Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

On the stability of discrete-time linear switched systems in block companion form

Vittorio De Iuliis * **Alessandro D'Innocenzo** *
Alfredo Germani * **Costanzo Manes** *

* *Dipartimento di Ingegneria e Scienze dell'Informazione, e Matematica, Università degli Studi dell'Aquila, Via Vetoio, 67100 Coppito (AQ), Italy (e-mail: vittorio.deiuliis@univaq.it).*

Abstract: Inspired by some insightful results on the delay-independent stability of discrete-time systems with time-varying delays, in this work we study the arbitrary switching stability for some classes of discrete-time switched systems whose dynamic matrices are in block companion form. We start from the special family of block companion matrices whose first block-row is made of permutations of nonnegative matrices, deriving a simple necessary and sufficient condition for its arbitrary switching stability. Then we relax both these assumptions, at the expense of introducing some conservatism. Some consequences on the computation of the Joint Spectral Radius for the aforementioned families of matrices are illustrated.

Keywords: Switched systems, Time-delay systems, Positive systems, Joint Spectral Radius.

1. INTRODUCTION

During the last decades an intensive research effort has been devoted to the stability analysis of dynamical systems. While the stability of linear time-invariant systems is a classic and relatively simple problem, for which long standing results are available, huge difficulties arise when time-varying systems or systems with time-delays are considered. In those cases, necessary and sufficient conditions are either not available or generally intractable from a computational viewpoint, and conservative results are often investigated. Interestingly enough, a remarkable simplification takes place when positive systems are considered. This has proven influential in linear delay systems of various classes (see e.g. Kerscher and Nagel (1984); Haddad and Chellaboina (2004); Liu et al. (2009, 2010)) and also for switched systems (see Blanchini et al. (2015) and references therein). Moreover, the so called comparison approach has provided useful results to export to non positive systems the aforementioned favourable properties, although at the expense of introducing some conservatism. In this work we proceed a step forward in this direction: first of all, we discuss how existing results for positive delay systems naturally yield an interesting result on the stability of discrete-time switched systems in block companion form, for which the arbitrary switching stability becomes a relatively trivial task. Then, we show how the positivity-based results for both delay and switched systems of the aforementioned classes can be exported to non positive systems of the same classes, yielding novel sufficient stability conditions.

Namely, we will focus on the stability of discrete-time switched systems described by:

$$x(k+1) = \mathcal{A}_{\sigma(k)} x(k) \tag{1}$$

where σ is an arbitrary switching sequence taking values in $\{1, \dots, p\}$ such that at time k the matrix $\mathcal{A}_{\sigma(k)} \in \mathbb{R}^{mn \times mn}$ is taken from a family $\mathcal{A} = \{\mathcal{A}_1, \dots, \mathcal{A}_p\}$ of block companion matrices, of the type

$$\mathcal{A}_j = \begin{bmatrix} A_{j_1} & A_{j_2} & \cdots & A_{j_{m-1}} & A_{j_m} \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}, \quad j = 1, \dots, p, \tag{2}$$

where $A_{j_i} \in \mathbb{R}^{n \times n}$ for all $i = 1, \dots, m$. Block companion switched systems arise in both theoretical and practical problems. For instance, they can serve as state-space representations of linear switched ARX models, whose estimation for control of large-scale systems is an active research area (see e.g. Smarra et al. (2020) and references therein). We will prove a number of stability results on this class of switching systems, starting from the case in which the first block row of $\mathcal{A}_j$ consists of permuting matrices, then relaxing this assumption.

The starting point of our analysis comes from the fact that system (1) with block companion $\mathcal{A}_j$ can be shown equivalent, under some conditions, to the discrete-time delay system

$$z(k+1) = \sum_{i=1}^{m} M_i z(k - \delta_i(k)) \tag{3}$$

where $\delta_i(k)$ takes value in $\mathbb{N}_0$. The enabling property that will be instrumental in our analysis is that for system (3), in the special case of nonnegative M_i matrices, the stability for all possible values of the time-varying delays is equivalent to the stability for a fixed set of constant delays.

The work is structured as follows. First, in Section 2, we will discuss in some detail the stability properties of the delay system (3), starting from the positive framework, then relaxing this assumption to consider not necessarily positive systems. Section 3 will relate the stability of switched systems in block companion form to the results illustrated in Section 2, presenting the main contributions of this work. Section 4 provides two numerical examples. Conclusions and ideas for future work close the paper.

Notation. $\mathbb{N}_0$ is the set of nonnegative integers. $\mathbb{R}_+$ is the set of nonnegative real numbers. $\mathbb{R}_+^n$ is the nonnegative orthant of $\mathbb{R}^n$. $\mathbb{R}_+^{m\times n}$ is the cone of nonnegative $m\times n$ matrices. I_n is the $n\times n$ identity matrix. Inequalities among vectors and matrices of the same dimensions have to be understood componentwise, i.e. $M \le N$ if $m_{ij} \le n_{ij}$ for all i, j. In this sense, a nonnegative matrix $M \in \mathbb{R}_+^{m\times n}$ is also denoted by $M \ge 0$, where 0 is the matrix of appropriate dimensions whose entries are all zero. $\sigma(M)$ and $\rho(M)$ respectively denote the spectrum and the spectral radius of a square matrix M, which is said to be *Schur-stable* if $\sigma(M) \subset \{z \in \mathbb{C} : |z| < 1\}$ or, equivalently, if $\rho(M) < 1$. For a set of matrices $\mathcal{M} = \{M_1, \dots, M_p\}$ the joint spectral radius of $\mathcal{M}$ is defined and denoted as

$$\rho^*(\mathcal{M}) = \rho^*(M_1, \dots, M_p) = \lim_{k\to\infty} \max_{B\in\mathcal{M}^k} \|B\|^{1/k},$$

where $\mathcal{M}^k$ is the set of all products of length k (allowing for repetitions) of matrices in $\mathcal{M}$. Clearly, for a single matrix M, $\rho^*(M) = \rho(M)$. See Jungers (2009) for further details.

2. THE ENABLING RESULTS ON DELAY SYSTEMS

Consider a discrete-time delay system governed by the difference equation:

$$\begin{aligned} z(k+1) &= \sum_{i=1}^{m} M_i z(k-\delta_i(k)), \quad && k \ge 0, \\ z(k) &= \phi(k), && k \in [-\delta, 0], \end{aligned} \tag{4}$$

where $\delta_i(k)$ is a time-varying delay with integer values in $[0, \delta]$, $z(k) \in \mathbb{R}^n$ is the state trajectory at time k and ϕ is the initial state function. Since the focus of our work is on stability, we do not model input and output functions to avoid unnecessary details.

It is instrumental to note that the trivial case of constant delays $\delta_i(k) = \delta_i \in \mathbb{N}_0$, with $0 \le \delta_i \le \delta$ can be rewritten as

$$z(k+1) = \sum_{j=0}^{\delta} \tilde{M}_j z(k-j), \quad k \ge 0, \tag{5}$$

where $\tilde{M}_j = \sum_{i\in I_j} M_i$, with $I_j = \{i : \delta_i = j\}$ accounting for possibly multiple delays with value j (note that I_j can be empty). It is clear that (5) admits an equivalent delay-free state-space representation

$$x(k+1) = \mathcal{A}x(k) \tag{6}$$

where $x(k) = \begin{bmatrix} z^T(k) & z^T(k-1) & \cdots & z^T(k-\delta)\end{bmatrix}^T$ and

$$\mathcal{A} = \begin{bmatrix} \tilde{M}_0 & \tilde{M}_1 & \cdots & \tilde{M}_{\delta-1} & \tilde{M}_\delta \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix} \tag{7}$$

which allows to conclude that the stability of a discrete-time delay system is a trivial question when constant delays are involved (i.e. $\rho(\mathcal{A}) < 1$ is a necessary and sufficient condition for the exponential stability).

This explains why most of the literature has focused on the general case of time-varying delays (4), for which the stability analysis is far from being trivial. In this respect, the literature has mainly investigated Lyapunov approaches, yielding LMIs of growing complexity in order to reduce conservatism. Some notable contributions can be found in: Gao and Chen (2007); Zhu and Yang (2008); Seuret et al. (2015).

A truly remarkable simplification is achieved if system (4) is positive, i.e. if nonnegative initial conditions (and nonnegative input, if modelled) can only produce nonnegative state at all time instants. This definition is trivially satisfied checking the system matrices, see Liu et al. (2009).

Lemma 1. The delay system (4) is positive if and only if M_i is componentwise nonnegative (i.e. $M_i \ge 0$) for all $i = 1, \dots, m$.

Then, if positivity is satisfied, checking the asymptotic stability of (4) is very simple, and a single check ensures that the system is delay-independent stable, i.e. is stable for all possible time-varying values of the delays $\delta_i(k)$, as shown in Liu et al. (2009).

Theorem 2. The delay system (4), with $M_i \ge 0$ for all i, is delay-independent asymptotically stable if and only if $\rho(\sum_{i=1}^m M_i) < 1$.

Remark 3. Note that the theorem states that the delay-independent stability can be verified checking the zero-valued delays stability (just substitute $\delta_i(k) \equiv 0$ in (4)), but this check is only the simplest way to say that the system is delay-independent stable if and only if it is stable for a given set of constant values of the delays. We will use this equivalence later in the work. We also note that the delay-independent stability is exponential for all possible constant delays, as shown in Liu and Lam (2013).

Now consider the special case of constant delays described by (5). Then the previous result implies that, if the system is positive ($\tilde{M}_j \ge 0$ for all j), its exponential stability is simply tested verifying that $\rho(\sum_{j=0}^{\delta} \tilde{M}_j) < 1$. Moreover, since the stability of (5) is equivalent to that of its augmented state-space representation (6)–(7), we easily infer the following result.

Lemma 4. Consider p nonnegative $n\times n$ matrices M_i. Then, the following result holds:

$$\rho\left(\begin{bmatrix} M_1 & M_2 & \cdots & M_{p-1} & M_p \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}\right) < 1 \iff \rho\left(\sum_{i=1}^{p} M_i\right) < 1 \tag{8}$$

Before deriving the main results of this work, presented in the next Section, we highlight the fact that a recently developed technique allows to export to non positive systems stability results that only hold for positive ones, although at the expense of introducing some conservatism. The general idea is to associate to a given non positive system an augmented positive representation whose state (and output, if modelled) trajectories always upper bound those of the original system. This method of analysis, which is described in the literature as the "Comparison principle" (see e.g. Niculescu (2001)), has been systematically implemented for a number of different classes of delay systems by means of the Internally Positive Representation (IPR) technique, see Conte et al. (2017); De Iuliis et al. (2017, 2019a,b, 2020a,b).

Avoiding unnecessary details which can be found in the referenced works, the IPR method can be readily applied to the class of systems described by (4) defining the following positive representations of a vector $v \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n\times n}$:

$$\pi(v) = \begin{bmatrix} v^+ \\ v^- \end{bmatrix}, \quad \Pi(M) = \begin{bmatrix} M^+ & M^- \\ M^- & M^+ \end{bmatrix}, \tag{9}$$

where v^+ denotes the componentwise positive part of v, i.e. $v_i^+ = \max(0, v_i)$, and v^- denotes its componentwise negative part, i.e. $v_i^- = \max(0, -v_i)$. The same can be said for M^+ and M^-. Notice that $\pi(v)$ and $\Pi(M)$ are positive (nonnegative, actually) representations of twice the dimensions of v and M. Moreover, $v = v^+ - v^-$ and $|v| = v^+ + v^-$ (and the same holds for M and $|M|$). Noteworthy, v can be obtained back from $\pi(v)$ simply defining the backwards operator $\Delta_n = [I_n \;\; -I_n]$, yielding $v = \Delta_n \pi(v)$, and $Mv = \Delta_n \Pi(M)\pi(v)$.

With these definitions in mind, it is easy to show that a simple IPR for system (4) is the following:

$$\begin{aligned} \mathcal{Z}(k+1) &= \sum_{i=1}^{m} \Pi(M_i)\mathcal{Z}(k-\delta_i(k)), \quad && k \geq 0, \\ \mathcal{Z}(k) &= \pi(\phi(k)), && k \in [-\delta, 0], \end{aligned} \tag{10}$$

The proof that (10) is a valid IPR for (4) consists of showing that the original system state trajectory $z(k)$, for a given initial condition ϕ can be obtained from the state trajectory $\mathcal{Z}(k)$ once started from $\pi(\phi)$, at every time step. We omit for brevity this straightforward proof, which is on the lines of that given for Theorem 6 in De Iuliis et al. (2017) and allows to conclude that $z(k) = \Delta_n \mathcal{Z}(k)$ for all $k \geq -\delta$.

Now, the following result is the core point of this section in order to export to non positive systems the strong delay-independent stability criterion of Theorem 2.

Lemma 5. The asymptotic stability of the IPR (10) implies the asymptotic stability of the original system (4).

Proof. The result is a simple consequence of the fact that the state trajectory of the IPR always dominates the state trajectory of the original system, since:

$$z(k) = \Delta_n \mathcal{Z}(k) \implies \|z(k)\| \leq \|\Delta_n\|\|\mathcal{Z}(k)\|. \tag{11}$$

Then, if the IPR is (asymptotically) stable, the original system is stable as well. □

Clearly, the previous result only gives a sufficient stability condition, since it can happen that the IPR of a stable system is not stable. This is due to the fact that $\Pi(M)$ properly contains the spectrum of M, but it also contains the spectrum of $|M|$ (see De Iuliis et al. (2019b) for details), i.e.

$$\begin{aligned} \sigma(\Pi(M)) &= \sigma(M^+ - M^-) \cup \sigma(M^+ + M^-) = \\ &= \sigma(M) \cup \sigma(|M|), \end{aligned} \tag{12}$$

and since $\rho(M) \leq \rho(|M|)$, one readily has that the added spectrum can consist of unstable modes.

Nevertheless, one can apply Theorem 2 to the IPR (10) and if the latter is proved stable, the original system (4) is stable as well. This leads to the following result, which concludes the section.

Theorem 6. The delay system (4) is delay-independent asymptotically stable if $\rho(\sum_{i=1}^m |M_i|) < 1$.

Proof. The proof consists of noting that Theorem 2 gives a necessary and sufficient stability condition for the IPR (10) which requires to check that $\rho(\sum_{i=1}^m \Pi(M_i)) < 1$. But since

$$\sigma\left(\sum_{i=1}^{m} \Pi(M_i)\right) = \sigma\left(\sum_{i=1}^{m} M_i\right) \cup \sigma\left(\sum_{i=1}^{m} |M_i|\right) \tag{13}$$

it would suffice to verify that both $\rho(\sum_{i=1}^m M_i) < 1$ and $\rho(\sum_{i=1}^m |M_i|) < 1$ in order to prove the stability of the IPR, which in turn implies the stability of the original system (4). Nevertheless, for matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$, such that $|P| \leq Q$, it holds that (see Meyer (2000)):

$$\rho(P) \leq \rho(|P|) \leq \rho(Q). \tag{14}$$

Applying (14) with $P = \sum_{i=1}^m M_i$ and $Q = \sum_{i=1}^m |M_i|$ one has:

$$\rho\left(\sum_{i=1}^{m} M_i\right) \leq \rho\left(\left|\sum_{i=1}^{m} M_i\right|\right) \leq \rho\left(\sum_{i=1}^{m} |M_i|\right), \tag{15}$$

and this means that $\rho(\sum_{i=1}^m |M_i|) < 1$ is a sufficient condition for the delay-independent asymptotic stability of (4), as stated in the theorem. □

Summing up, in this section we introduced the class of discrete-time systems with time-varying delays (4), illustrating how its stability analysis is difficult, in general, except for the case of constant delays, which can be expressed as a delay-free augmented system (6). Nevertheless, a huge simplification occurs when positive delay systems are considered, i.e. systems as in (4) satisfying $M_i \geq 0$ for all i. In this case, the stability is independent of delays, i.e. if the system is stable for given fixed values of the delays, it is stable for all their values, possibly time-varying. The result is readily checked considering the case of zero-valued delays, yielding the necessary and sufficient stability condition of Theorem 2. Finally, we have shown how this strong result can be exported also to non positive systems, at the expense of losing necessity, via the IPR technique. The outcome of this procedure is Theorem 6, which gives a sufficient condition for the delay-independent stability of (4) with no positivity constraint.

3. STABILITY OF BLOCK COMPANION SWITCHED SYSTEMS

In this section we consider the class of discrete-time switched systems presented in the Introduction, and described by (1)–(2).

The aim of the section is to unveil a number of peculiar stability results holding for this class of systems, starting from some special cases and then moving towards more general results.

3.1 First block permutations

We start addressing the special case in which the first block row of $\mathcal{A}_j$ consists, for each j, of a permutation (with no repetitions), of some matrices $M_1, \ldots, M_m$. Defining $\mathcal{P}$ as the set of all possible permutations of $\{1, \ldots, m\}$, we denote the particular permutation (among the possible $m!$) selected at time k with $\mathcal{P}^{\sigma(k)}$, and the single element with $\mathcal{P}_i^{\sigma(k)}$, for $i = 1, \ldots, m$. Then, $\mathcal{A}_{\sigma(k)}$ switches among the $m!$ matrices of the family $\mathcal{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_{m!}\}$ and the first block row of each $\mathcal{A}_j$ contains the matrices $A_{j_i} = M_{\mathcal{P}_i^j}$ for $i = 1, \ldots, m$.

Let us present a simple example in order to clarify the notation.

Example 7. Consider $m = 3$ matrices $M_1, M_2, M_3 \in \mathbb{R}^{n \times n}$. The set of the 3! possible permutations of $\{1, 2, 3\}$ is $\mathcal{P} = \{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \ldots, \mathcal{P}^6\} = \{\{1, 2, 3\}\}, \{1, 3, 2\}, \{2, 1, 3\}, \{2, 3, 1\}, \{3, 1, 2\}, \{3, 2, 1\}\}$. At each time k, one of the 6 permutations is possible. For example, consider at time $k = 2$, $\sigma(2) = 3$, such that $\mathcal{P}^3 = \{2, 1, 3\}$ is selected. Then, $\mathcal{P}_1^3 = 2$, $\mathcal{P}_2^3 = 1$, and $\mathcal{P}_3^3 = 3$. In this case, the first row of the $\mathcal{A}_3$ matrix will be made of $A_{3_i} = M_{\mathcal{P}_i^3}$, yielding:

$$\mathcal{A}_3 = \begin{bmatrix} M_2 & M_1 & M_3 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}. \tag{16}$$

At this point, bearing in mind the notation, the following result is readily obtained.

Theorem 8. Consider a switched system as in (1)–(2), where $\mathcal{A}_{\sigma(k)}$ is arbitrarily taken from the family $\mathcal{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_{m!}\}$ of matrices whose first block-row is a permutation of m nonnegative matrices $\mathcal{M} = \{M_1, \ldots, M_m\}$, i.e. $A_{j_i} = M_{\mathcal{P}_i^j}$. Then, the switched system is asymptotically stable for all possible switching sequences if and only if $\rho(\sum_{i=1}^m M_i) < 1$.

Proof. The result comes from the fact that, as detailed for the constant delay case (6)–(7), the first (block) state component of the switched system under examination, i.e. $x_1(k) \in \mathbb{R}^n$ is described at every time step k by the delay system:

$$z(k+1) = \sum_{i=1}^m M_i z(k - \tilde{\mathcal{P}}_i^{\sigma(k)}), \quad k > 0, \tag{17}$$

with $x_1(k) = z(k)$ and $\tilde{\mathcal{P}}_i^{\sigma(k)} = \mathcal{P}_i^{\sigma(k)} - 1$. Since M_i is nonnegative for all i, system (17) is a positive delay system with time-varying delays $\tilde{\mathcal{P}}_i^{\sigma(k)}$ with values in $\{0, 1, \ldots, m-1\}$. Then, by Theorem 2 we know that it is stable for all possible values of the delays if and only if $\rho(\sum_{i=1}^m M_i) < 1$, i.e. $z(k) \to 0$ as $k \to \infty$ for any initial condition ϕ, and since $x(k) = \begin{bmatrix} z^T(k) \; z^T(k-1) \; \cdots \; z^T(k-m+1) \end{bmatrix}^T$ one trivially has that $x(k) \to 0$ as $k \to \infty$ for any $x(0) \in \mathbb{R}^{mn}$. □

Remark 1. It is rather clear from the proof above that the case of permuting matrices is not the most general class of switching block companion systems which can be equivalently mapped with delay systems as in (4), since permutations (with no repetitions) do not cover the case of coinciding delays. Due to space limitation, the corresponding case of block companion systems with broader first row combinations is left to future work.

It is well known that testing the arbitrary switching stability for a discrete-time switched system as in (1) is equivalent to verifying that the joint spectral radius of the family $\mathcal{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_p\}$ is less than one (Jungers (2009)), a generally NP-hard problem. Nevertheless, for the class of nonnegative $\mathcal{A}_j$ matrices (not only in block companion form), some interesting relaxations have been proposed. Blondel and Nesterov (2005) introduced some noteworthy inequalities. The simplest of them is:

$$\frac{1}{p}\rho\left(\sum_{i=1}^p \mathcal{A}_i\right) \leq \rho^*(\mathcal{A}) \leq \rho\left(\sum_{i=1}^p \mathcal{A}_i\right) \tag{18}$$

while an arbitrary approximating inequality is formulated resorting to Kronecker lifting:

$$\frac{1}{p^{1/k}}\rho^{1/k}\left(\sum_{i=1}^p \mathcal{A}_i^{[k]}\right) \leq \rho^*(\mathcal{A}) \leq \rho^{1/k}\left(\sum_{i=1}^p \mathcal{A}_i^{[k]}\right) \tag{19}$$

where $[k]$ denotes the k-th order Kronecker power of a matrix, and the right inequality converges to the equality as $k \to \infty$.

It is clear that, while the aforementioned approximation holds for general nonnegative matrices, its computational burden increases dramatically with the required accuracy. In this respect, the simplification introduced by Theorem 8, even though restricted to the special class of block companion matrices with permuting entries, is noteworthy, as it directly implicates the following Corollary.

Corollary 9. Consider a set of nonnegative matrices $\mathcal{M} = \{M_1, \ldots, M_m\}$, and the family of block companion matrices $\mathcal{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_{m!}\}$ whose first block-row is a permutation of $\mathcal{M}$, i.e. $A_{j_i} = M_{\mathcal{P}_i^j}$. Then the following result holds:

$$\rho^*(\mathcal{A}) < 1 \iff \rho\left(\sum_{i=1}^m M_i\right) < 1. \tag{20}$$

3.2 Removing the positivity assumption

Bearing in mind Theorem 6, the results of Theorem 8 and Corollary 9 can easily be restated (introducing conservatism) for not necessarily positive systems (i.e. arbitrary set of matrices $\{M_1, \ldots, M_m\}$).

In this case, denoting $\mathcal{A}$ the family of block companion matrices satisfying the conditions of the aforementioned Theorem and Corollary, we can easily state the following result.

Theorem 10. Consider a set of arbitrary matrices $\mathcal{M} = \{M_1, \ldots, M_m\}$ and the family $\mathcal{A}$ of block companion matrices built from $\mathcal{M}$ under the conditions of Theorem 8. Then, the following result holds:

$$\rho\left(\sum_{i=1}^m |M_i|\right) < 1 \implies \rho^*(\mathcal{A}) < 1. \tag{21}$$

3.3 A more general result

It can be surely conceded that the previously illustrated results concern very special classes of switched systems. This restriction, however, should not be unexpected, since the arbitrary switching stability is known to be a very difficult problem in the general case. Nevertheless, in order to extend the previous results to a broader class of switched systems, we note that the comparison method readily gives us an idea to remove the special combinatorial structure of Theorem 8.

One just needs to note that, for nonnegative matrices P and Q such that $P \leq Q$ it follows that $\rho(P) \leq \rho(Q)$ and clearly $x(k+1) = Px(k)$ is upper-bounded by $x(k+1) = Qx(k)$. For switched systems, we readily have that if $P_k \leq Q_k$ for all k, the trajectory of $x(k+1) = Q_k x(k)$ always dominates that of $x(k+1) = P_k x(k)$. This trivially leads to the following Theorem.

Theorem 11. Consider a switched system $x(k+1) = \mathcal{A}_k x(k)$ with nonnegative block companion $\mathcal{A}_k$ as in (2), consisting of possibly distinct blocks A_{k_i} at every time instant. If there exist nonnegative matrices $M_1, \ldots, M_m$ such that $\rho(\sum_{i=1}^m M_i) < 1$, and the associated family $\bar{\mathcal{A}}$ as in Theorem 8 such that at every k there exists a $\bar{\mathcal{A}}_j$ in $\bar{\mathcal{A}}$ with $\mathcal{A}_k \leq \bar{\mathcal{A}}_j$, then the system $x(k+1) = \mathcal{A}_k x(k)$ is asymptotically stable.

Let us present a simple example in order to illustrate the previous result.

Example 12. Consider a switched system $x(k+1) = \mathcal{A}_k x(k)$ where

$$\mathcal{A}_k = \begin{bmatrix} A_{k_1} & A_{k_2} \\ I_n & 0 \end{bmatrix} \geq 0, \quad \forall k. \tag{22}$$

If there exist nonnegative M_1, M_2 such that, for all k, $A_{k_1} \leq M_i$ and $A_{k_2} \leq M_j$, $i \neq j$, then $\rho(M_1 + M_2) < 1$ ensures the asymptotic stability of the system under examination. Indeed, in this case the family $\bar{\mathcal{A}}$ associated to M_1, M_2 is simply the family $\{\bar{\mathcal{A}}_1, \bar{\mathcal{A}}_2\}$ built as in Theorem 8, with:

$$\bar{\mathcal{A}}_1 = \begin{bmatrix} M_1 & M_2 \\ I_n & 0 \end{bmatrix}, \quad \bar{\mathcal{A}}_2 = \begin{bmatrix} M_2 & M_1 \\ I_n & 0 \end{bmatrix}, \tag{23}$$

and $x(k+1) = \mathcal{A}_k x(k)$ is asymptotically stable because at each time k it holds $\mathcal{A}_k \leq \bar{\mathcal{A}}_j$, for at least one $j = 1, 2$.

Remark 13. The previous result can easily be extended to not necessarily positive systems (i.e. arbitrary block companion $\mathcal{A}_k$), just requiring that $|\mathcal{A}_k| \leq \bar{\mathcal{A}}_j$ in Theorem 11.

We conclude this section noting that for companion nonnegative matrices with scalar blocks, i.e. the first row of (2) is scalar with $A_{k_i} = a_{k_i} \in \mathbb{R}_+$, some interesting facts have been unveiled in Nesterov and Protasov (2013), reducing the arbitrary switching stability criterion to $\sum_{i=1}^m a_{k_i} < 1$ for all $k = 1, \ldots, p$, and proving that the maximum growth rate of $x(k)$ is obtained staying on the $\mathcal{A}_{\bar{k}}$ system with largest spectral radius. The result does not require special structures (permutations etc.) on the $a_k = (a_{k_1}, \ldots, a_{k_m})$ sequences, but is derived resorting to successive rank-one corrections of a common matrix in a given uncertainty set. Clearly, the generalization of this result to the class of block companion matrices considered in this work is not trivial, since multiple-rank corrections are involved.

4. NUMERICAL EXAMPLES

4.1 Example 1

We start from an easy example which illustrates Theorem 8 and Corollary 9. Consider the switched system $x(k+1) = \mathcal{A}_{\sigma(k)} x(k)$, where $\mathcal{A}_{\sigma(k)}$ switches in the family $\mathcal{A}$ of the $4! = 24$ matrices $\mathcal{A}_j$ whose first block-row is made of all the permutations of the following nonnegative matrices:

$$M_1 = \begin{bmatrix} 0.16 & 0.04 & 0.02 \\ 0.10 & 0.10 & 0.08 \\ 0.08 & 0.09 & 0.10 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.10 & 0.20 & 0 \\ 0.16 & 0.10 & 0 \\ 0 & 0.08 & 0.24 \end{bmatrix},$$
$$M_3 = \begin{bmatrix} 0.06 & 0.01 & 0.06 \\ 0.20 & 0.09 & 0.08 \\ 0.23 & 0.04 & 0.06 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0.04 & 0.05 & 0.01 \\ 0.01 & 0.09 & 0.10 \\ 0.08 & 0.01 & 0.21 \end{bmatrix}, \tag{24}$$

i.e.

$$\mathcal{A}_j = \begin{bmatrix} A_{j_1} & A_{j_2} & A_{j_3} & A_{j_4} \\ I_3 & 0 & 0 & 0 \\ 0 & I_3 & 0 & 0 \\ 0 & 0 & I_3 & 0 \end{bmatrix} \quad j = 1, \ldots, 4!, \tag{25}$$

with $A_{j_i} = M_{\mathcal{P}_i^j}$, see Example 7 (Sect. 3) for clarifications.

Then, we can prove that the switching system is stable for all possible switching sequences, since $\rho(\sum_{i=1}^4 M_i) = 0.9974 < 1$, allowing to conclude that $\rho^*(\mathcal{A}) < 1$.

Notice that a similar conclusion can not be attained with the Joint Spectral Radius toolbox for MATLAB (using standard settings), see Vankeerberghen et al. (2014), which after 34 minutes of computations on an Intel Core i5 460M returns the following bounds: $0.9990 \leq \rho^*(\mathcal{A}) \leq 1.0261$.

4.2 Example 2

To conclude, we provide a simple example for Theorem 11, applied to an arbitrary (not positive) switched system (see Remark 13).

Consider the system $x(k+1) = \mathcal{A}_k x(k)$, where

$$\mathcal{A}_k = \begin{bmatrix} A_{k_1} & A_{k_2} \\ I_2 & 0 \end{bmatrix}, \tag{26}$$

with:

$$A_{k_1} = \begin{bmatrix} -0.1 \sin\left(\frac{\pi}{2}k\right) & 0.3 \cos\left(\frac{\pi}{2}k\right) \\ -0.3 \cos\left(\frac{\pi}{2}k\right) & 0.1 \sin\left(\frac{\pi}{2}k\right) \end{bmatrix},$$
$$A_{k_2} = \begin{bmatrix} 0.2 \cos\left(\frac{\pi}{2}k\right) & -0.05 \\ -0.1 & -0.2 \cos\left(\frac{\pi}{2}k\right) \end{bmatrix}. \tag{27}$$

Then, the nonnegative matrices M_1, M_2:

$$M_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.15 & 0.3 \\ 0.3 & 0.15 \end{bmatrix}, \tag{28}$$

such that $\rho(M_1 + M_2) = 0.75 < 1$, can be used to construct the "dominating" family $\bar{\mathcal{A}} = \{\bar{\mathcal{A}}_1, \bar{\mathcal{A}}_2\}$, built with the two permutations of M_1 and M_2 on the first block-row:

$$\bar{\mathcal{A}}_1 = \begin{bmatrix} M_1 & M_2 \\ I_2 & 0 \end{bmatrix}, \quad \bar{\mathcal{A}}_2 = \begin{bmatrix} M_2 & M_1 \\ I_2 & 0 \end{bmatrix}. \tag{29}$$

Since:

$$\left.\begin{array}{l} |A_{k_1}| \leq M_1 \\ |A_{k_2}| \leq M_2 \end{array}\right\} \text{ i.e. } |\mathcal{A}_k| \leq \bar{\mathcal{A}}_1, \quad \text{for } k = 1, 3, 5, \ldots \tag{30}$$

$$\left.\begin{array}{l} |A_{k_1}| \leq M_2 \\ |A_{k_2}| \leq M_1 \end{array}\right\} \text{ i.e. } |\mathcal{A}_k| \leq \bar{\mathcal{A}}_2, \quad \text{for } k = 2, 4, 6, \ldots \tag{31}$$

we can conclude that the system is asymptotically stable, as confirmed computing the joint spectral radius of the family $\mathcal{A} = \{\mathcal{A}_k\}$: $\rho^*(\mathcal{A}) = 0.6276$.

5. CONCLUSION AND FUTURE WORK

This work, taking inspiration from some recent results on discrete-time positive delay systems, has illustrated how the arbitrary switching stability can easily be studied for some classes of switched systems in block companion form. Starting from positive systems whose first block-row has a special combinatorial structure, we removed both assumptions (even though at the expense of some conservatism). The consequences on the problem of computing the joint spectral radius of matrices in block companion form have been highlighted, describing how a generally hard problem ($\rho^* < 1$?) becomes very simple in the aforementioned cases.

For what concerns future work, we note that recently introduced switched identification methods via machine learning techniques naturally provide state-space models in block companion form, see Smarra et al. (2020), Smarra and D'Innocenzo (2020), whose stability analysis is a generally difficult problem, particularly when large scale systems with several operating modes are estimated. The results of this work are directly applicable for the stability analysis of such models.

ACKNOWLEDGEMENTS

The research leading to these results has received funding from the Italian Government under the PON Ricerca & Innovazione 2014-2020 (AIM1877124-Attività 1) and CIPE resolution n.135 (Dec. 21, 2012), project INnovating City Planning through Information and Communication Technologies (INCIPICT).

REFERENCES

Blanchini, F., Colaneri, P., Valcher, M.E., et al. (2015). Switched positive linear systems. *Foundations and Trends® in Systems and Control*, 2(2), 101–273.

Blondel, V.D. and Nesterov, Y. (2005). Computationally efficient approximations of the joint spectral radius. *SIAM Journal on Matrix Analysis and Applications*, 27(1), 256–272.

Conte, F., De Iuliis, V., and Manes, C. (2017). Internally Positive Representations and stability analysis of linear delay systems with multiple time-varying delays. In *Positive Systems: Theory and Applications (POSTA 2016) Rome, Italy, September 14-16, 2016*, 81–93. Springer.

De Iuliis, V., D'Innocenzo, A., Germani, A., and Manes, C. (2019a). Internally positive representations and stability analysis of linear differential systems with multiple time-varying delays. *IET Control Theory & Applications*, 13(7), 920–927.

De Iuliis, V., D'Innocenzo, A., Germani, A., and Manes, C. (2020a). On the stability of coupled differential-difference systems with multiple time-varying delays: a positivity-based approach. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, 37–42.

De Iuliis, V., Germani, A., and Manes, C. (2017). Internally positive representations and stability analysis of linear difference systems with multiple delays. *IFAC-PapersOnLine*, 50(1), 3099–3104.

De Iuliis, V., Germani, A., and Manes, C. (2019b). Internally positive representations and stability analysis of coupled differential-difference systems with time-varying delays. *IEEE Transactions on Automatic Control*, 64(6), 2514–2521.

De Iuliis, V., Germani, A., and Manes, C. (2020b). Stability analysis of linear delay systems via Internally Positive Representations: an overview. In *Incorporating constraints on the Analysis of Delay and Distributed Parameter Systems (to appear)*. Springer.

Gao, H. and Chen, T. (2007). New results on stability of discrete-time systems with time-varying state delay. *IEEE Transactions on Automatic Control*, 52(2), 328–334.

Haddad, W.M. and Chellaboina, V. (2004). Stability theory for nonnegative and compartmental dynamical systems with time delay. *Systems & Control Letters*, 51(5), 355 – 361.

Jungers, R. (2009). *The joint spectral radius: theory and applications*, volume 385. Springer Science & Business Media.

Kerscher, W. and Nagel, R. (1984). Asymptotic behavior of one-parameter semigroups of positive operators. In *Positive Semigroups of Operators, and Applications*, 297–309. Springer.

Liu, X. and Lam, J. (2013). Relationships between asymptotic stability and exponential stability of positive delay systems. *Int. J. General Systems*, 42(2), 224–238.

Liu, X., Yu, W., and Wang, L. (2009). Stability analysis of positive systems with bounded time-varying delays. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 56(7), 600–604.

Liu, X., Yu, W., and Wang, L. (2010). Stability analysis for continuous-time positive systems with time-varying delays. *IEEE Transactions on Automatic Control*, 55(4), 1024–1028.

Meyer, C.D. (ed.) (2000). *Matrix Analysis and Applied Linear Algebra*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA.

Nesterov, Y. and Protasov, V.Y. (2013). Optimizing the spectral radius. *SIAM Journal on Matrix Analysis and Applications*, 34(3), 999–1013.

Niculescu, S.I. (2001). *Delay effects on stability: a robust control approach*, volume 269. Springer Science & Business Media.

Seuret, A., Gouaisbaut, F., and Fridman, E. (2015). Stability of discrete-time systems with time-varying delays via a novel summation inequality. *IEEE Transactions on Automatic Control*, 60(10), 2740–2745.

Smarra, F., Di Girolamo, G.D., De Iuliis, V., Jain, A., Mangharam, R., and D'Innocenzo, A. (2020). Data-driven switching modeling for MPC using Regression Trees and Random Forests. *Nonlinear Analysis: Hybrid Systems*, 36, 100882.

Smarra, F. and D'Innocenzo, A. (2020). Learning Markov Jump Affine Systems via Regression Trees for MPC. In *IFAC 2020 - 21st IFAC World Congress, Berlin.*

Vankeerberghen, G., Hendrickx, J., and Jungers, R.M. (2014). JSR: A toolbox to compute the joint spectral radius. In *Proceedings of the 17th international conference on Hybrid systems: computation and control*, 151–156. ACM.

Zhu, X.L. and Yang, G.H. (2008). Jensen inequality approach to stability analysis of discrete-time systems with time-varying delay. In *2008 IEEE American Control Conference*, 1644–1649.