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On the Converse Passivity Theorems for LTI Systems ⋆

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Abstract: Converse passivity theorems are established for finite-dimensional (FD) linear time-invariant (LTI) systems. Consider an FD LTI system G_1 interconnected in positive feedback with another FD LTI system G_2. It is demonstrated that when the closed-loop system is (robustly) stable (in the sense of finite $\boldsymbol{L}_2$ gain) for arbitrary strictly passive G_2, then $-G_1$ must necessarily be passive. It is also demonstrated that when the closed-loop system is uniformly stable across the set of arbitrary passive G_2, then $-G_1$ must necessarily be strictly passive. The proofs are constructive; i.e., we show how to find a de-stabilizing FD LTI G_2 when G_1 violates the necessity condition of stability.

Keywords: Passivity, Converse Theorem, LTI Systems

1. INTRODUCTION

Passivity is pervasive in the field of systems and control [Willems 1972, Lozano et al. 2013, van der Schaft 2017]. It is a vitally important notion in the study of electronic circuits [Anderson and Vongpanitlerd 1973] and process control [Bao and Lee 2007]. The chief rationale behind its importance is that passivity is deep-rooted in physical systems modelling through the use of generalized energy and power, as is commonly witnessed in chemical processes and electromechanical systems. The mathematics underpinning passivity has found generalizations to the broad area of control, ranging from linear robust control to nonlinear system stabilization.

Passivity has also enjoyed great success when applied to the study of large-scale network systems. Its compositional property is particularly attractive in this regards — a negative feedback configuration is passive if both the subsystems are passive. This is often utilized in the derivation of distributed and scalable performance certifications for large-scale systems [Moylan and Hill 1978, Vidyasagar 1981, Arcak et al. 2016].

Despite the ubiquity of passivity in systems theory, converse results on passivity have rarely been investigated. One version of converse passivity result takes the following form: a negative feedback configuration is passive only if both the subsystems are passive. This is established in [Kerber and van der Schaft 2011] using state-space methods and subsequently in [Khong and van der Schaft 2017] from the input-output perspective. By contrast, this paper is concerned with the converse of passivity-based robust stability results. We prove in the multi-input-multi-output (MIMO), finite-dimensional (FD), linear time-invariant (LTI) setting that if a closed-loop system is stable for arbitrary strictly passive subsystem, then the other subsystem must be passive. Likewise, if the closed-loop system is uniformly stable for an arbitrary passive subsystem, then the other subsystem must be strictly passive. This kind of result has only been established for single-input single-output LTI systems in [Colgate and Hogan 1988], based on the Nyquist stability criterion. In contrast, we derive our results based on the equivalence between the small-gain property and passivity [Anderson 1972] via the chain-scattering formalism [Kimura 1997], which proves useful in many topics ranging from $\mathcal{H}_\infty$-synthesis [Green et al. 1990] to ν-gap metric characterization [Cantoni 2006, Khong and Cantoni 2013]. We also make use of the construction of a destabilizing controller in the proof of the necessity of the small-gain theorem [Doyle 1982]. Our main results consist of multiple versions of converse passivity theorems involving input/output strict passivity, and are summarized in Tables 1 and 2 in Section 4.

Converse passivity results admit fundamental importance in various applications. For instance, if a controlled robot is required to be stable when interacting with a passive but otherwise unknown environment, then the converse passivity result dictates that the robot must itself be strictly passive systems as seen from its interaction port. For a more detailed elaboration on such robotics motivations, the interested reader is referred to [Colgate and Hogan 1988, Stramigioli 2015].

While converse passivity theorems have been studied in the nonlinear time-varying setting in [Khong and van der Schaft 2018], the results therein cannot recover the LTI results in this paper. In particular, unlike the necessity

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proofs in [Khong and van der Schaft 2018], which rely on the S-procedure lossless theorem [Megretski and Treil 1993], the ones in this paper are based on the small-gain theorem and hence constructive. Furthermore, the uncertainty set against which the feedback system is robust is much larger (specifically, nonlinear and time-varying) in [Khong and van der Schaft 2018] than that considered in this paper, which plays a significant role in affecting the conservatism of the necessity directions of the results. Finally, we also want to emphasize that, although the converse passivity results we present are derived via well-known techniques and facts (i.e., the chain-scattering formalism and equivalence between "small gain" and "passivity") in the robust control community, they are not (clearly) documented in the existing literature to the best of our knowledge, and some, if not all, of the converse statements in Tables 1 and 2 do not appear to be straightforwardly obvious. This is precisely the reason that motivates our research in this matter.

The rest of the paper is organized as follows. The next section defines the notation to be used throughout the paper and reviews the definitions of passivity. Preliminaries on the small-gain theorem and its relation with passivity are presented in Section 3. The main converse passivity theorems are derived in Section 4. Section 5 concludes the paper and discusses future research directions.

2. NOTATION

$\mathbb{R}$ ($\mathbb{C}$), $\mathbb{R}^n$ ($\mathbb{C}^n$), $\mathbb{R}^{p\times m}$ ($\mathbb{C}^{p\times m}$) denote the sets of real (complex) numbers, n-dimensional real (complex) vectors, and $p\times m$ real (complex) matrices, respectively. The extended real set is denoted as $\bar{\mathbb{R}} := \mathbb{R}\cup\{\pm\infty\}$. Given a matrix M, the transpose and conjugate transpose are denoted by M^T and M^*, respectively. The maximum singular value of M is denoted by $\bar{\sigma}(M)$. The notation $M>0$ ($M\geq 0$) means that the matrix M is positive definite (positive semi-definite). The n-dimensional identity matrix and $n\times m$ zero matrix are denoted by I_n and $\mathbf{0}_{n\times m}$, respectively. The subscripts of these matrices are dropped when the their dimensions are clear from the context.

$\boldsymbol{L}_2^n$ denotes the space of $\mathbb{R}^n$-valued, square integrable functions on the non-negative reals, with the usual norm and inner product denoted by $\|\cdot\|_{\boldsymbol{L}_2}$ and $\langle\cdot,\cdot\rangle_{\boldsymbol{L}_2}$, respectively. The superscripts are dropped when the dimension is evident from the context. The extended $\boldsymbol{L}_2^n$ space is denoted as $\boldsymbol{L}_{2\mathrm{e}}^n$. This consists of functions f that satisfy $P_T f\in\boldsymbol{L}_2^n$, for all $T>0$, where P_T denotes the truncation operator, defined as:

$$(P_T f)(t) = \begin{cases} f(t) & \text{for } t\leq T \\ 0 & \text{otherwise} \end{cases}.$$

Let $G:\boldsymbol{L}_2\to\boldsymbol{L}_2$ be a linear operator. G is said to be causal if $P_T G P_T - P_T G = 0$ for all $T>0$. The induced norm of G is defined as

$$\|G\| = \sup_{u\in\boldsymbol{L}_2, u\neq 0}\frac{\|Gu\|_{\boldsymbol{L}_2}}{\|u\|_{\boldsymbol{L}_2}}.$$

G is said to be bounded if $\|G\|\leq\gamma$ for some $\gamma>0$. A causal and bounded G is said to be stable.

When a stable G commutes with the forward shift operator, it can be represented as multiplication in the frequency domain by a transfer function matrix in the $\mathcal{H}_\infty$ space, which consists of functions that are analytic and uniformly bounded in the right-half of the complex plane. In this case, G is said to be linear-time-invariant (LTI) and the transfer function matrix of G is denoted by $\hat{G}$. It is well-known that for such a G,

$$\|G\| = \sup_{\Re(s)>0}\bar{\sigma}(\hat{G}(s)) = \operatorname*{ess\,sup}_{\omega\in\bar{\mathbb{R}}}\bar{\sigma}(\hat{G}(\mathrm{j}\omega)).$$

It is also well-known that when G is finite-dimensional LTI with a real state space realization (A,B,C,D), then $\hat{G}(s) = C(sI-A)^{-1}B+D$ and belongs to the real rational subspace of $\mathcal{H}_\infty$, denoted by $\mathcal{RH}_\infty$. When the dimension of G is of some significance, we use the notation $\hat{G}\in\mathcal{RH}_\infty^{n\times m}$ to emphasize that G has m inputs and n outputs.

A stable LTI system G is called *passive* if $\langle u, Gu\rangle_{\boldsymbol{L}_2}\geq 0$ for any $u\in\boldsymbol{L}_2$. It is called *input strictly passive* if there exists $\epsilon>0$ such that $\langle u, Gu\rangle_{\boldsymbol{L}_2}\geq\epsilon\|u\|_{\boldsymbol{L}_2}^2$ for any $u\in\boldsymbol{L}_2$, and *output strictly passive* if there exists $\epsilon>0$ such that $\langle u, Gu\rangle_{\boldsymbol{L}_2}\geq\epsilon\|Gu\|_{\boldsymbol{L}_2}^2$ for any $u\in\boldsymbol{L}_2$. It is well-known that G is passive if and only if (iff)

$$\hat{G}(\mathrm{j}\omega)+\hat{G}(\mathrm{j}\omega)^*\geq 0 \quad \text{for all } \omega\in\bar{\mathbb{R}},$$

G is input strictly passive iff there exists $\epsilon>0$ such that

$$\hat{G}(\mathrm{j}\omega)+\hat{G}(\mathrm{j}\omega)^*\geq \epsilon I \quad \text{for all } \omega\in\bar{\mathbb{R}},$$

and G is output strictly passive iff there exists $\epsilon>0$ such that

$$\hat{G}(\mathrm{j}\omega)+\hat{G}(\mathrm{j}\omega)^*\geq \epsilon\hat{G}(\mathrm{j}\omega)^*\hat{G}(\mathrm{j}\omega) \quad \text{for all } \omega\in\bar{\mathbb{R}}.$$

Note that input strict passivity implies output strict passivity, since

$$\begin{aligned} &\exists\varepsilon>0 \text{ s.t. } \hat{G}(\mathrm{j}\omega)+\hat{G}(\mathrm{j}\omega)^*\geq\varepsilon I,\ \forall\omega\in\bar{\mathbb{R}} \\ \Longrightarrow\ &\hat{G}(\mathrm{j}\omega)+\hat{G}(\mathrm{j}\omega)^*\geq\frac{\varepsilon}{\|G\|^2}\hat{G}(\mathrm{j}\omega)^*\hat{G}(\mathrm{j}\omega),\ \forall\omega\in\bar{\mathbb{R}}. \end{aligned}$$

In the sequel, the set of all stable, finite-dimensional (FD), linear time-invariant (LTI), input strictly passive systems, output strictly passive systems, and passive systems are denoted by $\mathcal{P}_I$, $\mathcal{P}_O$, and $\mathcal{P}$, respectively. Clearly, it holds that

$$\mathcal{P}_I\subset\mathcal{P}_O\subset\mathcal{P}. \tag{1}$$

Note that both the inclusions are *strict*.

3. PRELIMINARY RESULTS

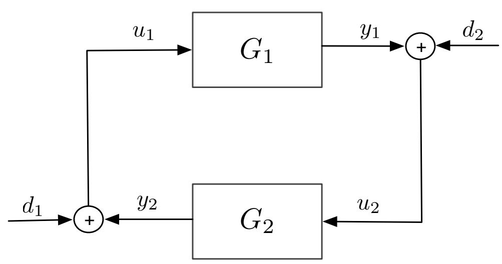

Fig. 1. Positive feedback interconnection of G_1 and G_2.

Consider the feedback interconnection of FD LTI causal systems G_1 and G_2 mapping $\boldsymbol{L}_{2\mathrm{e}}$ to $\boldsymbol{L}_{2\mathrm{e}}$, as illustrated in Figure 1. Algebraically, we have

$$\begin{cases} u_1 = y_2+d_1 \\ u_2 = y_1+d_2 \end{cases} \quad \begin{cases} y_2 = G_2u_2 \\ y_1 = G_1u_1 \end{cases}. \tag{2}$$

Definition 1. The feedback system shown in Figure 1 is said to be *well-posed* if the map $([\begin{smallmatrix}u_1\\u_2\end{smallmatrix}],[\begin{smallmatrix}y_2\\y_1\end{smallmatrix}]) \mapsto [\begin{smallmatrix}d_1\\d_2\end{smallmatrix}]$ defined by (2) has a causal inverse on $\boldsymbol{L}_{2\mathrm{e}}$. It is *stable* if it is well-posed and the inverse is bounded.

Remark 1. Note that when the feedback system is well-posed, the closed-loop map $[\begin{smallmatrix}d_1\\d_2\end{smallmatrix}] \mapsto [\begin{smallmatrix}u_1\\u_2\end{smallmatrix}]$ can be expressed as

$$\begin{bmatrix} I & -G_2 \\ -G_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-G_2G_1)^{-1} & (I-G_2G_1)^{-1}G_2 \\ G_1(I-G_2G_1)^{-1} & I+G_1(I-G_2G_1)^{-1}G_2 \end{bmatrix}.$$

Suppose G_1 and G_2 are both stable. In this case, the above identity leads to the following conclusion: $[G_1, G_2]$ is stable if and only if $(I-\hat{G}_2\hat{G}_1)^{-1} \in \mathcal{RH}_\infty$. Furthermore, it can be shown that $(I-\hat{G}_2\hat{G}_1)^{-1} \in \mathcal{RH}_\infty$ if and only if $\det(I-\hat{G}_2(s)\hat{G}_1(s)) \neq 0$ for all s in the closed right-side complex plane. Moreover, it can be readily verified that the map $[\begin{smallmatrix}d_1\\d_2\end{smallmatrix}] \mapsto [\begin{smallmatrix}u_1\\u_2\end{smallmatrix}]$ and the map $[\begin{smallmatrix}d_1\\d_2\end{smallmatrix}] \mapsto [\begin{smallmatrix}y_2\\y_1\end{smallmatrix}]$ are summed to the identity operator. Thus, the map $[\begin{smallmatrix}d_1\\d_2\end{smallmatrix}] \mapsto ([\begin{smallmatrix}u_1\\u_2\end{smallmatrix}],[\begin{smallmatrix}y_2\\y_1\end{smallmatrix}])$ is well-defined and of a finite gain if and only if the map $[\begin{smallmatrix}d_1\\d_2\end{smallmatrix}] \mapsto [\begin{smallmatrix}u_1\\u_2\end{smallmatrix}]$ is of the same properties. Henceforth, we use $[G_1, G_2]$ to denote the closed-loop map $[\begin{smallmatrix}d_1\\d_2\end{smallmatrix}] \mapsto [\begin{smallmatrix}u_1\\u_2\end{smallmatrix}]$ and to represent the feedback system shown in Figure 1.

Suppose one of the sub-systems, say G_2, is taken from a set $\mathcal{U}$. We have the following definition regarding robustness and uniformity in stability.

Definition 2. Consider a set $\mathcal{U}$ and suppose $G_2 \in \mathcal{U}$. Then $[G_1, G_2]$ is said to be *robustly* stable over $\mathcal{U}$ if $[G_1, G_2]$ is stable for all $G_2 \in \mathcal{U}$. Moreover, $[G_1, G_2]$ is said to be *uniformly* stable over $\mathcal{U}$ if $[G_1, G_2]$ is stable for all $G_2 \in \mathcal{U}$, and there exists $\gamma > 0$ such that $\sup\limits_{G_2\in\mathcal{U}} \|[G_1, G_2]\| \leq \gamma$.

The following result on the relation between passivity and small gain can be found in [Anderson 1972] and [Green and Limebeer 1995, Section 3.5.3]. A simple proof is given here for completeness.

Lemma 1. Consider a stable FD LTI system G. It holds that G is passive if and only if $(I+G)^{-1}$ is stable and $\|(I+G)^{-1}(I-G)\|_{\boldsymbol{L}_2} \leq 1$.

Proof. Since G is passive, it is well-known that $[G, -I]$ is stable and therefore $(I+\hat{G})^{-1} \in \mathcal{RH}_\infty$. Furthermore, we have $\hat{G}(\mathrm{j}\omega) + \hat{G}(\mathrm{j}\omega)^* \geq 0$ for all $\omega \in \bar{\mathbb{R}}$. Note that

$$\begin{aligned}
&(I+\hat{G}(\mathrm{j}\omega))^{-1}(I-\hat{G}(\mathrm{j}\omega))(I-\hat{G}(\mathrm{j}\omega))^*(I+\hat{G}(\mathrm{j}\omega))^{-*} - I \\
&= (I+\hat{G}(\mathrm{j}\omega))^{-1}\Big((I-\hat{G}(\mathrm{j}\omega))(I-\hat{G}(\mathrm{j}\omega))^* \\
&\qquad -(I+\hat{G}(\mathrm{j}\omega))(I+\hat{G}(\mathrm{j}\omega))^*\Big)(I+\hat{G}(\mathrm{j}\omega))^{-*} \\
&= -2(I+\hat{G}(\mathrm{j}\omega))^{-1}(\hat{G}(\mathrm{j}\omega)+\hat{G}(\mathrm{j}\omega)^*)(I+\hat{G}(\mathrm{j}\omega))^{-*}.
\end{aligned}$$

Thus,

$$(I+\hat{G}(\mathrm{j}\omega))^{-1}(I-\hat{G}(\mathrm{j}\omega))(I-\hat{G}(\mathrm{j}\omega))^*(I+\hat{G}(\mathrm{j}\omega))^{-*} \leq I.$$

if and only if $\hat{G}(\mathrm{j}\omega) + \hat{G}(\mathrm{j}\omega)^* \geq 0$. That is,

$$\sup_{\omega\in\bar{\mathbb{R}}} \bar{\sigma}((I+\hat{G}(\mathrm{j}\omega))^{-1}(I-\hat{G}(\mathrm{j}\omega))) \leq 1$$

if and only if $\hat{G}(\mathrm{j}\omega)+\hat{G}(\mathrm{j}\omega)^* \geq 0$ for all $\omega \in \bar{\mathbb{R}}$. The former inequality is equivalent to $\|(I+G)^{-1}(I-G)\|_{\boldsymbol{L}_2} \leq 1$. This concludes the proof. $\square$

Remark 2. Clearly, the lemma also holds when the inequalities are strict; i.e. G is input strictly passive if and only if $\|(I+G)^{-1}(I-G)\|_{\boldsymbol{L}_2} < 1$. This immediately leads to the following equivalent relationships: $\mathcal{P}_I = \mathcal{G}$ and $\mathcal{P} = \bar{\mathcal{G}}$, where

$$\mathcal{G} := \{G : G \text{ is FD LTI, stable, } (I+G)^{-1} \text{ is stable, and } \|(I+G)^{-1}(I-G)\|_{\boldsymbol{L}_2} < 1\}$$

and

$$\bar{\mathcal{G}} := \{G : G \text{ is FD LTI, stable, } (I+G)^{-1} \text{ is stable, and } \|(I+G)^{-1}(I-G)\|_{\boldsymbol{L}_2} \leq 1\}.$$

Lemma 1 establishes that any passive system induces a system whose gain is smaller than one. The next lemma shows that the opposite is also true.

Lemma 2. Consider a stable FD LTI system G. It holds that $\|G\| \leq 1$ (< 1) and $(I+G)^{-1}$ is stable if and only if $(I-G)(I+G)^{-1}$ is (input strictly) passive.

Proof. The proof follows from Lemma 1 and Remark 1 by observing that $H = (I-G)(I+G)^{-1}$ if and only if $G = (I+H)^{-1}(I-H)$. $\square$

The next lemma is the FD LTI version of the well-known small gain theorem. It is included here for the sake of completeness.

Lemma 3. Consider the feedback interconnected system $[G_1, G_2]$ as shown in Fig. 1, where G_1 and G_2 are FD LTI and stable. Let $\gamma > 0$. Then $[G_1, G_2]$ is

(a) stable for all G_2 satisfying $\|G_2\| < 1/\gamma$ if and only if $\|G_1\| \leq \gamma$.
(b) uniformly stable over all G_2 satisfying $\|G_2\| \leq 1/\gamma$ if and only if $\|G_1\| < \gamma$.
(c) uniformly stable over all G_2 satisfying $\|G_2\| < 1/\gamma$ if and only if $\|G_1\| < \gamma$.

Proof. The proof of the necessity of (a) and (b) can be found in [Zhou et al. 1996, Theorem 9.1], where an explicit construction of a destabilizing G_2 is provided when the conditions are violated. For the necessity of (c), one can use the same construction of G_2 in (b), and then let $\{\alpha_i\}$ be a sequence of positive numbers that converges to 1 from below. It then follows that $\|\alpha_i G_2\| < 1/\gamma$ for all i and $\|[G_1, \alpha_i G_2]\| \to \infty$ as $i \to \infty$, which means $[G_1, G_2]$ is not uniformly stable over all G_2 satisfying $\|G_2\| < 1/\gamma$.

The sufficiency proof of (a) can be found in [Zhou et al. 1996, Theorem 9.1]. We note that even though the uniformity stated in (b) and (c) is not discussed in Theorem 9.1 of [Zhou et al. 1996], the proof clearly shows that the small gain condition stated in (b) or (c) implies that $\|(I-G_2G_1)^{-1}\|$ is bounded by $\gamma/(\gamma - \|G_1\|) < \infty$. This in turn implies that $\|[G_1, G_2]\|$ is upper bounded by some positive constant for all G_2 satisfying $\|G_2\| \leq 1/\gamma$. $\square$

4. MAIN RESULTS: CONVERSE PASSIVITY THEOREMS

In this section, we derive the main results of this manuscript. First we show that passivity is necessary for robust stability over the set of input strictly passive systems.

Theorem 1. Consider the feedback interconnected system $[G_1, G_2]$ shown in Figure 1, where $\hat{G}_1,\ \hat{G}_2 \in \mathcal{RH}_\infty$. The system $[G_1, G_2]$ is robustly stable for all input strictly passive G_2 if and only if $-G_1$ is passive.

Proof. Sufficiency is well-known in the literature; see [Green and Limebeer 1995, van der Schaft 2017]. For necessity, suppose $-G_1$ is not passive and thus for some $\tilde{\omega}$,

$$-(\hat{G}_1(\mathrm{j}\tilde{\omega}) + \hat{G}_1(\mathrm{j}\tilde{\omega})^*) \not\geq 0.$$

We consider two possible scenarios. First, if $I - \hat{G}_1(\mathrm{j}\tilde{\omega})$ is not invertible, by Remark 1 we see that $[G_1, I]$ is not stable. Note that the identity system is input strictly passive; therefore, we have found a strictly passive system G_2 such that $[G_1, G_2]$ is not stable.

If $I - \hat{G}_1(\mathrm{j}\tilde{\omega})$ is invertible, let

$$M = -(I - \hat{G}_1(\mathrm{j}\tilde{\omega}))^{-1}(I + \hat{G}_1(\mathrm{j}\tilde{\omega})).$$

By Lemma 1, $\bar{\sigma}(M) > 1$. By the construction described in the proof of Theorem 9.1 of [Zhou et al. 1996], one can find a $\hat{\Delta} \in \mathcal{RH}_\infty$ such that $\|\Delta\| < 1$ and $I - M\hat{\Delta}(\mathrm{j}\tilde{\omega})$ is not invertible. Observe that since $\|\Delta\| < 1$, $I + \hat{\Delta}(\mathrm{j}\omega)$ is invertible for all $\omega \in \bar{\mathbb{R}}$. Also observe that (for notational simplicity, we omit the $\tilde{\omega}$-dependency in the following derivation)

$$\begin{aligned} I - M\hat{\Delta} &= (I - \hat{G}_1)^{-1}(I - \hat{G}_1 + (I + \hat{G}_1)\hat{\Delta}) \\ &= (I - \hat{G}_1)^{-1}(I + \hat{\Delta} - \hat{G}_1(I - \hat{\Delta})) \\ &= (I - \hat{G}_1)^{-1}(I - \hat{G}_1(I - \hat{\Delta})(I + \hat{\Delta})^{-1})(I + \hat{\Delta}). \end{aligned}$$

Thus

$$\det(I - M\hat{\Delta}) = 0 \Leftrightarrow \det(I - \hat{G}_1(I - \hat{\Delta})(I + \hat{\Delta})^{-1}) = 0.$$

Let $G_2 = (I - \Delta)(I + \Delta)^{-1}$. Note that G_2 is stable, as $\|\Delta\| < 1$. Moreover, by Lemma 2, G_2 is input strictly passive. Hence, we have found an input strictly passive G_2 such that $[G_1, G_2]$ is not stable. □

The same conclusion holds when G_2 is output strictly passive, as opposed to input strictly passive, which requires a nonzero feedthrough term in the system.

Theorem 2. Consider the feedback interconnected system $[G_1, G_2]$ shown in Figure 1, where $\hat{G}_1,\ \hat{G}_2 \in \mathcal{RH}_\infty$. The system $[G_1, G_2]$ is robustly stable for all output strictly passive G_2 if and only if $-G_1$ is passive.

Proof. Sufficiency is well known in the literature; see [Bao and Lee 2007, Green and Limebeer 1995, van der Schaft 2017]. Necessity follows from Theorem 1, since $\mathcal{P}_I \subset \mathcal{P}_O$ as noted in (1). □

Moreover, the following sufficient condition can also be established, by similar arguments via Theorems 1 and 2.

Theorem 3. Consider the feedback interconnected system $[G_1, G_2]$ shown in Figure 1, where $\hat{G}_1,\ \hat{G}_2 \in \mathcal{RH}_\infty$. The system $[G_1, G_2]$ is robustly stable for all passive G_2 if $-G_1$ is output strictly passive.

Proof. For sufficiency, note that the stability of $[G_1, G_2]$ is equivalent to $(I - \hat{G}_2\hat{G}_1)^{-1} \in \mathcal{RH}_\infty$, which is in turn equivalent to $(I - \hat{G}_1\hat{G}_2)^{-1} \in \mathcal{RH}_\infty$. Therefore, if $-G_1$ is passive and G_2 is output strictly passive implies $[G_1, G_2]$ is stable as in Theorem 2, then $[G_1, G_2]$ must be stable when $-G_1$ is output strictly passive and G_2 is passive.

Remark 3. Given the premise that G_1 and G_2 are stable, it is not yet clear whether the necessity direction of Theorem 3 is true. That is, we do not know whether $[G_1, G_2]$ is robustly stable for all (FD, LTI, stable) passive G_2 implies that $-G_1$ is output strictly passive. **The statement can be proven true** if G_1 and G_2 are single-input-single-output (SISO). To see this, note that $[G_1, G_2]$ is robustly stable for all passive G_2 implies it is robustly stable for all output strictly passive G_2. It follows from Theorem 2 that $-G_1$ must be passive. Now suppose $-G_1$ is not output strictly passive but is only passive. Then $-\hat{G}_1(\mathrm{j}\tilde{\omega}) - \hat{G}_1(\mathrm{j}\tilde{\omega})^* = 0$ for some $\tilde{\omega}$. One can readily verify that setting $G_2 = -G_1/\|\hat{G}_1(\mathrm{j}\tilde{\omega})\|^2$ would destabilize $[G_1, G_2]$, as $1 - \hat{G}_2(\mathrm{j}\omega)\hat{G}_1(\mathrm{j}\omega)$ is equal to 0 at $\omega = \tilde{\omega}$.

In the multiple-input-multiple-output setting, the statement can also be proven true if we allow G_2 to **have poles on the imaginary axis**. In this case, the proof of necessity is almost identical to that of Theorem 1, and we note that the corresponding Δ satisfies $\|\Delta\| \leq 1$ and $G_2 := (I - \Delta)(I + \Delta)^{-1}$ is passive but might have poles on the imaginary axis.

The inclusion relationship $\mathcal{P}_I \subset \mathcal{P}_O \subset \mathcal{P}$ gives rise to the following necessary or sufficient conditions, which follow immediately from Theorems 1 to 3.

Corollary 1. If system $[G_1, G_2]$ is robustly stable for all passive G_2, then $-G_1$ is passive.

Corollary 2. If $-G_1$ is output strictly passive, then $[G_1, G_2]$ is robustly stable for all output strictly passive, or input strictly passive G_2.

Corollary 3. If $-G_1$ is input strictly passive, then $[G_1, G_2]$ is robustly stable for all passive, output strictly passive, or input strictly passive G_2.

On the other hand, we note that

- $-G_1$ being output strictly passive is *not necessary* for $[G_1, G_2]$ to be robustly stable for all output strictly passive G_2.
- $-G_1$ being output strictly passive is *not necessary* for $[G_1, G_2]$ to be robustly stable for all input strictly passive G_2.
- $-G_1$ being input strictly passive is *not necessary* for $[G_1, G_2]$ to be robustly stable for all output strictly passive G_2.
- $-G_1$ being input strictly passive is *not necessary* for $[G_1, G_2]$ to be robustly stable for all input strictly passive G_2.
- $-G_1$ being input strictly passive is *not necessary* for $[G_1, G_2]$ to be robustly stable for all passive G_2.

These statements simply follow from Theorems 1 and 2 and the fact that $\mathcal{P}_I$ and $\mathcal{P}_O$ are *strictly* contained in $\mathcal{P}$; for example, the zero system is output strictly passive but not input strictly passive, and hence the statement of the last three bullet points are true.

Lastly, we note that $-G_1$ being passive is *not sufficient* for $[G_1, G_2]$ to be robustly stable over the set of all passive G_2. To see this, take

$$G_1 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \quad \text{and} \quad G_2 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

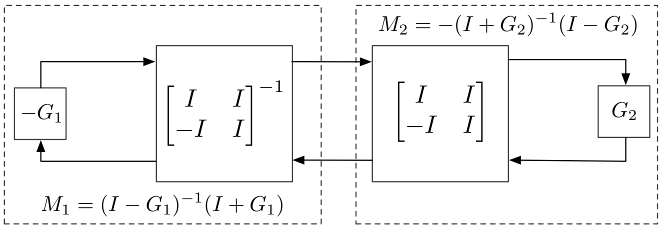

Fig. 2. Chain-scattering transformation on G_1 and G_2.

One can readily verify that both G_1 and G_2 are passive and $I - G_1G_2 = 0$. Therefore $[G_1, G_2]$ is not even well-posed, let alone stable.

Tables 1 summarizes all the robust stability conditions discovered.

	$-G_1 \in \mathcal{P}$	$-G_1 \in \mathcal{P}_O$	$-G_1 \in \mathcal{P}_I$
$\forall G_2 \in \mathcal{P}$	N$\cancel{\text{S}}$	(N)S	$\cancel{\text{N}}$S
$\forall G_2 \in \mathcal{P}_O$	NS	$\cancel{\text{N}}$S	$\cancel{\text{N}}$S
$\forall G_2 \in \mathcal{P}_I$	NS	$\cancel{\text{N}}$S	$\cancel{\text{N}}$S

Table 1. Conditions on robust stability of $[G_1, G_2]$ over a set of G_2.

N / $\cancel{\text{N}}$: the condition in the top row is (necessary / not necessary) for feedback stability over the set in the first column.
S / $\cancel{\text{S}}$: the condition in the top row is (sufficient / not sufficient) for feedback stability over the set in the first column.
(N): See Remark 3.

By imposing uniformity on stability, one can derive the following necessary and sufficient condition using arguments similar to those in the proof of Theorem 1 .

Theorem 4. Consider the feedback interconnected system $[G_1, G_2]$ shown in Figure 1, where $\hat{G}_1,\ \hat{G}_2 \in \mathcal{RH}_\infty$. The system $[G_1, G_2]$ is uniformly stable for all passive G_2 if and only if $-G_1$ is input strictly passive.

Proof. Again, it is well-known that when G_2 is passive and $-G_1$ is input strictly passive, the feedback system $[G_1, G_2]$ is stable. To see that $[G_1, G_2]$ is uniformly stable for all G_2 with passivity, we note by the chain-scattering transformation [Kimura 1997] the feedback system $[G_1, G_2]$ is equivalent to $[M_1, M_2]$, where $M_1 = (I - G_1)^{-1}(I+G_1)$ and $M_2 = -(I+G_2)^{-1}(I-G_2)$; see Figure 2 for an illustration. By Lemma 1 and Remark 2, we see that $\|M_1\| < 1$. Thus by Lemma 3, we have that $[M_1, M_2]$ is uniformly stable for all M_2 with $\|M_2\| \leq 1$, which is the case because G_2 is passive.

For necessity, suppose $-G_1$ is not input strictly passive, which leads to two possibilities: either $-G_1$ is not passive or $-G_1$ is passive but not input strictly passive. If $-G_1$ is not passive, the proof of Theorem 1 has shown that there exists an input strictly passive G_2 such that $[G_1, G_2]$ is unstable. Suppose $-G_1$ is passive but not input strictly passive. Then again the stability of $[G_1, G_2]$ with passive G_2 is equivalent to that of $[M_1, M_2]$, where $M_1 = (I - G_1)^{-1}(I + G_1)$ and $M_2 = -(I + G_2)^{-1}(I - G_2)$. By the proof of Lemma 1, we see that $\|M_1\| = 1$. Thus by Lemma 3, $[M_1, M_2]$ cannot be uniformly stable for all M_2 with $\|M_2\| \leq 1$, which in turn implies $[G_1, G_2]$ is not uniformly stable for all passive G_2. This completes the proof. $\square$

Theorem 5. Consider the feedback interconnected system $[G_1, G_2]$ shown in Figure 1, where $\hat{G}_1,\ \hat{G}_2 \in \mathcal{RH}_\infty$. The system $[G_1, G_2]$ is uniformly stable for all input strictly passive G_2 if and only if $-G_1$ is input strictly passive.

Proof. Sufficiency follows from Theorem 4, since $\mathcal{P}_I \subset \mathcal{P}$. For necessity, suppose $-G_1$ is not input strictly passive, which leads to two possibilities: either $-G_1$ is not passive or $-G_1$ is passive but not input strictly passive. If $-G_1$ is not passive, the proof of Theorem 1 has shown that there exists an input strictly passive G_2 such that $[G_1, G_2]$ is unstable. Suppose $-G_1$ is passive but not input strictly passive. Then again the stability of $[G_1, G_2]$ with passive G_2 is equivalent to that of $[M_1, M_2]$, where $M_1 = (I - G_1)^{-1}(I + G_1)$ and $M_2 = -(I + G_2)^{-1}(I - G_2)$. By the proof of Lemma 1, we see that $\|M_1\| = 1$. Thus by Lemma 3, $[M_1, M_2]$ cannot be uniformly stable for all M_2 with $\|M_2\| < 1$, which in turn implies $[G_1, G_2]$ is not uniformly stable for all input strictly passive G_2. $\square$

Again by the strict inclusion relationship $\mathcal{P}_I \subset \mathcal{P}_O \subset \mathcal{P}$, the following necessary or sufficient conditions are immediate consequences of Theorems 4 and 5.

Corollary 4. If $-G_1$ is input strictly passive, then system $[G_1, G_2]$ is uniformly stable for all output strictly passive G_2.

Corollary 5. If system $[G_1, G_2]$ is uniformly stable for all passive G_2, then $-G_1$ is passive, and in fact, output strictly passive and input strictly passive.

Corollary 6. If system $[G_1, G_2]$ is uniformly stable for all output strictly passive G_2, then $-G_1$ is passive, and in fact, output strictly passive and input strictly passive.

Corollary 7. If system $[G_1, G_2]$ is uniformly stable for all input strictly passive G_2, then $-G_1$ is passive, and in fact, output strictly passive and input strictly passive.

To see Corollaries 5 to 7, note that $[G_1, G_2]$ uniformly stable for all passive G_2 or for all output strictly passive G_2 implies that for all input strictly passive G_2, as $\mathcal{P}_I$ is strictly contained in $\mathcal{P}_O$ and $\mathcal{P}$. Then by Theorem 5, $-G_1$ must be input strictly passive, and hence it must be output strictly passive and passive, again due to $\mathcal{P}_I \subset \mathcal{P}_O \subset \mathcal{P}$.

Even although $-G_1$ being output strictly passive is sufficient for $[G_1, G_2]$ to be robustly stable over the set of all input strictly passive G_2 (cf. Corollary 2), the condition is not sufficient if we require uniform stability. To see this, note that the zero system is output strictly passive, and $[0, G_2]$ is not uniformly stable for all input strictly passive G_2. This is because $[0, G_2]$ is equal to G_2, and G_2 can be input strictly passive while having an arbitrarily large gain. Moreover, this observation and the inclusion relationship $\mathcal{P}_I \subset \mathcal{P}_O \subset \mathcal{P}$ immediately leads to the following conclusions:

- $-G_1$ being output strictly passive is *not sufficient* for $[G_1, G_2]$ to be uniformly stable for all passive, output strictly passive, or even, input strictly passive G_2.
- $-G_1$ being passive is *not sufficient* for $[G_1, G_2]$ to be uniformly stable for all passive, output strictly passive, or even, input strictly passive G_2.

Tables 2 in the next page summarizes all the uniform stability conditions discovered.

	$-G_1 \in \mathcal{P}$	$-G_1 \in \mathcal{P}_O$	$-G_1 \in \mathcal{P}_I$
$\forall G_2 \in \mathcal{P}$	N$\not S$	N$\not S$	NS
$\forall G_2 \in \mathcal{P}_O$	N$\not S$	N$\not S$	NS
$\forall G_2 \in \mathcal{P}_I$	N$\not S$	N$\not S$	NS

Table 2. Conditions on uniform stability of $[G_1, G_2]$ over a set of G_2.

N / $\not N$: the condition in the top row is (necessary / not necessary) for uniform feedback stability over the set in the first column.
S / $\not S$: the condition in the top row is (sufficient / not sufficient) for uniform feedback stability over the set in the first column.

5. CONCLUSIONS

We derived several versions of converse passivity theorems in this paper. The proofs are based on the renowned small-gain theorem and thus constructive. Roughly speaking, our main results state that if a feedback interconnection is stable when one of the open-loop systems is an arbitrary passive system, then the other open-loop system must also exhibit the passivity property. The results have implications on the field of robotics, where the stability of a robot's interaction with a passive but otherwise unknown environment is of crucial significance.

The work of generalizing the converse results described in this paper to the unifying framework of integral quadratic constraints [Megretski and Rantzer 1997, Megretski et al. 2010] is currently ongoing and will be reported elsewhere [Khong and Kao 2019]. It admits the potentials of producing a useful list of converse robustness results, such as those involving 'mixed' frequency-weighted small-gain and passivity, as well as passivity indices [Kottenstette et al. 2014] commonly employed to characterizing passivity surplus/deficit and their tradeoff in ensuring robust closed-loop stability.

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