New relaxed stability and stabilization conditions for differential linear repetitive processes [★]

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Abstract: The paper develops new results on the stability analysis of differential linear repetitive processes. These processes are a distinct class of two-dimensional systems that arise in the modelling of physical processes and also the existing systems theory for them can be used to effect in solving control problems for other classes of systems, including iterative learning control design. This paper uses a version of the Kalman-Yakubovich-Popov Lemma to develop relaxed conditions for the stability property in terms of linear matrix inequalities. The main result is reduced conservatism in applying tests for the stability property with an extension to control law design. A numerical example to illustrate the application of the new results is also given.

Keywords: Repetitive processes, stability along the pass, linear matrix inequalities, finite frequency domain

1. INTRODUCTION

Linear repetitive processes repeat the same finite duration operation over and over again, where each repetition is termed a pass and the finite duration the pass length Rogers et al. (2007). A physical example is metal rolling operations see, e.g, Rogers et al. (2015) which, in turn, cites the original work, where in essence deformation of the workpiece takes place between two rolls, multiple times until the desired shape of the workpiece is obtained (or to within an acceptable tolerance). The notation for variables in this paper is of the form $y_k(t)$, $0 \leq t \leq \alpha$ where y is the vector or scalar-valued variable under consideration, $\alpha < \infty$ is the pass length and $k \geq 0$ is the pass number.

Let $\{y_k\}_k$ denote the sequence of pass profiles generated by a repetitive process, where the profile on each pass, e.g., $y_i(t)$, acts as a forcing function on the dynamics of $y_{i+1}(t)$ and so on. The result can be oscillations that increase in amplitude with k. Moreover, this behaviour cannot be regulated by applying feedback control action on the current pass. Instead, recognising their 2D systems structure, i.e., along the passes (t) and from pass-to-pass (k), a control law must augment feedback control on the current pass with feedforward control action from the previous pass.

A stability theory for linear repetitive processes has been developed Rogers et al. (2007) using a setting that includes all linear time-invariant processes as special cases. Given the unique control problem this theory requires that a bounded initial pass profile y_0 produces a bounded sequence of pass profiles $\{y_k\}_k$, where the boundedness property is defined in terms of the norm on the underlying function space. This stability theory enforces the bounded-input bounded-output property either over the finite and fixed pass length or uniformly with respect to α, where the former property is known as asymptotic stability and the latter stability along the pass.

Stability along the pass, which can be analyzed mathematically by considering $\alpha \to \infty$, is the stronger property. Moreover, asymptotic stability is a necessary condition for stability along the pass. This stability theory has been applied to problems in other areas of control theory, e.g., iterative learning control (ILC) law design, see, e.g. Paszke et al. (2016) and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle Roberts (2002). Applications include differential processes, where the along the pass dynamics are governed by a linear ordinary differential equation and also discrete dynamics where the along the pass dynamics are governed by an ordinary difference equation. The finite pass length and the structure of the boundary conditions are the main differences with other classes of 2D linear systems. See Rogers et al. (2007) for a full treatment of this aspect, including repetitive process dynamics to which

the existing theory for other classes of 2D linear systems is not applicable even at the stability testing stage.

In the case of differential (and discrete) linear repetitive processes, testing for stability can be problematic computational wise due to the need to determine the locations of the eigenvalues of a frequency response matrix relative to the unit circle in the complex plane and also makes control law design less transparent. This paper develops new stability tests by dividing the entire frequency domain into several sub-intervals and then applying the Kalman-Yakubovich-Popov (KYP) Lemma to each frequency sub-interval and builds on preliminary research on this approach in Rogers et al. (2016); Boski et al. (2018). The major outcome is reduced conservatism by the introduction of additional decision variables in the final linear matrix inequality (LMI) forms. Also the new conditions are extended to allow control law design and two numerical examples are given to demonstrate the effectiveness of the new results.

Throughout this paper the null and identity matrices, respectively, with compatible dimensions are denoted by 0 and I. For a square matrix M, $\mathrm{sym}(M)$ denotes $M + M^T$ and $\rho(\cdot)$ the spectral radius of its matrix argument. Furthermore, $M \succ 0$ ($M \prec 0$) means that the symmetric matrix M is positive definite (negative definite). Finally, $(\star)$ denotes a block entry in a symmetric matrix and the superscript $*$ denotes the complex conjugate transpose of a matrix. Finally, $\otimes$ denotes the Kronecker product.

The new results in this paper are formulated in terms of LMIs and hence the following lemmas are useful in transforming non-LMI formulations into LMI form, where the first is the KYP lemma, the second the Projection Lemma and the third is a version of the bounding inequality.

Lemma 1. Iwasaki and Hara (2005) Let A, B_0 and Θ be given. Then if $\det(\mathrm{j}\omega I - A) \neq 0$ for all $\omega \in [0, \infty)$ the following conditions are equivalent:

i) The frequency domain inequality

$$\begin{bmatrix} (\mathrm{j}\omega I - A)^{-1}B_0 \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (\mathrm{j}\omega I - A)^{-1}B_0 \\ I \end{bmatrix} \prec 0 \qquad (1)$$

holds $\forall \omega \in \Omega$ where Ω is the frequency range, i.e. ω belongs to a subset of real numbers denoted by Ω and specified as in Table 1.

ii) There exist symmetric matrices $Q \succ 0$ and P such that

$$\begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix}^* (\Psi \otimes Q + \Phi \otimes P) \begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix} + \Theta \prec 0, \qquad (2)$$

where

$$\Psi = \begin{bmatrix} \tau & \upsilon \\ \upsilon^* & \varsigma \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad (3)$$

The values of τ, υ and ς for specified choices of $\omega \in \Omega$ are shown in Table 1 and LF, MF and HF denote, respectively, the low, middle and high frequency ranges.

Table 1. Frequency ranges of interest

Ω	LF	MF	HF
Ω	$|\omega| < \omega_l$	$\omega_1 \leq \omega \leq \omega_2$	$|\omega| > \omega_h$
τ	-1	-1	1
υ	0	$\mathrm{j}\frac{\omega_1+\omega_2}{2}$	0
ς	ω_l^2	$-\omega_1\omega_2$	$-\omega_h^2$

Lemma 2. Gahinet and Apkarian (1994) Given matrices $\Gamma = \Gamma^T \in \mathbb{R}^{p \times p}$ and two matrices Λ, Σ of column dimension p, there exists an unstructured matrix W that satisfies

$$\Gamma + \mathrm{sym}\{\Lambda^T W \Sigma\} \prec 0, \qquad (4)$$

if, and only if

$$\Lambda_\perp{}^T \Gamma \Lambda_\perp \prec 0, \text{ and } \Sigma_\perp{}^T \Gamma \Sigma_\perp \prec 0, \qquad (5)$$

where $\Lambda_\perp$ and $\Sigma_\perp$ are arbitrary matrices whose columns form, respectively, a basis of the nullspaces of Λ and Σ. Hence $\Lambda \Lambda_\perp = 0$ and $\Sigma \Sigma_\perp = 0$.

2. STABILITY OF DIFFERENTIAL LINEAR REPETITIVE PROCESSES

The differential linear repetitive processes considered in this paper are described by the following state-space model over $0 \leq t \leq \alpha$, $k \geq 0$,

$$\begin{aligned} \dot{x}_{k+1}(t) &= A x_{k+1}(t) + B_0 y_k(t) + B u_{k+1}(t), \\ y_{k+1}(t) &= C x_{k+1}(t) + D_0 y_k(t) + D u_{k+1}(t), \end{aligned} \qquad (6)$$

where $x_k(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^m$ and $y_k(t) \in \mathbb{R}^p$, respectively, denote the process state, input and pass profile (output) vectors at time instant t on pass k. The boundary conditions for these processes are the state initial vector on each pass and the initial pass profile. In the analysis of this paper, no loss of generality arises from assuming that $x_{k+1}(0) = 0$, $\forall k \geq 0$, and $y_0(t) = f(t)$, $0 \leq t \leq \alpha$, where the entries in the vector $f(t) \in \mathbb{R}^p$ are known functions of t and no further explicit mention of the boundary conditions is made in the rest of this paper. See Rogers et al. (2007) for other forms of boundary conditions for repetitive processes that cannot be represented by 2D systems models of the Roesser (Roesser, 1975) or Fornasini Marchesini (Fornasini and Marchesini, 1978) forms.

The following result is the starting point for the analysis in this paper.

Lemma 3. Rogers et al. (2007) A differential linear repetitive process described by (6) is stable along the pass if and only if

i) $\rho(D_0) < 1$,

ii) all eigenvalues of the matrix A lie in the open left-half of the complex plane, and

iii) all eigenvalues of $G(s) = C(sI - A)^{-1}B_0 + D_0$, $s = \mathrm{j}\omega$, $\forall \omega \geq 0$, have modulus strictly less than unity.

The first two conditions in Lemma 3 pose no computational difficulties, where the first is the necessary and sufficient condition for asymptotic stability. Also $\rho(D_0) < 1$ describes the direct feedthrough from the previous pass profile to the next. Moreover, asymptotic stability is independent of the current pass state dynamics and this is a direct consequence of the finite pass length, over which duration even an unstable example can only produce a bounded output. In this respect, the second condition is to be expected but, see Rogers et al. (2007), examples exist that demonstrate this condition is also only necessary for stability along the pass and the third condition is also required. This condition requires frequency attenuation of the previous pass profile dynamics over the complete frequency range and condition i) enforces this property at the start of each pass. Hence this stability theory has a well defined physical basis.

The key difficulty for stability testing and control law design is condition iii), which is equivalent to

$$\rho\left(G(j\omega)\right) < 1, \ \forall \omega \in [0, \infty). \tag{7}$$

Alternatively, the this last result requires, for each $\omega \in [0, \infty)$, the existence of a $R(j\omega) \succ 0$ such that the following Lyapunov inequality holds

$$G(j\omega)^* R(j\omega) G(j\omega) - R(j\omega) \prec 0, \ \forall \omega \in [0, \infty).$$

but stability along the pass is then characterized by a convex feasibility test over an infinite-dimensional space. Furthermore, the function $R(j\omega)$ depends on ω and hence this inequality cannot be easily solved. However, simple multipliers, e.g., $R(j\omega) = R$ or $R(j\omega) = I$ can be used to avoid computational problems when multipliers with direct dependence on ω are considered.

These simple multipliers allow application of the results of Lemma 1 and this converts the problem to a finite-dimensional convex optimization problem over constraints in terms of LMIs that are comparatively easier to solve and directly lead to the control law design algorithms. In the remainder of this section, the results of Lemmas 1 and 2 are used to develop conditions for stability along the pass of differential linear repetitive processes in the form of LMI conditions. The starting point is to divide the complete frequency range into N intervals such that

$$[0, \infty) = \bigcup_{i=1}^{N} [\omega_{i-1}, \omega_i), \tag{8}$$

where $\omega_0 = 0$ and $\omega_N = \infty$ and then the result of Lemma 1 is applied to each interval. Furthermore, this allows the use of piecewise constant multipliers over a priori chosen frequency ranges and the following new result can be established.

Theorem 4. Suppose that the entire frequency range is arbitrarily divided into N different frequency intervals as given in (8). Then, a differential linear repetitive process described by (6) is stable along the pass if there exist matrices $S \succ 0$, $P_{2i} \succ 0$, $Q_i \succ 0$ and a symmetric P_{1i} such that the following LMIs

$$[I \ A](\Phi \otimes S)[I \ A]^T \prec 0, \tag{9}$$

$$\begin{bmatrix} A & B_0 \\ I & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \Upsilon_{1i} & \begin{bmatrix} 0 \\ C^T P_{2i} D_0 \end{bmatrix} \\ (\star) & D_0^T P_{2i} \mathcal{D}_0 - P_{2i} \end{bmatrix} \begin{bmatrix} A & B_0 \\ I & 0 \\ 0 & I \end{bmatrix} \prec 0, \tag{10}$$

where

$$\Upsilon_{1i} = (\Psi_i \otimes Q_i + \Phi \otimes P_{1i}) + \begin{bmatrix} 0 & 0 \\ 0 & C^T P_{2i} C \end{bmatrix}$$

are feasible for all $i = 1, \dots, N$ where τ_i, υ_i and ς_i are specified in Table 2

Table 2. Values of τ_i, υ_i and ς_i

i	1	$1 < i < N$	N
τ_i	-1	-1	1
υ_i	0	$j\frac{\omega_{i-1}+\omega_i}{2}$	0
ς_i	ω_0^2	$-\omega_{i-1}\omega_i$	$-\omega_{N-1}$

Remark 1. The entries in Table 2 are those in Table 1 but in this case they also depend on the interval number i, which takes values from 0 to N as in (8).

Proof. Suppose that there exist matrices $S \succ 0$, $P_{2i} \succ 0$, $Q_i \succ 0$ and P_{1i} for all $i = 1, \dots, N$, such that the LMIs (9)-(10) are feasible. Also, feasibility of (9)-(10) implies that

$$D_0^T P_{2i} D_0 - P_{2i} \prec 0, \ AS + SA^T \prec 0.$$

Equivalently, conditions i) and ii), respectively, of Lemma 3 hold. Next, by routine manipulations each LMI in (10) can be rewritten as

$$\begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix}^T (\Psi \otimes P_{1i} + \Phi_i \otimes Q_i) \begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D_0 \\ 0 & I \end{bmatrix}^T (\Pi \otimes P_{2i}) \begin{bmatrix} C & D_0 \\ 0 & I \end{bmatrix} \prec 0, \tag{11}$$

where Φ is given in (3), $\Pi = \text{diag}\{1, -1\}$ and Ψ_i is of the form (3), where τ, υ and ς, respectively, are replaced by τ_i, υ_i and ς_i. Direct application of the result of Lemma 1 to each frequency interval with the choice of piecewise constant matrices P_{1i}, $P_{2i} \succ 0$, $Q_i \succ 0$ $\forall i = 1, \dots, N$ gives

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi \otimes P_{2i}) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \prec 0, \tag{12}$$

where G is defined by iii) in Lemma 3. Moreover, (12) can be written as

$$G(j\omega)^* P_{2i} G(j\omega) - P_{2i} \prec 0,$$

where the existence of $P_{2i} \succ 0$ directly implies that $\rho(G(j\omega)) < 1$, $\forall \omega \geq 0$, i.e., feasibility of (10) guarantees that condition iii) of Lemma 3 holds and the proof is complete.

In the case of $i = 0$ (the low frequency range starting from $\omega = 0$, i.e. $\omega_0 = 0$) then, following the development in Iwasaki and Hara (2005), the matrix Ψ_i in (11) has to be chosen from the second column ($i = 1$) in Table 2. Also, for $i = N$ (the high frequency range ending with $\omega = \infty$, i.e. $\omega_N = \infty$) the matrix Ψ_i in (11) has to be chosen from last column in Table 2.

Application of this last result requires a systematic method to specify the number of frequency ranges and also the range of frequencies each contains. Moreover, increasing the number of ranges makes the condition less conservative and as $N \to \infty$ the necessary and sufficient condition is approached.

Remark 2. Setting $\Psi = 0$ recovers the standard LMI-based stability along the pass result given in Rogers et al. (2007).

3. NOVEL LMI-BASED CONDITIONS FOR STABILITY ALONG THE PASS

The conditions of Theorem 4 can be improved, i.e., less conservativeness, by introducing additional slack matrix variables. To proceed, introduce the notation

$$\mathbb{A} = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix}, \ \Gamma_{11i} = \begin{bmatrix} \tau_i Q_i & 0 \\ 0 & 0 \end{bmatrix}, \ \Gamma_{22i} = \begin{bmatrix} \varsigma_i Q_i & 0 \\ 0 & -P_{2i} \end{bmatrix}.$$

Then the theorem established next reformulates the results of Theorem 4 to give new LMI-based conditions for stability along the pass.

Theorem 5. Suppose that the entire frequency range is arbitrarily divided into N possible different frequency intervals as in (8). Then, a differential linear repetitive process described by (6) is stable along the pass if there

exist matrices $P_{2i} \succ 0$, $Q_i \succ 0$, $S \succ 0$, F_{1i}, F_{2i}, F_{3i}, W_1, W_2, W_3 and a symmetric P_{1i} such that the LMIs (9) and

$$\begin{bmatrix} \Gamma_{11i} - \text{sym}\{W_1\} & (\star) & (\star) \\ \Gamma_{21i} + \mathbb{A}^T W_1^T - W_2 & \Gamma_{22i} + \text{sym}\{W_2 \mathbb{A}\} & (\star) \\ F_{30i} - W_3 & -F_{12i}^T + W_3 \mathbb{A} & P_{2i} - \text{sym}\{F_{3i}\} \end{bmatrix} \prec 0, \quad (13)$$

where

$$\Gamma_{21i} = \begin{bmatrix} P_{1i} + v_i^* Q_i & F_{1i} \\ 0 & F_{2i} \end{bmatrix}, F_{12i} = \begin{bmatrix} F_{1i} \\ F_{2i} \end{bmatrix}, F_{30i} = \begin{bmatrix} 0 & F_{3i} \end{bmatrix}$$

are feasible for all $i = 1, \dots, N-1$.

Proof. Assume that the LMIs defined in (9) and (13) are feasible for all $i = 1, \dots, N$. Then it is immediate that the feasibility of (9) ensures the stability of the state matrix A. Next, the LMIs in (13) can be rewritten as

$$\Gamma_i + \text{sym} \left\{ \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} \begin{bmatrix} -I & \mathbb{A} & 0 \end{bmatrix} \right\} \prec 0, \quad (14)$$

where

$$\Gamma_i = \begin{bmatrix} \Gamma_{11i} & \Gamma_{21i}^T & F_{30i}^T \\ \Gamma_{21i} & \Gamma_{22i} & -F_{12i} \\ F_{30i} & -F_{12i}^T & P_{2i} - \text{sym}\{F_{3i}\} \end{bmatrix}.$$

Introduce the matrices

$$\Lambda = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, W = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}, \Sigma = \begin{bmatrix} -I & \mathbb{A} & 0 \end{bmatrix}$$

and then (14) can be reformulated by application of Lemma 2 as the second inequality in (5), i.e.

$$\Sigma_\perp{}^T \Gamma \Sigma_\perp \prec 0, \quad (15)$$

where by construction the matrix $\Sigma_\perp$ is

$$\Sigma_\perp = \begin{bmatrix} \mathbb{A} & 0 \\ I & 0 \\ 0 & I \end{bmatrix}.$$

Since $\Lambda = I$ then $\Lambda_\perp = 0$ and hence the first inequality in (5) holds. Furthermore, after some routine matrix manipulations the inequality (15) can be rewritten as

$$\Gamma_{1i} + \text{sym} \left\{ \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} F_{1i} \\ F_{2i} \\ F_{3i} \end{bmatrix} \begin{bmatrix} C & D_0 & -I \end{bmatrix} \right\} \prec 0, \quad (16)$$

where

$$\Gamma_{1i} = \begin{bmatrix} \tau_i A^T Q_i A + \varsigma_i Q_i + \text{sym}\{P_{1i} A + v_i^* Q_i A\} \\ \tau_i B_0^T Q_i A + v_i B_0^T Q_i + B_0^T P_{1i} \\ 0 \end{bmatrix}$$
$$\begin{matrix} \tau_i A^T Q_i B_0 + v_i^* Q_i B_0 + P_{1i} B_0 & 0 \\ \tau_i B_0^T Q_i B_0 - P_{2i} & 0 \\ 0 & P_{2i} \end{matrix} \end{bmatrix} \prec 0$$

and by Lemma 2, feasibility of (16) implies that the inequality

$$\Sigma_{1_\perp}^T \Gamma_1 \Sigma_{1_\perp} \prec 0$$

must hold where

$$\Sigma_{1_\perp} = \begin{bmatrix} I & 0 \\ 0 & I \\ C & D_0 \end{bmatrix}.$$

Finally, this last inequality is equivalent to (10) and by Theorem 4 stability along the pass is ensured.

4. CONTROL LAW DESIGN

In this section, the stability results of the previous section are extended to the design of a control law with the structure

$$u_{k+1}(t) = K_1 x_{k+1}(t) + K_2 y_k(t), \quad (17)$$

where K_1 and K_2 are the control law matrices to be found. Application of this control law results in the controlled process state-space model

$$\dot{x}_{k+1}(t) = (A + BK_1) x_{k+1}(t) + (B_0 + BK_2) y_k(t),$$
$$y_{k+1}(t) = (C + DK_1) x_{k+1}(t) + (D_0 + DK_2) y_k(t). \quad (18)$$

In addition to stability, the control law should also satisfy some specifications on the transient behaviour along the passes, e.g., fast and well-damped transient response and reasonable (implementable) control law gains, where the former requirement is closely related to the locations of the eigenvalues of $A + BK_1$. In this section, the problem solved is how to place the eigenvalues of this matrix to a subregion of the open left-half of the complex plane, where the choice of this region is a matter for judgement based on knowledge of the particular application considered.

Suppose that the region of interest is the interior of the circle of radius $r > 0$ with center at c denoted by $C(c, r)$, i.e.,

$$C(c, r) := \{x + jy \in \mathbb{C} : |x + jy - c| < r\}.$$

To guarantee that the interior of this circle is located in open the left-half of the complex plane requires $c < 0$ and $|c| > r$. Let $\lambda = -\zeta \omega_n \pm j \omega_d$ be a pair of eigenvalues of $A + BK_1$, where $0 < \zeta < 1$ is the damping ratio, ω_n is the undamped natural frequency, and $\omega_d := \omega_n \sqrt{1 - \zeta^2}$ is the damped natural frequency. Further, suppose that each λ is placed in $C(c, r)$. Then

$$\zeta > \sqrt{1 - \left(\frac{r}{c}\right)^2}, \ \omega_d < r, \ -c - r < \omega_n < -c + r.$$

The LMI formulation for the poles of $A + BK_1$ to lie in the region $C(c, r)$ is characterized as the following inequality, which is an extension of the LMI (9).

$$\begin{bmatrix} (A + BK_1)^T & I \end{bmatrix} \left(\begin{bmatrix} 1 & -c \\ -c & |c|^2 - r^2 \end{bmatrix} \otimes S \right) \begin{bmatrix} A + BK_1 \\ I \end{bmatrix} \prec 0, \quad (19)$$

where $S \succ 0$ is a matrix variable, see again LMI (9).

Summarizing, the aim is to develop LMI-based results that enable the computation of the control law matrices of (17) for chosen frequency partitioning and all eigenvalues of $A + BK_1$ placed in the circle of radius r with center at $(-c, 0)$. To proceed, introduce the following notation

$$\mathbb{B} = \begin{bmatrix} B \\ D \end{bmatrix}, \ \mathbb{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

and then the results of the previous section can be directly used to develop control law design algorithms.

Theorem 6. Suppose that a control law of the form (17) is applied to a differential linear repetitive process described by (6). Suppose also that the entire frequency range is arbitrarily divided into N different frequency intervals as in (8). Then the resulting controlled process (18) is stable along the pass and all eigenvalues of $A + BK_1$ are located in the circle of radius r with center at $(-c, 0)$ if there exist matrices $\hat{P}_{2i} \succ 0$, $\hat{Q}_i \succ 0$, $\hat{S} \succ 0$, $\hat{F}_{1i}$, $\hat{F}_{2i}$, $\hat{F}_{3i}$, $\hat{W}_1$, $\hat{W}_2$,

N_1, N_2, a symmetric $\hat{P}_{1i}$ and real scalars p and q such that the following LMIs

$$\begin{bmatrix} \hat{S} & -c\hat{S} \\ (\star) & (|c|^2-r^2)\hat{S} \end{bmatrix} + \text{sym}\left\{ \begin{bmatrix} -\hat{W}_1^T \\ (A\hat{W}_1+BY_1)^T \end{bmatrix} \begin{bmatrix} qI & -pI \end{bmatrix} \right\} \prec 0, \quad (20)$$

$$\begin{bmatrix} \hat{\Gamma}_{11i} - \text{sym}\{\hat{W}\} & (\star) \\ \hat{\Gamma}_{21i} + (\mathbb{A}\hat{W}+\mathbb{B}N-\hat{W})^T & \hat{\Gamma}_{22i} + \text{sym}\left\{\mathbb{A}\hat{W}+\mathbb{B}N\right\} \\ \hat{F}_{30i}-\hat{W}_3 & -\hat{F}_{12i}^T+[0\ I]\mathbb{A} \end{bmatrix}$$

$$\begin{bmatrix} (\star) \\ (\star) \\ \hat{P}_{2i}-\hat{F}_{3i}-\hat{F}_{3i}^T \end{bmatrix} \prec 0, \quad (21)$$

where the pair of scalars p and q satisfies $p^2 - 2cpq + q^2(|c|^2-r^2) < 0$ and

$$\hat{W} = \text{diag}\{\hat{W}_1, \hat{W}_2\}, \ \hat{W}_3 = [0\ \hat{W}_2], \ N = [N_1\ N_2],$$

$$\Gamma_{21i} = \begin{bmatrix} \hat{P}_{1i}+v_i^*\hat{Q}_i & \hat{F}_{1i} \\ 0 & \hat{F}_{2i} \end{bmatrix}, \ \hat{\Gamma}_{11i} = \begin{bmatrix} \tau_i\hat{Q}_i & 0 \\ 0 & 0 \end{bmatrix}, \ \hat{\Gamma}_{22i} = \begin{bmatrix} \varsigma_i\hat{Q}_i & 0 \\ 0 & -\hat{P}_{2i} \end{bmatrix}$$

are feasible for all $i = 1, \ldots, N - 1$. Also, if these LMIs are feasible, the required control law matrices K_1 and K_2 of (17) can be calculated as

$$[K_1\ K_2] = N\hat{W}^{-1}. \quad (22)$$

Proof. Suppose that the LMIs in (20) and (21) hold. Then a feasible solution of these inequalities implies that $\hat{P}_{2i} \succ 0$, $\hat{Q}_i \succ 0$, $\hat{S} \succ 0$ and the matrices $\hat{W}_1$ and $\hat{W}_2$ are nonsingular. Next, let $\mathcal{W}_1 = \text{diag}\left\{\hat{W}_1^{-1}, \hat{W}_1^{-1}\right\}$; then pre- and post-multiplying (20) by $\mathcal{W}_1^T$ and $\mathcal{W}_1$, respectively, results in a version of the first inequality of Lemma 2 where

$$\Gamma = \begin{bmatrix} S & -cS \\ -cS & (|c|^2-r^2)S \end{bmatrix}, \Lambda = \begin{bmatrix} -I & (A+BK_1)^T \end{bmatrix}, \Sigma = \begin{bmatrix} qI & -pI \end{bmatrix}$$

and $K_1 = N_1\hat{W}_1$, $S = \hat{W}_1^{-T}\hat{S}\hat{W}_1^{-1}$. Since $\Sigma_\perp^T\Gamma\Sigma_\perp \prec 0$ holds for any p, q satisfying $p^2 - 2cpq + q^2(|c|^2-r^2) < 0$ then the equivalence between (20) and (19) follows from Lemma 2.

Next, apply the congruence transformation specified by $\text{diag}\left\{\hat{W}^{-1}, \hat{W}^{-1}, \hat{W}_2^{-1}\right\}$ to (21). Setting $W_1 = W_2 = \hat{W}^{-1}$, $Q_i = \hat{W}_1^{-T}\hat{Q}_i\hat{W}_1^{-1}$, $P_{1i} = \hat{W}_1^{-T}\hat{P}_{1i}\hat{W}_1^{-1}$, $P_{2i} = \hat{W}_2^{-T}\hat{P}_{2i}\hat{W}_2^{-1}$, $F_{1i} = \hat{W}_1^{-T}\hat{F}_{1i}\hat{W}_2^{-1}$, $F_{2i} = \hat{W}_2^{-T}\hat{F}_{2i}\hat{W}_2^{-1}$ and $F_{3i} = \hat{W}_2^{-T}\hat{F}_{3i}\hat{W}_2^{-1}$ transforms the LMI (21) into a version of (13) for the process given by (18). Finally, by employing the same steps used as those in proving (21), it follows that feasibility of (20) implies feasibility of (9). Therefore the controlled process is stable along the pass and the proof is complete.

Remark 3. Comparing Theorems 5 and 6, it is immediate that slack matrix variables W_1 and W_2 must be the same, i.e. $W_1 = W_2$ in control law design. Moreover, these matrix variables must be block diagonal and this may introduce a level of conservatism into the design.

Remark 4. The design conditions (21) are LMIs that can be easily and effectively solved via numerical software. In addition, optimal values of the scalar parameters p and q can be sought to reduce the conservatism of the solutions.

5. SIMULATION BASED CASE STUDY

As an illustrative example of the new results in this paper, the design of the   control law (17) for a metal rolling process is considered. Following Rogers et al. (2007), the (simplified) state space model of the metal rolling process written in the form of (6) is

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & 0 \end{bmatrix}, \ B_0 = \begin{bmatrix} 0 \\ -b_0 + a_0 b_2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ c_0 \end{bmatrix},$$

$$C = [1\ 0], \ D_0 = -b_2, \ D = 0$$

and

$$a_0 = \frac{\alpha_1\alpha_2}{M(\alpha_1+\alpha_2)}, b_0 = \frac{-\alpha_1\alpha_2}{M(\alpha_1+\alpha_2)},$$

$$b_2 = \frac{-\alpha_2}{\alpha_1+\alpha_2}, \quad c_0 = \frac{-\alpha_1}{M(\alpha_1+\alpha_2)},$$

where $\alpha_1 = 600N/m$ is the stiffness of the adjustment mechanism spring, $\alpha_2 = 2000N/m$ is the hardness of the metal strip and $M = 100kg$ denotes the lumped mass of the roll-gap adjusting mechanism. By choosing the above parameter values the state-space model matrices of (6) are

$$\left[\begin{array}{c|c|c|c} A & B & B_0 \\ \hline C & D & D_0 \end{array}\right] = \left[\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ -4.6154 & 0 & -0.0023 & 1.0651 \\ \hline 1 & 0 & 0 & 0.7692 \end{array}\right].$$

This repetitive process is unstable along the pass since condition *iii*) of Lemma 3 is not satisfied and hence the LMIs in (13) are not feasible for any frequency partitioning. The plot of $|\rho(G(j\omega))|$ is given in Fig. 1 confirms that $|\rho(G(j\omega))| > 1$ for some frequencies.
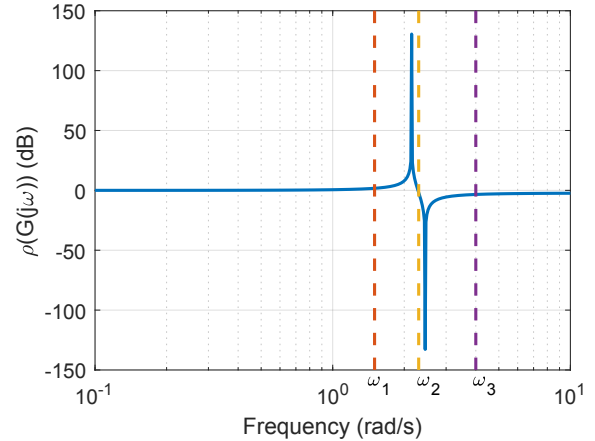

Fig. 1. Plot of $|\rho(G(j\omega))|$ for the nominal model.

Assume now that the entire frequency range $[0, \infty)$ is arbitrarily divided into 4 frequency intervals, which are depicted with dashed lines in Figures 1 and 2

$$[0, 1.5) \cup [1.5, 2.3) \cup [2.3, 4) \cup [4, \infty) = [0, \infty).$$

Hence, $\omega_0 = 0rad$, $\omega_1 = 1.5rad$, $\omega_2 = 2.3rad$, $\omega_3 = 4rad$ and $\omega_4 = \infty$. Then executing the design procedure given in Theorem 6 for $p = -11$, $q = 5$ and assuming all eigenvalues of $A+BK_1$ to be located in the circle of radius 3 with center at $(-4, 0)$ gives a controlled process that is stable along the pass. Moreover, based on the solution of the LMIs (20), (21) and using (22) the following control law matrices are obtained

$$K_1 = 10^3 \times [-0.0606\ 1.7996], K_2 = 483.6303$$

and the eigenvalues of $A+BK_1$ are $\{-2.0765 \pm 0.4047j\}$, which lie within the region $C(-4, 3)$. The stability along the pass property is confirmed in Fig. 2 where it is seen

that spectral radius of the controlled process is less than 1 in all frequency ranges, i.e., $|\rho(G_C(j\omega))|$, $\omega \in [0, \infty)$ where

$$G_C(j\omega) = (C + DK_1)(j\omega I - (A + BK_1))^{-1}(B_0 + BK_2) + D_0 + DK_2$$

and hence the design specifications are met. Note that the design procedure developed in Boski et al. (2018) fails to generate a feasible solution. This means that the previous methods for computing the control law matrices of (17) over finite frequency ranges fail but the new design developed in this paper allows this computation. The
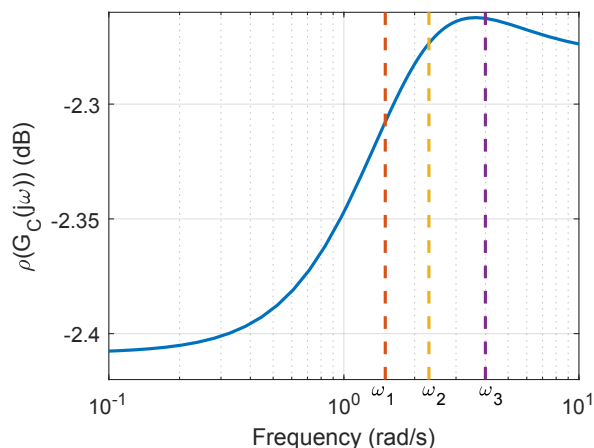

Fig. 2. Spectral radius associated with the controlled dynamics.

controlled dynamics were simulated over 20 passes where the initial pass profile ($k = 0$) is chosen as

$$y_0(t) = \sin(2\pi t/10) + 0.5\sin(4\pi t/10), \text{ for } 0 \le t \le 20.$$

Fig. 3 shows the pass profile values as a function of t and k and confirms that the controlled dynamics are stable along the pass.
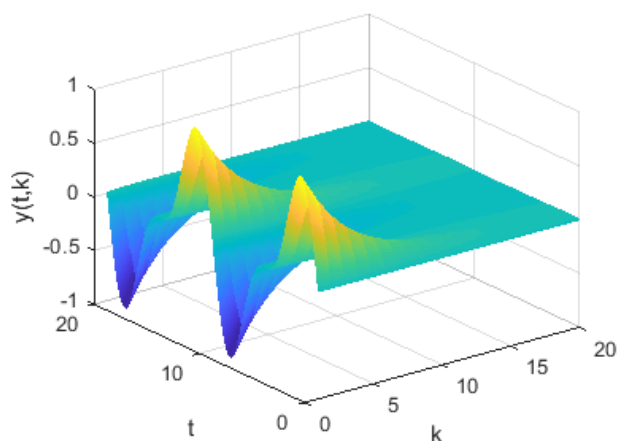

Fig. 3. Response of the controlled process.

6. CONCLUSIONS

This paper has developed novel conditions for stability along the pass and stabilization of differential linear repetitive processes. These conditions are given in terms of LMIs and therefore they are numerically trackable. The major

benefit of the new results in this paper is that stability tests extend in a direct manner to give control law design algorithms. The advantage over current results in this last aspect is the avoidance of product terms between the state-space model matrices and some Lyapunov/LMI decision matrices. This decoupling has been achieved through the use of slack matrix variables and therefore a reduction in the conservatism of the stability tests is possible. Moreover, differential repetitive processes are of major interest in terms of eventual application since there will be cases where control law design in the analog domain, e.g., in the iterative learning control area, is the preferred or only possible setting. All of the new results in this paper are the subject of ongoing research with the eventual aim of developing control laws for the case when there is uncertainty associated with repetitive process models and/or control law design in the presence of constraints on, e.g., the input and/or output signals.

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