scheme with asymmetric adaptive terminal sets is developed for the regulation of large-scale systems with a distributed structure. Similar to typical MPC schemes, a structured Lyapunov matrix and a distributed terminal controller, respecting the distributed structure of the system, are computed offline. However, in this scheme, a distributed positively invariant terminal set is computed online and updated at each time instant taking into consideration the current state of the system. In particular, we consider ellipsoidal terminal sets as they are easy to compute for large-scale systems. The size and center of these terminal sets, together with the predicted state and input trajectories, are considered as decision variables in the online phase. The efficacy of the proposed scheme is illustrated in simulation by comparing it to a recent distributed MPC scheme with adaptive terminal sets.

Keywords: Predictive Control, Invariance, Distributed Control, Large-scale Systems, Adaptive Control

1. INTRODUCTION

Thanks to its flexibility, versatility and strong theoretical properties (Kouvaritakis and Cannon, 2016), Model Predictive Control (MPC) has been used over the past years in many practical applications such as robotics (Klančar and Škrjanc, 2007) and energy management (Scherer et al., 2014). Besides, many MPC variants have been developed including, but not limited to, robust MPC (Bemporad and Morari, 1999) and stochastic MPC (Mesbah, 2016). MPC is typically designed in a centralized fashion with one optimization problem solved for the whole controlled plant. For large-scale distributed systems such as power systems and water networks, centralized MPC may lead to communication and computational complications (Christofides et al., 2013). To overcome these difficulties, distributed MPC schemes have been developed to decompose the large-scale system into several smaller subsystems and design a local controller for each.

Due to the increasing interest in MPC, various efforts have been devoted to ensure the closed loop stability of plants controlled using MPC (Mayne et al., 2000). A well-known method for ensuring asymptotic stability and recursive feasibility is the addition of a terminal cost and/or a terminal constraint. This method has been extensively used for centralized MPC, see, for example, Rawlings and Muske (1993). It has also been extended to distributed MPC, by using a quadratic terminal cost and an ellipsoidal terminal set (Conte et al., 2012, 2016). In most cases, the terminal set is computed without taking the system's current state into account, possibly resulting in small regions of attraction. Recently, a distributed MPC scheme with adaptive terminal sets is proposed in Darivianakis et al. (2019). In this scheme, an ellipsoidal terminal set is determined and updated online based on the current state of the system, yielding a larger domain of attraction.

In this work, a novel distributed MPC scheme with asymmetric adaptive terminal sets is developed for regulating constrained large-scale linear time-invariant systems. One advantage of this approach over the one introduced in Darivianakis et al. (2019) is that the terminal set is not centered at the origin. Instead, the center of the terminal set, together with its size, are assumed to be decision variables to be determined online. The online computation of the terminal set center results generally in enlarging the feasible region. The terminal set invariance and constraint satisfaction are guaranteed through the addition of extra constraints formulated as linear matrix inequalities (LMIs) in the online optimal control problem. Unlike Darivianakis et al. (2019), the LMIs are derived using the linear state and input constraints directly without converting them to quadratic constraints which are more conservative. The effectiveness of this approach is evaluated by means of a simulation example.

In Section II, we formulate the distributed MPC problem. In Section III, we present the distributed MPC scheme with asymmetric adaptive terminal sets. In Section VI, we show how the proposed MPC problem is solved using distributed optimization techniques. Finally, a numerical simulation illustrates the efficacy of the proposed scheme in Section V.

Notation: Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}_+$ be the sets of real numbers, non-negative real numbers and non-negative natural numbers, respectively. Denote the transpose of a vector v by $v^\top$ and its norm by $||v||$. Let $||v||_P = \sqrt{v^\top P v}$ be the weighted norm of the vector v using the matrix P. The matrix $P = \text{diag}(P_1, ..., P_M)$ denotes

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a diagonal matrix with the submatrices P_i, $i \in \{1,...,M\}$, along its diagonal. Let $\mathcal{X} \times \mathcal{Y}$ denote the cartesian product of the two sets $\mathcal{X}$ and $\mathcal{Y}$ and $\times_{i\in\{1,...,M\}}\mathcal{X}_i$ the cartesian product of the sets $\mathcal{X}_i$ for all $i \in \{1,...,M\}$.

2. PROBLEM FORMULATION

We consider a large-scale dynamical system which admits a separable structure and thus, can be decompsed into M subsystems. For each subsystem $i \in \{1,...,M\}$, a set $\mathcal{N}_i$ of neighbours is defined comprising subsystem i itself as well as all other subsystems coupled with subsystem i through the dynamics and/or the constraints. Each subsystem i is described as a discrete-time linear time-invariant system given by

$$x_i(t+1) = A_{\mathcal{N}_i} x_{\mathcal{N}_i}(t) + B_i u_i(t), \tag{1}$$

where $t \in \mathbb{N}_+$ is the time index, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $x_{\mathcal{N}_i} \in \mathbb{R}^{n_{N_i}}$ are the state vector of subsystem i, the input vector of subsystem i and the state vector of the neighbours of subsystem i respectively. The system matrices $A_{\mathcal{N}_i} \in \mathbb{R}^{n_i \times n_{N_i}}$ and $B_i \in \mathbb{R}^{n_i \times m_i}$ are assumed to be known. The state and input constraint sets of each subsystem are given by

$$\begin{aligned} x_{\mathcal{N}_i}(t) \in \mathcal{X}_{\mathcal{N}_i} &= \{x_{\mathcal{N}_i} \in \mathbb{R}^{n_{N_i}} : G_{\mathcal{N}_i} x_{\mathcal{N}_i} \leq g_{\mathcal{N}_i}\}, \\ u_i(t) \in \mathcal{U}_i &= \{u_i \in \mathbb{R}^{m_i} : H_i u_i \leq h_i\}, \end{aligned} \tag{2}$$

where the constraints matrices $G_{\mathcal{N}_i} \in \mathbb{R}^{q_i \times n_{N_i}}$, $H_i \in \mathbb{R}^{r_i \times m_i}$ and vectors $g_{\mathcal{N}_i} \in \mathbb{R}^{q_i}$, $h_i \in \mathbb{R}^{r_i}$ are assumed to be known. The origin is assumed to be contained in the interior of the constraint set. We also assume that the inputs of the different subsystems are coupled neither through the dynamics, nor through the constraints; indeed this assumption can be imposed without loss of generality, because inputs can always be decoupled by introducing new auxiliary variables (Darivianakis et al., 2019).

Our main aim is to regulate the system to the origin. We therefore impose a quadratic cost function in the states and the inputs. To maintain the distributed structure of the optimal control problem, the local cost function of subsystem i is assumed to be a function of the states of the neighbours of subsystem i and the inputs of subsystem i. Let $T \in \mathbb{N}_+$ be the prediction horizon. Therefore, the local cost function of subsystem i is designed to be

$$J_i = \sum_{t=0}^{T-1} \left[x_{\mathcal{N}_i}(t)^\top Q_{\mathcal{N}_i} x_{\mathcal{N}_i}(t) + u_i(t)^\top R_i u_i(t) \right] + x_i(T)^\top P_i x_i(T), \tag{3}$$

where $Q_{\mathcal{N}_i} \in \mathbb{R}^{n_{N_i} \times n_{N_i}}$ and $R_i \in \mathbb{R}^{m_i \times m_i}$ are the local cost function matrices and $P_i \in \mathbb{R}^{n_i \times n_i}$ is the local terminal cost matrix.

Denoting the global state and input vectors of the whole system as $x = [x_1^\top, ..., x_M^\top]^\top \in \mathbb{R}^n$ and $u = [u_1^\top, ..., u_M^\top]^\top \in \mathbb{R}^m$ respectively, the mappings $U_i \in \{0,1\}^{n_i \times n}$, $W_{\mathcal{N}_i} \in \{0,1\}^{n_{N_i} \times n}$ and $V_i \in \{0,1\}^{m_i \times m}$ can be defined to relate the local variables of subsystem i to the global variables as follows,

$$x_i = U_i x, \quad x_{\mathcal{N}_i} = W_{\mathcal{N}_i} x, \quad u_i = V_i u. \tag{4}$$

Note that the cost function matrices $Q_{\mathcal{N}_i}$, R_i and P_i are selected as in Darivianakis et al. (2019) where $Q = \sum_{i=1}^M W_{\mathcal{N}_i}^\top Q_{\mathcal{N}_i} W_{\mathcal{N}_i}$ is positive semidefinite and the pair (A, Q) is observable, $R = \sum_{i=1}^M U_i^\top R_i U_i$ is positive definite, $P = \sum_{i=1}^M U_i^\top P_i U_i$ is positive definite and $A = [(A_{\mathcal{N}_1} W_{\mathcal{N}_1})^\top, ..., (A_{\mathcal{N}_M} W_{\mathcal{N}_M})^\top]^\top$.

To ensure the asymptotic stability of the closed-loop system and the recursive feasibility of the proposed distributed MPC, the final state $x_i(T)$ of each subsystem i is constrained to lie in an ellipsoidal terminal set as follows,

$$x_i(T) \in \mathcal{X}_{f,i} = \{x_i \in \mathbb{R}^{n_i} : (x_i - c_i)^\top P_i (x_i - c_i) \leq \alpha_i\}, \tag{5}$$

where $\alpha_i \in \mathbb{R}$ represents the size of the terminal set and $c_i \in \mathbb{R}^{n_i}$ represents the center of the terminal set. This ellipsoidal terminal set is required to be invariant under the terminal controller $u_{f,i} = K_{\mathcal{N}_i} x_{\mathcal{N}_i}$. Thus, assuming that $\mathscr{X}_{f,i}(K_{\mathcal{N}_i})$ is the set of ellipsoidal terminal sets which are invariant under the terminal controller $u_{f,i}$, we impose the constraint

$$\mathcal{X}_{f,i} \in \mathscr{X}_{f,i}(K_{\mathcal{N}_i}). \tag{6}$$

We assume that the terminal control gain $K_{\mathcal{N}_i}$ and the matrix P_i have been designed offline and we seek c_i and α_i online such that $\mathcal{X}_{f,i}$ satisfies (6).

In conclusion, the global cooperative online optimal control problem is formulated as

$$\begin{aligned} &\min \sum_{i=1}^M J_i \\ &s.t. \begin{cases} \left.\begin{aligned} &x_i(t+1) = A_{\mathcal{N}_i} x_{\mathcal{N}_i} + B_i u_i, \\ &x_{\mathcal{N}_i}(t) \in \mathcal{X}_{\mathcal{N}_i}, \ u_i(t) \in \mathcal{U}_i, \end{aligned}\right\} \begin{aligned} &\forall t \in \{0,...,T\}, \\ &\forall i \in \{1,...,M\}, \end{aligned} \\ \left.\begin{aligned} &x_i(0) = x_{i,0}, \ x_i(T) \in \mathcal{X}_{f,i}, \\ &\mathcal{X}_{f,i} \in \mathscr{X}_{i,f}(K_{\mathcal{N}_i}), \end{aligned}\right\} \forall i \in \{1,...,M\}, \end{cases} \end{aligned} \tag{7}$$

where $x_{i,0} \in \mathbb{R}^{n_i}$ is the current state of subsystem i. The decision variables of this optimal control problem are the predicted state trajectory $x_i(t)$ for all $i \in \{1,...,M\}$ and $t \in \{0,...,T\}$, the predicted input trajectory $u_i(t)$ for all $i \in \{1,...,M\}$ and $t \in \{0,...,T-1\}$, the terminal set size α_i for all $i \in \{1,...,M\}$ and the terminal set center c_i for all $i \in \{1,...,M\}$. Note that the terminal set depends on the system state because it is computed in the online optimal control problem whose solution is a function of the current state of the system. The last constraint in (7) is ensured by means of convex optimization tools in the next section.

Note that, in the above MPC formulation, the systems matrices $A_{\mathcal{N}_i}$, B_i, the constraint matrices $G_{\mathcal{N}_i}$, H_i, the constraint vectors $g_{\mathcal{N}_i}$, h_i, the cost function matrices $Q_{\mathcal{N}_i}$, R_i are all known for all $i \in \{1,...,M\}$. Note also that the terminal cost matrix P_i and the stabilizing terminal controller $u_{f,i}$ need to be computed appropriately offline to ensure asymptotic stability and recursive feasibility. To compute these terminal ingredients, we follow the method in Conte et al. (2012, 2016) where the terminal cost matrices P_i are computed such that $P = \text{diag}(P_1, ..., P_M) \in \mathbb{R}^{n \times n}$ is a Lyapunov matrix of (1) under the terminal controller $u_{f,i}$.

3. DISTRIBUTED MPC SCHEME

In this section, we modify the online optimal control problem (7) by replacing the last constraint with a set of LMIs involving the terminal set size and center to ensure the positive invariance of the terminal set and consequently, the asymptotic stability of the closed-loop system. The following proposition shows the conditions to ensure the positive invariance of terminal sets.

Proposition 1. (Darivianakis et al. (2019)). Define the sets $\mathcal{X}_{f,\mathcal{N}_i} = \times_{j\in\mathcal{N}_i}\mathcal{X}_{f,j}$. Each local terminal set $\mathcal{X}_{f,i}$ is positively invariant if for each $i \in \{1,...,M\}$ and for all $x_{\mathcal{N}_i} \in \mathcal{X}_{f,\mathcal{N}_i}$,

$$(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i}) x_{\mathcal{N}_i} \in \mathcal{X}_{f,i}, \tag{8a}$$

$$x_{\mathcal{N}_i} \in \mathcal{X}_{\mathcal{N}_i}, \tag{8b}$$

$$K_{\mathcal{N}_i} x_{\mathcal{N}_i} \in \mathcal{U}_i. \tag{8c}$$

Consequently, the global terminal set $\mathcal{X}_f = \times_{i\in\{1,...,M\}}\mathcal{X}_{f,i}$ is positively invariant.

Condition (8a) ensures that the terminal set $\mathcal{X}_{f,i}$ is invariant. Whereas, conditions (8b) and (8c) show that all the state and input constraints are satisfied inside the terminal set respectively. In the sequel, LMIs are derived for each of the conditions in Proposition 1. Embedding these LMIs in the online optimal control problem (7) guarantees the positive invariance of the terminal set. The derived LMIs depend on the following quantities: $\alpha = \text{diag}(\alpha_1 I_{n_1},..,\alpha_i I_{n_i},..,\alpha_M I_{n_M})$, $c = [c_1^\top,..,c_i^\top,..,c_M^\top]^\top$, $\alpha_{\mathcal{N}_i} = W_{\mathcal{N}_i}\alpha W_{\mathcal{N}_i}^\top$ and $c_{\mathcal{N}_i} = W_{\mathcal{N}_i} c$.

Condition (8a) can be represented using an LMI as shown in the following proposition; the inequality (10) to which we refer in this proposition is found overleaf in single column.

Proposition 2. For each subsystem $i \in \{1,...,M\}$, the terminal set invariance condition

$$[(A_{\mathcal{N}_i}+B_iK_{\mathcal{N}_i})x_{\mathcal{N}_i} - c_i]^\top P_i[(A_{\mathcal{N}_i}+B_iK_{\mathcal{N}_i})x_{\mathcal{N}_i} - c_i] \le \alpha_i, \quad \forall j \in \mathcal{N}_i,\ x_j \ni (x_j - c_j)^\top P_j (x_j - c_j) \le \alpha_j, \tag{9}$$

holds if there exist $\lambda_{ij} \ge 0$ such that (10) holds.

Proof. The proof is found in the Appendix.

Condition (8b) can be represented as an LMI as shown in the following proposition.

Proposition 3. Denote the k^{th} row of the matrix $G_{\mathcal{N}_i}$ by $G_{\mathcal{N}_i}^k$ and the k^{th} element of the vector $g_{\mathcal{N}_i}$ by $g_{\mathcal{N}_i}^k$. For each subsystem $i \in \{1,...,M\}$, the state constraint $k \in \{1,2,...,q_i\}$

$$G_{\mathcal{N}_i}^k x_{\mathcal{N}_i} \le g_{\mathcal{N}_i}^k, \quad \forall j \in \mathcal{N}_i,\ x_j \ni (x_j - c_j)^\top P_j (x_j - c_j) \le \alpha_j, \tag{11}$$

holds if there exist $\sigma_{ij}^k \ge 0$ such that

$$\begin{bmatrix} \sum_{j\in\mathcal{N}_i} \sigma_{ij}^k P_{ij} & \frac{1}{2}\alpha_{\mathcal{N}_i}^{1/2} G_{\mathcal{N}_i}^{k\top} \\ \frac{1}{2} G_{\mathcal{N}_i}^k \alpha_{\mathcal{N}_i}^{1/2} & g_{\mathcal{N}_i}^k - G_{\mathcal{N}_i}^k c_{\mathcal{N}_i} - \sum_{j\in\mathcal{N}_i} \sigma_{ij}^k \end{bmatrix} \ge 0. \tag{12}$$

Proof. The proof is found in the Appendix.

Condition (8c) can be represented as an LMI as shown in the following proposition.

Proposition 4. Denote the l^{th} row of the matrix $H_{\mathcal{N}_i}$ by $H_{\mathcal{N}_i}^l$ and the l^{th} element of the vector $h_{\mathcal{N}_i}$ by $h_{\mathcal{N}_i}^l$. For each subsystem $i \in \{1,...,M\}$, the input constraint $l \in \{1,2,...,r_i\}$

$$H_{\mathcal{N}_i}^l K_{\mathcal{N}_i} x_{\mathcal{N}_i} \le h_{\mathcal{N}_i}^l,\ \forall j \in \mathcal{N}_i,\ x_j \ni (x_j - c_j)^\top P_j (x_j - c_j) \le \alpha_j,$$

holds if there exist $\beta_{ij}^l \ge 0$ such that

$$\begin{bmatrix} \sum_{j\in\mathcal{N}_i} \beta_{ij}^l P_{ij} & \frac{1}{2}\alpha_{\mathcal{N}_i}^{1/2} K_{\mathcal{N}_i}^\top H_{\mathcal{N}_i}^{l\top} \\ \frac{1}{2} H_{\mathcal{N}_i}^l K_{\mathcal{N}_i} \alpha_{\mathcal{N}_i}^{1/2} & h_{\mathcal{N}_i}^l - H_{\mathcal{N}_i}^l K_{\mathcal{N}_i} c_{\mathcal{N}_i} - \sum_{j\in\mathcal{N}_i} \beta_{ij}^l \end{bmatrix} \ge 0. \tag{13}$$

Proof. The proof follows that of Proposition 3 by replacing σ_{ij}^k, $g_{\mathcal{N}_i}^k$ and $G_{\mathcal{N}_i}^k$ with β_{ij}^l, $h_{\mathcal{N}_i}^l$ and $H_{\mathcal{N}_i}^l K_{\mathcal{N}_i}$ respectively.

Notice that the center c_i and the square root of the size α_i of each local terminal set are considered as decision variables without affecting the convexity of the problem. However, it is not possible to achieve convex conditions, and thus a convex optimization problem, when considering the terminal control gain $K_{\mathcal{N}_i}$ as a decision variable. This fact is due to the existence of the bilinear terms $K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{0.5}$ and $K_{\mathcal{N}_i} c_{\mathcal{N}_i}$ which would result in a nonconvex problem if the gain $K_{\mathcal{N}_i}$ is assumed to be a decision variable. Thus, the terminal control gain is computed offline using the method in Conte et al. (2012).

Note that the S-Lemma parameters introduced in Propositions 2, 3 and 4 have to be non-negative for these propositions to hold. Thus, for all $i \in \{1,...,M\},\ j \in \mathcal{N}_i,\ k \in \{1,...,q_i\}$ and $l \in \{1,...,r_i\}$, the following constraints are imposed in the online optimal control problem,

$$\lambda_{ij} \ge 0,\ \sigma_{ij}^k \ge 0,\ \beta_{ij}^l \ge 0. \tag{14}$$

Finally, recall that the final state $x_i(T)$ has to satisfy the constraint $(x_i(T) - c_i)^\top P_i (x_i(T) - c_i) \le \alpha_i$. By means of the Schur complement (Boyd et al. (1994)), an equivalent form to this constraint can be formulated as

$$\begin{bmatrix} P_i^{-1}\alpha_i^{1/2} & x_i(T) - c_i \\ (x_i(T) - c_i)^\top & \alpha_i^{1/2} \end{bmatrix} \ge 0. \tag{15}$$

In conclusion, the online optimal control problem of the proposed distributed MPC scheme is given by

$$\begin{aligned} &\min \sum_{i=1}^{M} J_i \\ &s.t. \begin{cases} (1),(2)\ \forall t \in \{0,1,...,T\},\ \forall i \in \{1,...,M\}, \\ x_i(0) = x_{i,0},\ (10),\ (15),\ \forall i \in \{1,...,M\}, \\ (12)\ \forall k \in \{1,...,q_i\},\ \forall i \in \{1,...,M\}, \\ (13)\ \forall l \in \{1,...,r_i\},\ \forall i \in \{1,...,M\}, \\ (14)\ \forall i \in \{1,..,M\},\ j \in \mathcal{N}_i,\ k \in \{1,..,q_i\},\ l \in \{1,..,r_i\}. \end{cases} \end{aligned} \tag{16}$$

To maintain the convexity of (16), we consider the square root of the terminal set size (and not the terminal set size itself) as a decision variable. The following theorem shows the recursive feasibility of the proposed MPC scheme and the asymptotic stability of the closed-loop system.

Theorem 5. The distributed MPC problem with asymmetric adaptive terminal sets is recursively feasible and the closed-loop system under this MPC controller is asymptotically stable.

Proof. The proof follows that of Theorem 3 in Darivianakis et al. (2019).

Normally, the online optimal control problem is formulated as a quadratic program (QP) with a polytopic terminal set or a quadratically-constrained-quadratic program (QCQP) with an ellipsoidal terminal set. The feasible region in this case is enlarged by choosing longer predicition horizons leading to an increase in the number of decision variables and constraints. In addition, the number of consensus variables to be agreed on by the neighbours also increases. On the other side, although the resulting optimal control problem of the proposed approach is a semidefinite program (SDP), shorter prediction horizons can be possibly used. Note also that, following Darivianakis et al. (2019), the terminal set can be computed in the first timestep only and then enforced for the next timesteps. In this case, there is no need to recalculate the terminal set in each timestep. Thus, an SDP is solved in the first timestep, whereas a QCQP is solved for the rest of the simultion. Furthermore, if the initial condition is known a priori, the SDP can be solved offline once and the online optimal control problem becomes a QCQP which

$$\begin{bmatrix} P_i^{-1}\alpha_i^{1/2} & (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2}+B_iK_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2}) & [(A_{\mathcal{N}_i}+B_iK_{\mathcal{N}_i})c_{\mathcal{N}_i}-c_i] \\ (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2}+B_iK_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2})^\top & \sum\limits_{j\in\mathcal{N}_i}\lambda_{ij}P_{ij} & 0 \\ (A_{\mathcal{N}_i}+B_iK_{\mathcal{N}_i})c_{\mathcal{N}_i}-c_i]^\top & 0 & \alpha_i^{1/2}-\sum\limits_{j\in\mathcal{N}_i}\lambda_{ij} \end{bmatrix} \geq 0. \quad (10)$$

is the same program used when fixed ellipsoidal terminal sets are considered.

4. DISTRIBUTED IMPLEMENTATION

The global cooperative online optimal control problem (16) can be solved using distributed optimization techniques. In this section, we show how this problem can be solved using the alternating direction method of multipliers (ADMM). This alogrithm is iterative and each iteration is mainly composed of three steps which are summarized in this section. For more details about this algorithm, see Boyd et al. (2011).

Let $x_{j|i}$, $\alpha_{j|i}$ and $c_{j|i}$ be local copies of the state, terminal set size and center of the subsystems $j \in \mathcal{N}_i$ computed by subsystem i respectively. Let $z_x \in \mathbb{R}^n$, $z_\alpha \in \mathbb{R}^n$ and $z_c \in \mathbb{R}^n$ be global copies comprising the state, terminal set size and center of the whole system respectively. Let the local copy $(\cdot)_{j|i}$ correspond to the component $z_{(\cdot)_{\delta(i,j)}}$ of the global copy using the map $\delta(\cdot,\cdot)$. This map is used because several local copies may correspond to the same component of the global copy (e.g. $\delta(i,j)=\delta(k,j)$ if $\{i,k\}\subseteq\mathcal{N}_j$). The goal is that the components of the global copy and the corresponding local copies reach consensus and converge to the centralized solution of (16). We also define the lagrange multipliers $\gamma_{x_{j|i}}$, $\gamma_{\alpha_{j|i}}$ and $\gamma_{c_{j|i}}$. In the first step of iteration $\tau+1$, each subsystem i computes $u_i^{\tau+1}(t)$, $x_{j|i}^{\tau+1}(t)$, $\alpha_{j|i}^{\tau+1}$, $c_{j|i}^{\tau+1}$, $\lambda_{ij}^{\tau+1}$, $\sigma_{ij}^{k^{\tau+1}}$, $\beta_{ij}^{l^{\tau+1}}$ $\forall j\in\mathcal{N}_i$, $k\in\{1,..,q_i\}$, $l\in\{1,..,r_i\}$ by solving the optimization problem

$$\min J_i + \sum_{j\in\mathcal{N}_i}\Bigg[\sum_{t=0}^{T}\Big(\gamma^\tau_{x_{j|i}}(t)x_{j|i}(t)+\frac{\rho}{2}(x_{j|i}(t)-z^\tau_{x_{\delta(i,j)}}(t))^2\Big)+$$
$$\gamma^\tau_{\alpha_{j|i}}\alpha_{j|i}^{\frac{1}{2}}+\gamma^\tau_{c_{j|i}}c_{j|i}+\frac{\rho}{2}\left(\alpha_{j|i}^{\frac{1}{2}}-\sqrt{z^\tau_{\alpha_{\delta(i,j)}}}\right)^2+\frac{\rho}{2}(c_{j|i}-z^\tau_{c_{\delta(i,j)}})^2\Bigg]$$
$$s.t. \begin{cases} (1),\ (2)\ \forall t\in\{0,1,...,T\}, \\ x_i(0)=x_{i,0},\ (10),\ (15), \\ (12),\ (13),\ (14)\ \forall j\in\mathcal{N}_i,\ k\in\{1,...,q_i\},\ l\in\{1,...,r_i\}, \end{cases}$$

where ρ is the step size and x_j, α_j and c_j are replaced by $x_{j|i}$, $\alpha_{j|i}$ and $c_{j|i}$ in (1), (2), (10), (12), (13), (14), (15) and the definition of J_i. In the second step, subsystem i updates the component $z_{c_{\delta(i|i)}}$ of the global copy as follows,

$$z^{\tau+1}_{c_{\delta(i,i)}} = \text{argmin}\left(\sum_{j\in\mathcal{N}_i}-\gamma^\tau_{c_{i|j}}z_{c_{\delta(i,i)}}+\frac{\rho}{2}\sum_{j\in\mathcal{N}_i}(c^{\tau+1}_{i|j}-z_{c_{\delta(i,i)}})^2\right).$$

The same applies to $z_{x_{\delta(i,i)}}(t)$ and $\sqrt{z_{\alpha_{\delta(i,i)}}}$. Finally, subsystem i updates the lagrange multipliers corresponding to its local variables as follows,

$$\gamma^{\tau+1}_{c_{j|i}} = \gamma^\tau_{c_{j|i}}+\rho(c^{\tau+1}_{j|i}-z^{\tau+1}_{c_{\delta(i,j)}}).$$

The same applies to $\gamma_{x_{j|i}}(t)$ and $\gamma_{c_{j|i}}$. Note that $z^0_{(\cdot)_{\delta(i,j)}}$ and $\gamma^0_{(\cdot)_{j|i}}$ are initialized so that the first step of the first iteration can be implemented. Note also that we terminate the program after a fixed number of iterations in this work even though there exist more advanced techniques which can be used for termination.

5. SIMULATION RESULTS

In this section, the effectiveness of the proposed distributed MPC scheme with asymmetric adaptive terminal set (16) (denoted by D-ASYM) is illustrated by means of a simulation example. This scheme is also compared to the distributed MPC scheme with adaptive terminal set (denoted by D-ADAP) developed in Darivianakis et al. (2019).

We consider the discrete-time linear time-invariant system

$$x_1^+ = 2x_1+0.5x_2,\quad x_M^+ = 0.5x_{M-1}+2x_M,$$
$$x_i^+ = 0.5x_{i-1}+2x_i+0.5x_{i+1}, \forall\ i\in\{2,...,M-1\}.$$

The state and input constraints of this system are represented as

$$-5\leq x_i\leq 5,\quad -0.25\leq u_i\leq 1,\quad \forall\ i\in\{1,...,M\}.$$

The system and constraint matrices in (1) and (2) can be derived accordingly. The cost function matrices are selected such that $Q=I_n$ and $R=I_m$. The terminal cost and controller can then be computed based on the work in Conte et al. (2012). Two test scenarios are considered in this paper. In the first one, we consider the case in which $M=2$ to compare D-ASYM and D-ADAP, whereas we consider the case in which $M=9$ in the second one to demonstrate the efficacy of the proposed approach for relatively large-scale systems.

Figure 1 shows the result of the first test scenario. In particular, the predicted state trajectory (refered to as PT) and the terminal set (refered to as TS) of the two distributed MPC schemes; D-ASYM and D-ADAP, are shown with a prediction horizon $T=2$ for three different initial conditions when the optimization problem is solved once. Notice that the terminal set is described by a rectangle and not an ellipsoid because it is the product of two ellipsoidal sets in one dimension.

The online optimal control problem is initially feasible for both schemes and the optimal solutions are the same when the initial condition is $x_0=[-0.1\ \ -0.4]^\top$. The online optimal control problem is still feasible for both schemes when $x_0=[-0.7\ \ 0.45]^\top$. However, the optimal solutions are different reflecting the conservativeness of D-ADAP. Finally, for $x_0=[-0.6\ \ -0.6]^\top$, only D-ASYM is initially feasible showing that its domain of attraction is possibly larger than that of D-ADAP.

Table 1 shows the cost function value for the different schemes and initial conditions when the optimization problem is solved recursively for 10 timesteps. When the initial condition is $x_0=[-0.1\ -0.4]^\top$, the cost of both schemes is the same because the optimal state and input trajectories are the same independent of the scheme applied. In the case of $x_0=[-0.7\ 0.45]^\top$, the cost of D-ADAP is higher than that of D-ASYM showing the realtive suboptimality of D-ADAP with respect to D-ASYM. Finally, for $x_0=[-0.6\ \ -0.6]^\top$, no value is shown for D-ADAP since the online optimal control problem is infeasible.

Figure 2 shows the result of the second test scenario when $M=9$. In the left figure, the state trajectories of the third and sixth subsystems are shown when D-ASYM is implemented recursively for ten timesteps with a prediction horizon $T=2$.

Table 1. Value of cost function for different schemes and initial conditions

Initial Conditions	$x_0 = \begin{bmatrix}-0.1\\-0.4\end{bmatrix}$	$x_0 = \begin{bmatrix}-0.7\\0.45\end{bmatrix}$	$x_0 = \begin{bmatrix}-0.6\\-0.6\end{bmatrix}$
D-ADAP	0.2528	1.1562	-
D-RLXD	0.2528	1.1071	1.8185

As shown, the system states converge to the origin illustrating Theorem 5. In the middle figure, the ADMM trajectories of the states of the third and sixth subsystems are shown. In particular, the trajectories of the variables $z_{x_{\delta(3,3)}}(1)$ and $z_{x_{\delta(6,6)}}(1)$ at the first timestep are shown. Note that the states computed using distributed optimization (i.e. ADMM) converge to the same values obtained by solving the whole optimization problem centrally. In other words, the optimal state trajectories are reached using distributed optimization. In the right figure, the ADMM trajectories of the terminal set size and center of the fifth subsystem at the first timestep is shown. In particular, the local copies of these variables computed by all the neighbours of the 5^{th} subsystem and the corresponding global copies are shown. Although more iterations are required so that the local and global copies converge to the steady-state value, these copies reach consensus. Note that the program can be safely terminated in this case since the optimal trajectories are reached and the terminal set size and center reach consensus which means that all the constraints are satisfied.

6. CONCLUSION

A novel distributed MPC scheme is proposed where the terminal set size and center are determined online. The terminal set positive invariance is ensured by imposing additional constraints in the MPC problem. The proposed approach is compared to a recently-proposed one in the literature. Extensions to nonlinear systems are to be considered for future work.

APPENDIX

Proof of Proposition 2

The inequalities (18), (19) and (20) to which we refer in this proof are found overleaf in single column

Define an auxiliary vector $s_i \in \mathbb{R}^{n_i}$ for each subsystem's state vector x_i as follows,

$$x_i = c_i + \alpha_i^{1/2} s_i. \tag{17}$$

By concatenation, the relation $x_{\mathcal{N}_i} = c_{\mathcal{N}_i} + \alpha_{\mathcal{N}_i}^{1/2} s_{\mathcal{N}_i}$ also holds, By substituting these auxiliary vectors in (9), the invariance condition is written as

$$\begin{aligned} & s_{\mathcal{N}_i}^\top (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2})^\top P_i (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2}) s_{\mathcal{N}_i} \\ & + 2[(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i]^\top P_i (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2}) s_{\mathcal{N}_i} \\ & + [(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i]^\top P_i [(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i] \le \alpha_i, \\ & \qquad \forall j \in \mathcal{N}_i,\ s_j \ni s_j^\top P_j s_j \le 1. \end{aligned}$$

Using the mapping equations in (4) and multiplying the above equation by $\alpha_i^{-1/2}$ gives the condition (18) where $P_{ij} = W_i U_j^\top P_j U_j W_i^\top$. By applying the S-procedure (Boyd et al., 1994) to (18), the invariance condition for each subsystem $i \in \{1,...,M\}$ holds if there exist $\lambda_{ij} \ge 0, j \in \mathcal{N}_i$ such that (19) holds. Equation (19) can be rearranged as shown in (20). Applying Schur's complement (Boyd et al., 1994) to (20) leads to the linear matrix inequality (10).

Proof of Proposition 3

Consider the auxiliary vectors s_i defined in (17) and the concatenated auxiliary vectors $s_{\mathcal{N}_i}$. Substituting these auxiliary vectors in (11), the state constraints become

$$G_{\mathcal{N}_i}^k (c_{\mathcal{N}_i} + \alpha_{\mathcal{N}_i}^{1/2} s_{\mathcal{N}_i}) \le g_{\mathcal{N}_i}^k, \quad \forall j \in \mathcal{N}_i,\ s_j \ni s_j^\top P_j s_j \le 1.$$

Using the mapping equations in (4), the above implication can be expressed as

$$G_{\mathcal{N}_i}^k \alpha_{\mathcal{N}_i}^{1/2} s_{\mathcal{N}_i} + G_{\mathcal{N}_i}^k c_{\mathcal{N}_i} \le g_{\mathcal{N}_i}^l, \quad \forall j \in \mathcal{N}_i,\ s_{\mathcal{N}_i}^\top P_{ij} s_{\mathcal{N}_i} \le 1.$$

Applying the S-procedure Boyd et al. (1994) to the above implication yields

$$\sum_{j \in \mathcal{N}_i} \sigma_{ij}^k \begin{bmatrix} P_{ij} & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2}\alpha_{\mathcal{N}_i}^{1/2} G_{\mathcal{N}_i}^{k\top} \\ \frac{1}{2} G_{\mathcal{N}_i}^k \alpha_{\mathcal{N}_i}^{1/2} & G_{\mathcal{N}_i}^k c_{\mathcal{N}_i} - g_{\mathcal{N}_i}^k \end{bmatrix} \ge 0.$$

Rearranging the above LMI results in (12).

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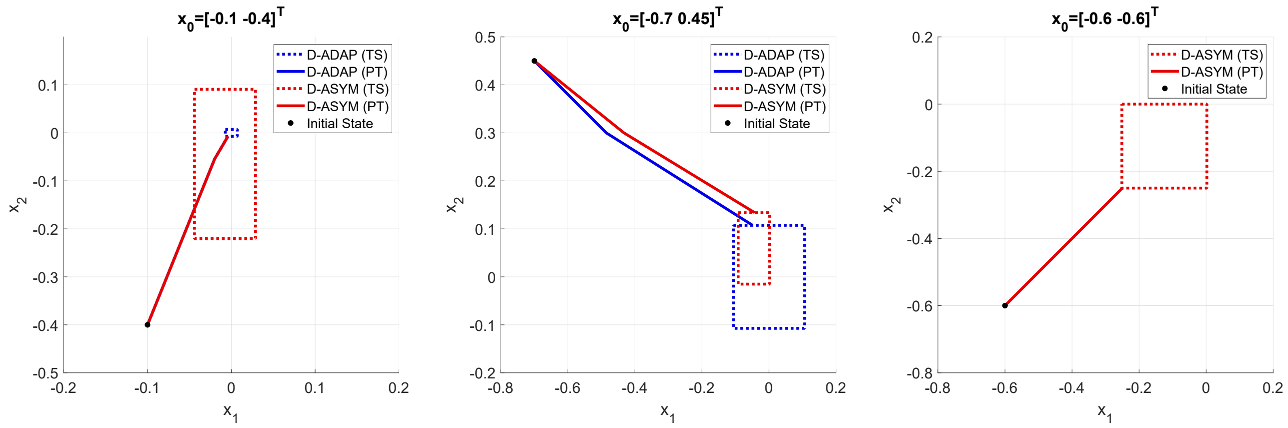

Fig. 1. Predicted state trajectories (PT) and terminal sets (TS) of two distributed MPC schemes; D-ADAP (Blue) and D-ASYM (Red) for three different initial conditions and a prediction horizon of $T = 2$ when solving the optimization problem once

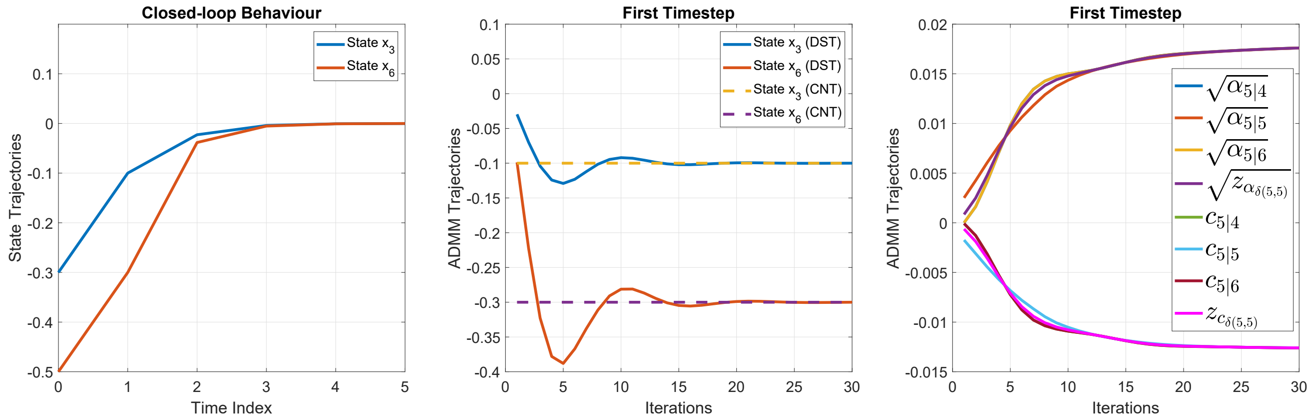

Fig. 2. (left) Optimal state trajectories of the third and sixth subsystems starting from $x_0 = [-0.3\ -0.5\ -0.3\ -0.5\ -0.3\ -0.5\ -0.3\ -0.5\ -0.3]^\top$ with a prediction horizon $T = 2$ when solving the optimization problem recursively, (middle) ADMM iterations of the global copies of the states of the third and sixth subsystems at the first timestep and the corresponding centralized solutions, (right) ADMM iterations of the local and global copies of the terminal set size and center of the fifth subsystem at the first timestep

$$
\begin{aligned}
& s_{\mathcal{N}_i}^\top (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2})^\top P_i \alpha_i^{-1/2} (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2}) s_{\mathcal{N}_i} \\
& \quad + 2[(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i]^\top P_i \alpha_i^{-1/2} (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2}) s_{\mathcal{N}_i} \\
& \qquad + [(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i]^\top P_i \alpha_i^{-1/2} [(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i] \leq \alpha_i^{1/2}, \quad \forall j \in \mathcal{N}_i,\ s_{\mathcal{N}_i} \ni s_{\mathcal{N}_i}^\top P_{ij} s_{\mathcal{N}_i} \leq 1.
\end{aligned} \tag{18}
$$

$$\Downarrow$$

$$
\sum_{j \in \mathcal{N}_i} \lambda_{ij} \begin{bmatrix} P_{ij} & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2})^\top P_i \alpha_i^{-1/2} (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2}) & (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2})^\top P_i \alpha_i^{-1/2} [(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i] \\ [(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i]^\top P_i \alpha_i^{-1/2} (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2}) & [(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i]^\top P_i \alpha_i^{-1/2} [(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i] - \alpha_i^{1/2} \end{bmatrix} \geq 0. \tag{19}
$$

$$\Updownarrow$$

$$
\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \lambda_{ij} P_{ij} & 0 \\ 0 & \alpha_i^{1/2} - \sum_{j \in \mathcal{N}_i} \lambda_{ij} \end{bmatrix} - \begin{bmatrix} (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2})^\top \\ (A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i]^\top \end{bmatrix} P_i \alpha_i^{-1/2} \left[(A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2})\ \ [(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i] \right] \geq 0. \tag{20}
$$