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Observers of Vlasov-Poisson system

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Abstract: This work focuses on observer's for one-dimensional (1D) Vlasov-Poisson (VP) system. Thanks to the discontinuous Galerkin method (DGM) to put the system into a suitable and explicit state space representation form. Then we construct a state observer of finite dimension that assures asymptotic convergence under weak conditions. Indeed, we introduce a useful Linear Parameter Varying System formulation to compute the observer gain matrix from a Linear Matrix Inequality. Moreover, since matrices obtained by the DGM are tridiagonal, we show that only a reduced order observer is necessary to estimate the whole state of the system. In the noise context, extension to H_∞ state estimation is also established.

Keywords: Observers, Discontinuous Galerkin method, Non-linear PDE.

1. INTRODUCTION

The study of the stability and the balance of the charged particles is one of the applications in plasma physics, in particular for the controlled fusion which one seeks to realize to provide energy in civilian projects such as ITER (International Thermonuclear Experimental Reactor) at CEA [1] in Cadarache or military such as Laser MegaJoule (LMJ) at CESTA in Bordeaux. One can also be interested, for example for the damage of space materials subjected to charged particle beams, to the study of plasmas and beams of particles present in the space. Another application is the construction and study of particle accelerators.

It is recalled that a plasma, often called the fourth state of matter,is in fact a globally neutral ionized gas, consisting of neutral and charged particules that react both with each other and with the presence of an electromagnetic field. Our of thermodynamic equilibrium which is reached thanks to the collisions between particles, the behavior of a plasma can't always be assimilated to that of a fluid. Plasmas and charged particle beams are modeled by a statistical function. This distribution function represents the probability of presence of particles at a point in the phase space. This function is then a solution of the Vlasov equation which involves an electromagnetic field created by charged particles, itself a solution of Maxwell's equations. Under certain assumptions, the model can be reduced to the probleme of Vlasov equation coupled to a Poisson equation. It is from the latter that we will be interested in this paper. Numerous numerical methods have been developed to solve the Vlasov-Poisson system. We have mainly three families of classical numerical methods fort these equations: PIC [2] methods Birdsall and Langdon (1991)Tskhakaya and Schneider (2007), Eulerian methods (such as the discontinuous Galerkin method or the finite volume or different finite method) Nicolas Crouseilles and Sonnendrücker (2009) and semi-Lagrangian methods Cheng and Knorr. (1976) Eric Sonnendrücker and Ghizzo (1998).

Among these methods, that of discontinuous Galerkin is of paramount importance for our research work. It is a variational type method (like the finite element method) where functions are approched by piecewise polynomial functions and where the discontinuity between each element of the mesh requires the definition of a flux (like the finite volume method). See Blanca Ayuso De Dios and Shu. (2012) Eric Madaule and Sonnendrücker (2014) for the work that has been done on (VP). This discontinuous Galerkin methodology has the advantage of being both conservative on the macroscopic quantities (charge, current), dissipative in norm L^2 and of being able to deal naturally with complex geometries. Although subject to a stability constraint of the CFL type, the method gains efficiency by being of high order in speed and space and being parallelizable.

All this work to come to study the observer of this system. The observer of (VP) is very little studied, at the limit not at all. In the absence of bibliography we present to some methods applicable to nonlinear systems.

The state observation of a non-linear system is a little more delicate and there is no universal method for observer synthesis. The possible approaches are either an extension of linear algorithms or specific nonlinear algorithms. In the first case, the extension is based on a linearization of the model around an operating point. For the case of specific nonlinear algorithms, the numerous researches carried out on this subject (see Misawa and Hedrick. ((1989) B. L. Walcott and Zak. (1987)) gave rise to numerous observation algorithms.

Non linear transformation methods: This technique uses a coordinate change to transform a nonlinear system into a linear system. Once such a transformation is made, the use of a Luenberger-type observer will be sufficient to estimate the state of the transformed system, and thus the state of

[1] The Commissariat for Atomic Energy and Alternative Energies
[2] Particle In Cell

the original system using the inverse coordinate change (see Krener and Isidori (1983) Krener and Respondek. (1985) and also M. Ghattassi (2018)).

In this paper for the synthesis of the observers we adopt an approach based on the use of the Differential Mean Value Theorem (DMVT). This is to transform the estimation error dynamics into an Linear parameter varying (LVP) system.In this paper for the synthesis of the observers we adopt an approach based on the use of the theorem of the finished increases (DMVT). This is to transform the estimation error dynamics into an LPV system. LPV calculation techniques allow us to obtain stability conditions in the form of linear matrix inequality (LMI). One of the main features, thanks to the obtained tridiagonal structure of the system, is to show that only a reduced order observer is necessary to estimate the whole state of the system. Extension to H_∞ state estimation, in the presence of noises, is also established

Notation:

- For a bounded domain $B \subset \mathbb{R}^2$ we denote by $H^n(B)$ the L^2-Sobolev space of order $n \geqslant 0$.
- $\|.\|_{n,B}$ and $|.|_{n,B}$ Sobelev norm and seminorm respectively.
- $L_0^2(B)$ the space of $L^2(B)$ functions having zero average over B.
- $\mathcal{E} = \{e_s(i) \mid e_s(i) = (0,\ldots,0,1,0,\ldots,0)^T,\ i = 1,\ldots,s\}$ canonical basis of $\mathbb{R}^s$ for all $s \geqslant 1$.

We recall the Vlasov-Poisson system 1D: let $x \in \Omega_x = [0,1]$ et $t \in \Omega_T = [0,T]$,
$v \in \Omega_v = [-V_c, V_c]$. We denote by $f(x,v,t)$ the electron distribution function in the phase space (with the mass normalized to one and the charge of an electron to plus one) and by $E(x,t)$ the self-consistent electric field. The Vlasov equation is written as follows

$$f_t + v f_x - \phi_x f_v = 0, \quad (x,v,t) \in \Omega_x \times \Omega_v \times \Omega_T. \tag{1}$$

In order to describe the motion of charged particles in plasmas, we must calculate the force field from the macroscopic density of the particles

$$\rho(x,t) = \int_{\Omega_v} f(x,v,t)\,dv. \tag{2}$$

We calculate the force field from the Poisson equation,

$$-\phi_{xx} = \rho(x,t) - 1, \quad (x,t) \in \Omega_x \times \Omega_T, \tag{3}$$

where

$$-E_x(t,x) = \int_{\Omega_v} f(t,x,v)dv - 1,\ (x,t) \in \Omega_x \times \Omega_T, \tag{4}$$

with a positive initial data,

$$f(x,v,0) = f_0(x,v),\ (x,v) \in \Omega_x \times \Omega_v, \tag{5}$$

We impose periodic boundary conditions in x:

$$\phi(0,t) = \phi(1,t),\ E(0,t) = E(1,t) \tag{6}$$

and

$$f(t,0,v) = f(t,1,v),\ (v,t) \in \Omega_v \times \Omega_T, \tag{7}$$

a condition of compatibility signifying the "global neutrality" of the plasma

$$\int_x \int_v f(t,x,v)dvdx = 1,\ t \in \Omega_T. \tag{8}$$

To uniquely determine the electric field $E(t,x)$, we add a condition of zero mean

$$\int_{\Omega_x} E_x(t,x)dx = 0,\ t \in \Omega_T, \tag{9}$$

which amounts to supposing that the electric potential is periodic. We will denote

$$Q(t) = 1 + \sup\{|v| : \exists \in [0,1]\ et\ \tau \in\ [0,t] \\ \text{tel que } f(x,v,\tau) \neq 0\},$$

for all $t \in [0,\infty)$ a mesure of the support of the distribution function.

Theorem 1 (Well-poseedness of the continuous 1D VP J. Cooper (1983)). *Given $f_0 \in C^1(\mathbb{R}_x \times \mathbb{R}_v)$, 1-periodic in x and compactly suppoted in v, $Q(0) \leqslant Q_0$ with $Q_0 > 1$. Then the periodic Vlasov-Poisson system* (1)-(4) *has a unique classical solution (f,E), that is 1-periodic in x for all time $t \in [0,T]$ for all $T > 0$.*

We assume that the initial data f_0 satisfies the hypotheses in Theorem(1),and thus,the unique classical solution to the periodic Vlasov-Poisson system $(1) - (4)$ satisfies that there exists $V_c > 0$ dépending of f_0, T et Q_0 such that $supp(f(t)) \subseteq \Omega$ for all $t \in [0,T]$. The weak formulation of the problem continues (1) gives: find (f,E) such that

$$\int_{\Omega_x}\int_{\Omega_v} f_t \phi dxdv - \int_{\Omega_x}\int_{\Omega_v} v f \phi_x dxdv + \\ \int_{\Omega_x}\int_{\Omega_v} E f \phi_v dxdv = 0\ \forall \phi \in C_0^\infty(\Omega_x \times \Omega_v). \tag{10}$$

2. FORMULATION OF THE DISCONTINUOUS GALERKIN METHOD

Let $\{\mathcal{T}_h\}$ be family of partitions of our coumputational. Let phase space $\Omega = \Omega_x \times \Omega_v$ and we assume to be regularCiarlet (1991). Each cartesian mesh $\mathcal{T}_h$ is defined as $\mathcal{T}_h := \{T_{ij} = I_i \times J_j,\ 1 \leqslant i \leqslant N_x,\ 1 \leqslant j \leqslant N_v\}$ where

$$I_i = [x_{i-1/2}, x_{i+1/2}]\ \forall\ i = 1,\ldots N_x \\ J_j = [v_{j-1/2}, v_{j+1/2}]\ \forall\ j = 1,\ldots N_v,$$

and the mesh sizes h_x and h_v relative to the partition are defined as

$$h_x = \max_{1\leqslant i \leqslant N_x} h_i := x_{i+1/2} - x_{i-1/2}, \\ 0 < h_v = \max_{1\leqslant j \leqslant N_v} h_j := v_{j+1/2} - v_{j-1/2},$$

where h_i et h_j are the cell lengths of I_i et J_j. $h = \max(h_x, h_v)$ is the mesh size of the partition. We olso assume that $v = 0$ corresponds to a node, $v_{j-1/2} = 0$ for some j of the partition of Ω_v. The set all vertical edge is denoted by Γ_x and respectively, we will refer to Γ_v as set of all horizontal edge:

$$\Gamma_x := \bigcup_{i,j}\{x_{i-1/2}\} \times J_j,\ \Gamma_v := \bigcup_{i,j} I_i \times \{v_{j-1/2}\}, \\ \Gamma_h = \Gamma_x \cup \Gamma_v.$$

We shall denote by $\{\Omega_x^h\}$ the family of partition of the interval $\mathcal{I}$;

$$\Omega_x^h := \{I_i\ :\ 1 \leqslant i \leqslant N_x\}\ \ \gamma_x := \bigcup_i \{x_{i-1/2}\}.$$

We define the discontinuous finite element spaces V_h^k and Z_h^k and a corfirming finite element space, W_h^{k+1}, for $k \geqslant 0$,

$$V_h^k = \{\psi \in L^2(\Omega_x) :\ \psi \in \mathbb{P}^k(I_i), \\ \forall\ x\ \in\ I_i\ i = 1,\ldots N_x\}$$

$$Z_h^k := \{z \in L^2(\Omega_x \times \Omega_v) :\ z \in \mathbb{Q}^k(T_{ij}), \\ \forall\ (x,v)\ \in\ T_{ij} = I_i \times\ J_j,\ \forall\ i,j\}$$

$$W_h^{k+1} := \{\chi \in C^0(\Omega_x) : \ \chi \in \ \mathbb{P}^{k+1}(I_i), \ \forall \ x \ \in I_i, \\ i = 1, \dots N_x\} \cap L^2(\Omega_x)/\mathbb{R},$$

where $\mathbb{P}^k(I_i)$ is the polynomials (in one dimension)of the degree up to k, and $\mathbb{Q}^k(T_{ij})$ the space of polynomials of degree at most k in each variable.

Projection Operators : Let $k \geqslant 0$, and let $\mathcal{P}_h : L^2(\Omega_x \times \Omega_v) \longleftrightarrow \mathcal{Z}_h^k$ the standard L^2-projection defined by $\mathcal{P}_h(w) = (P_x^k \otimes P_v^k)(w)$; i.e., for all i et j,

$$\int_{I_i}\int_{J_j} (\mathcal{P}_h(w(x,v)) - w(x,v))\varphi_h(x,v)dvdx = 0 \\ \forall \varphi_h \in \mathbb{P}^k(I_i) \otimes \mathbb{P}^k(J_j). \tag{11}$$

This projection is stable in $L^2(\mathcal{T})$,

$$\|\mathcal{P}_h(w)\|_{L^p(\mathcal{T}_h)} \leqslant C\|w\|_{L^p(\Omega_x\times\Omega_v)}, \ 1 \leqslant p \leqslant \infty. \tag{12}$$

3. DISCONTUNOUS GALERKIN APPROXIMATION OF VLASOV EQUATION

Due to the spicial structure of the transport equation: v is independent of x and E is independent of v, the overall formulation of the DG method of the equation (1) is: find $(E_h, f_h) : \Omega_T \longrightarrow (W_h, Z_h^k)$ such that

$$\sum_{i=1}^{N_x}\sum_{j=1}^{N_v} a_{ij}^h(E_h; f_h, \varphi_h) = 0 \ \ \forall \varphi_h \in Z_h^k, \tag{13}$$

where the bilinear form $a_{ij}^h(E_h; f_h, \varphi_h)$ is defined for each $i, \ j$ and $\varphi_h \in Z_h^k$, as:

$$\begin{aligned} a_{ij}(E_h; f_h, \varphi_h) &= \int_{T_{ij}} \frac{\partial f_h}{\partial t}\varphi_h dvdx \\ &- \int_{T_{ij}} v f_h \frac{\partial \varphi_h}{\partial x} dvdx + \int_{T_{ij}} E_h^i f_h \frac{\partial \varphi_h}{\partial v} dvdx \\ &+ \int_{J_j} \left[((\widehat{vf_h})\varphi_h^-)_{i+1/2,v} - ((\widehat{vf_h})\varphi_h^+)_{i-1/2,v} \right] dv \\ &- \int_{I_i} \left[((\widehat{E_h^i f_h})\varphi_h^-)_{x,j+1/2} - ((\widehat{E_h^i f_h})\varphi_h^+)_{x,j-1/2} \right] dx \end{aligned} \tag{14}$$

The numerical fluxes are the boundary conditions and they are the approximation of the functions vf and Ef at the vertical and horizontal boundaries Γ_x and Γ_v respectively.

$$\widehat{vf_h} = \begin{cases} vf_h^- \ \text{ si } v \geqslant 0 \\ vf_h^+ \ \text{ si } v < 0 \end{cases} \quad \widehat{E_h^i f_h} = \begin{cases} E_h^i f_h^+ \ \text{ si } E_h^i \geqslant 0 \\ E_h^i f_h^- \ \text{ si } E_h^i < 0 \end{cases} \tag{15}$$

We define the numerical flux at the boundary $\partial\left(\Omega_x \times \Omega_v\right)$ par

$$(\widehat{vf_h})_{1/2,v} = (\widehat{vf_h})_{N_x+1/2,v}, \\ (\widehat{E_h^i f_h})_{x,1/2} = (\widehat{E_h^i f_h})_{x,N_v+1/2}, \ \forall \ (x,v) \in \Omega_x \times \Omega_v, \tag{16}$$

so that the periodicity in x and compactness in v are reflected. the approximation of the initial data is given by $f_h(0) = \mathcal{P}_h(f_0)$. We can write (13) in the following matrix form:

$$\mathbb{M}_h\dot{\mathbb{F}}_h = \mathbb{A}_h\mathbb{F}_h + \mathbb{B}(E_h)\mathbb{F}_h + Flux \tag{17}$$

We can rewrite it (17):

$$\mathbb{M}_h\dot{\mathbb{F}}_h = \mathbb{A}_h\mathbb{F}_h + \mathbf{G}_h(\mathbb{F}_h), \tag{18}$$

with

$$\mathbf{G}(\mathbb{F}_h) = \mathbb{B}(E_h)\mathbb{F}_h + Flux$$

where $\mathbb{M}_h, \mathbb{A}_h, \mathbb{B}_h$ and $Flux$ are the global mass matrix, the global gradient matrix following v, the global gradient matrix following E, the flow matrix respectively. See Pietro and Ern (2011)Hesthaven and Warburton (2008) for the assembly of the matrix of the DG method. Note that the non-linearity of this equation lies in the fact that the electrostatic field depends on the distribution function. The vecteur $\mathbb{F}_h$ is defined by:

$$\mathbb{F}_h := (f_{h,1}^1, \cdots, f_{h,N_d}^1, \cdots, f_{h,1}^N, \cdots, f_{h,N_d}^N)^T \\ := [(\mathbb{F}_h)_{T_{1,1}}, (\mathbb{F}_h)_{T_{1,2}} \cdots, (\mathbb{F}_h)_{T_{2,1}}, \cdots, (\mathbb{F}_h)_{T_{N_x,N_v}}]^T$$

where $N := card(\mathcal{T}_h)$, $N_d := (k+1).(k+1)$ is the degree of freedom in each element T_{ij}.

4. DISCOUNTINUOUS GALERKIN APPROXIMATION OF POISSON EQUATION

Consider the problems (3) and (4).

$$\begin{cases} E_h &= \partial_x\varphi_h \\ -\partial_x E_h &= \rho_h - 1, \quad x \in \Omega_x \\ \varphi_h(0,t) &= \varphi_h(1,t) \end{cases} \tag{19}$$

The overall formulation of the DG method of equation (19) is as follows: find $(E_h, \varphi_h) \in Z_h^k \times Z_h^k$ such that for all i:

$$\int_{I_i} E_h\tau dx = -\int_{I_i} \varphi_h\tau_x dx \\ +[(\widehat{\varphi_h}\tau^-)_{i+1/2} - (\widehat{\varphi_h}\tau^+)_{i-1/2}] \ \forall \tau \in Z_h^k \tag{20}$$

$$\int_{I_i} E_h\nu_x - [(\widehat{E_h}\nu^-)_{i+1/2} - (\widehat{E_h}\nu^+)_{i-1/2}] = \int_{I_i}(\rho_h - 1)\nu dx \\ \forall \nu \in Z_h^k, \tag{21}$$

where $(\widehat{\varphi_h})_{i-1/2}$ and $(\widehat{E_h})_{i-1/2}$ are the numerical fluxes. We consider the following family of DG-schemes:

$$\begin{cases} (\widehat{\varphi_h})_{i-1/2} := \{\varphi_h\}_{i-1/2} - c_{12}[\![\varphi_h]\!]_{i-1/2} + c_{22}[\![E_h]\!]_{i-1/2} \\ (\widehat{E_h})_{i-1/2} := \{E_h\}_{i-1/2} + c_{12}[\![E_h]\!]_{i-1/2} + c_{11}[\![\varphi_h]\!]_{i-1/2} \end{cases} \tag{22}$$

where the parameters c_{11}, c_{22} et c_{22} dépend solely on $x_{i-1/2} \ \forall \ i$. At the bouundary nodes due to periodicity in x we impose

$$(\widehat{\varphi_h})_{1/2} = (\widehat{\varphi_h})_{N_x+1/2}, \ \ (\widehat{E_h})_{1/2} = (\widehat{E_h})_{N_x+1/2}.$$

Following P. Castillo and Sch (2000) we define

$$a(E_h, z) := \sum_i \int_{I_i} E_h z dx + \sum_i c_{22}[\![E_h]\!]_{i-1/2}[\![z]\!]_{i-1/2},$$

$$b(\varphi_h, z) := \sum_i \int_{I_i} \varphi_h z_x dx + \sum_i (\{\varphi_h\} - c_{12}[\![\varphi_h]\!])[\![z]\!]_{i-1/2},$$

$$c(\varphi_h, p) := \sum_i c_{11}[\![\varphi_h]\!]_{i-1/2}[\![p]\!]_{i-1/2},$$

and

$$\mathcal{B}((E_h,\varphi_h);(z,p)) = a(E_h,z) + b(\varphi_h,z) - b(p,E_h) + c(\varphi_h,p). \tag{23}$$

Thus, problem (20)-(22) can be rewritten as:

$$\mathcal{B}((E_h,\varphi_h);(z,p)) = \sum_i \int_{I_i} (\rho_h - 1)dx \ \ \forall \ (z,p) \in Z_h^k \times Z_h^k. \tag{24}$$

The parameters of (22) are calculated by different methods. Following Blanca Ayuso De Dios and Shu (2011), we choose $c_{22} \equiv 0$ and $c_{11} \geqslant \frac{(k+1)^2}{h_x}$.

5. SYNTHESIS METHODS OF VLASOV-POISSON

The numerical method of DG presented allows us to obtain a finite-dimensional model written in the form of an explicit state representation allowing a We recall the semi-discrete scheme of $(1)-(4)$ obtained by the approximation of DG see (19).

$$\begin{cases} \mathbb{M}_h\dot{\mathbb{F}}_h = \mathbb{A}_h\mathbb{F}_h + \mathbf{G}_h(\mathbb{F}_h) \\ Y_h = \mathbb{C}_h\mathbb{F}_h \end{cases} \tag{25}$$

where

$$\mathbf{G}(\mathbb{F}_h) = \mathbb{B}(E_h)\mathbb{F}_h + Flux$$

and E_h is given by (24). The matrices $\mathbb{M}_h$ and $\mathbb{A}_h$ are symmetrical tridiagonal square matrices (Lagrangian basis). $\mathbb{M}_h$ is positive difinite see **?**. $\mathbb{M}_h, \mathbb{A}_h \in \mathcal{M}_N$ are consist of matrices $M_h, A_h, S_h, R_h \in \mathcal{M}_n(\mathbb{R})$ respectively with $N = n \times n_0$ where $n = N_x \times N_v$ et n_0 number of matrix blocks see Hajian and Dios (preprint, 2012). They are of the form:

$$\mathbb{M}_h = \begin{pmatrix} S_h & M_h & 0 & \cdots & \cdots & 0 \\ M_h^T & S_h & M_h & \ddots & & \vdots \\ 0 & M_h^T & S_h & M_h & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & M_h^T & S_h & M_h \\ 0 & \cdots & \cdots & 0 & M_h^T & S_h \end{pmatrix},$$

$$\mathbb{A}_h = \begin{pmatrix} R_h & A_h & 0 & \cdots & \cdots & 0 \\ A_h^T & R_h & A_h & \ddots & & \vdots \\ 0 & A_h^T & R_h & A_h & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & A_h^T & R_h & A_h \\ 0 & \cdots & \cdots & 0 & A_h^T & R_h \end{pmatrix}.$$

$\mathbb{F}_h \in \mathbb{R}^N$ is the vector of the system, $Y_h \in \mathbb{R}^p$ is the output vector which presents the number of sensors placed at the boundary phase space $\partial(\Omega_x \times \Omega_v)$. $\mathbb{C}_h \in \mathcal{M}_{p,N}$ is the output matrix, composed of 1 and 0 and we assume that $p \leqslant n$. She has the form:

$$\mathbb{C}_h = [C_h^1, 0] \;\; avec \; C_h^1 \in \mathcal{M}_{p,n}(\mathbb{R}).$$

Direct observation to determine the distribution function of such a state vector is quite expensive for the computation times. It is therefore necessary to estimate the state vector of the system (25): it's the role of the observer. The objective is to build a distribution function using the sensors on the surface of the domain (here it will be the phase space boundary).
Consider the observer associated with the system (25):

$$\mathbb{M}_h\dot{\tilde{\mathbb{F}}}_h = \mathbb{A}_h\tilde{\mathbb{F}}_h + \mathbf{G}(\tilde{\mathbb{F}}_h) + \mathbb{K}(Y_h - \mathbb{C}\tilde{\mathbb{F}}_h), \tag{26}$$

where $\tilde{\mathbb{F}}_h$ represents the estimate of $\mathbb{F}$ and $\mathbb{K}$ must be constructed in such a way that the estimation error $\mathfrak{e}_r = \mathbb{F}_h - \tilde{\mathbb{F}}_h$ converge asymptotically to zero for any initial condition $\tilde{\mathbb{F}}_{0h}$ et $\mathbb{F}_{0h}$. We can give the dynamics of the estimation error by:

$$\mathbb{M}_h\dot{\mathfrak{e}}_r = (\mathbb{A}_h - \mathbb{K}\mathbb{C}_h)\mathfrak{e}_r + [\mathbf{G}(\mathbb{F}_h) - \mathbf{G}(\tilde{\mathbb{F}}_h)]. \tag{27}$$

We can now study the synthesis of state observers of the system 25.

5.1 DMVT-based approach

Transformation into LPV: The matrix $\mathbb{M}$ is symmetric and positive definite so it is invertible. The system 27 becomes:

$$\dot{\mathfrak{e}}_r = \mathbb{M}_h^{-1}(\mathbb{A}_h - \mathbb{K}\mathbb{C}_h)\mathfrak{e}_r + \mathbb{M}_h^{-1}[\mathbf{G}(\mathbb{F}_h) - \mathbf{G}(\tilde{\mathbb{F}}_h)]. \tag{28}$$

In the following we assume that $\mathbf{G}_h$ is differentiable and satisfies the following hypothesis:

$$\underline{b}_{ij} \leqslant \frac{\partial \mathbf{G}_h^i(x)}{\partial x_j} \leqslant \bar{b}_{ij} \;\; \forall i,j \in [1,\ldots,N] \; avec \; \underline{b}_{ij} \leqslant \bar{b}_{ij} \leqslant 0 \tag{29}$$

where $\underline{b}_{ij}$ and $\bar{b}_{ij}$ are constant numbers and N the number of components of $\mathbf{G}_h$. $\mathbf{G}_h$ is thus Lipschitz. By the DMVT theorem in A. Zemouche (2008) it exist $z \in Co(\mathbb{F}_h, \tilde{\mathbb{F}}_h)$ such as

$$\mathbf{G}_h(\mathbb{F}_h) - \mathbf{G}_h(\tilde{\mathbb{F}}_h) = \Lambda(\eta)e,$$

where

$$\Lambda(\eta) = \sum_{i,j=1}^{N} \mathbf{G}_h^{ij} H_h^{ij} \text{ and } \mathbf{G}_h^{ij} = \frac{\partial \mathbf{G}_h^i(z)}{\partial x_j} \tag{30}$$

$$H_h^{ij} = e_N(i)e_N^T(j) \quad \eta = (\mathbf{G}_h^{11}, \mathbf{G}_h^{22}, \ldots, \mathbf{G}_h^{NN})$$

According to the hypothesis (29) the parameter η belongs in a convex domain

$$\varpi_N = \{\eta = (\mathbf{G}_h^{11}, \mathbf{G}_h^{22}, \ldots, \mathbf{G}_h^{NN}) \mid \mathbf{G}_{ij} \in \{\underline{b}_{ij}, \bar{b}_{ij}\} \; \forall i = \{1,\ldots,N\}\}.$$

The Jacobian matrix Λ is diagonal and semi-negative. The dynamics of the estimation error (27) becomes:

$$\mathbb{M}_h\dot{\mathfrak{e}}_r = (\mathbb{A}_h + \Lambda(\eta) - \mathbb{K}\mathbb{C}_h)\mathfrak{e}_r \tag{31}$$

Theorem 2. *The observer* (27) *is asymptotically stable if there exists a gain matrix* $\mathbb{K}$ *of appropriate size such that*
$$(\mathbb{A}_h + \Lambda(\eta))^T + (\mathbb{A}_h + \Lambda(\eta)) - \mathbb{C}_h^T\mathbb{K}^T - \mathbb{K}\mathbb{C}_h < 0, \;\; \eta \in \varpi_N. \tag{32}$$

Proof. We consider the following Lyapunov function:

$$V(\mathfrak{e}_r) = \mathfrak{e}_r^T\mathbb{M}_h\mathfrak{e}_r > 0.$$

By derivation of the Lyapunov function along the trajectory of (29), we obtain

$$\dot{V}(\mathfrak{e}_r) = \mathfrak{e}_r^T\Psi(\eta)\mathfrak{e}_r,$$

where

$$\Psi(\eta) = (\mathbb{A}_h + \Lambda(\eta) - \mathbb{K}\mathbb{C}_h)^T + (\mathbb{A}_h + \Lambda(\eta) - \mathbb{K}\mathbb{C}_h).$$

Condition $\dot{V} < 0$ for all $\eta \neq 0$ if and only if

$$\Psi(\eta) < 0 \;\; \forall \; \eta \in \varpi_N. \tag{33}$$

According to the principle of convexity S. Boyd and Balakrishnan. (1994), we deduce that the inequality (33) is satisfied if and only if the LMI (32) is satisfied for all $\eta \in \varpi_N$, this means that there is an observation gain $\mathbb{K}$ such that $\dot{V} < 0 \; \forall \; \mathfrak{e}_r \neq 0$. □

Proposition 3. *Let the matrix* $\mathbb{K}$ *such as the following LMI is verified:*

$$\mathbb{A}_h^T + \mathbb{A}_h - \mathbb{C}_h^T\mathbb{K}^T - \mathbb{K}\mathbb{C}_h < 0 \tag{34}$$

then the observation gain $\mathbb{K}$ *satisfies* (32).

Proof. $\mathbb{M}_h\Lambda$ is semi negative definite because the Jacobian Λ is semi negative definite. In addition to the hypothesis (34) we obtain that the gain $\mathbb{K}$ satisfies (32). □

5.2 Reduced order Observer

Now we will consider a matrix block to reduce the computation time and this gives the following system:

$$\begin{cases} \mathbb{M}_h^\star \dot{\mathbb{F}}_h^\star = (\mathbb{A}_h^\star + \Lambda^\star)\mathbb{F}_h^\star \\ Y_h^\star = \mathbb{C}_h^\star \mathbb{F}_h^\star \end{cases} \tag{35}$$

where

$$\mathbb{M}_h^\star = \begin{pmatrix} S_h & M_h \\ M_h^T & S_h \end{pmatrix},\ \mathbb{A}_h^\star = \begin{pmatrix} R_h & A_h \\ A_h^T & R_h \end{pmatrix},$$

$$\mathbb{C}_h^\star = [C_h^1, 0] \text{ et } \Lambda^\star = \chi\mathbb{I} \in \mathcal{M}_{2n}(\mathbb{R})$$

with

$$\chi = \max\{|S_p(\mathcal{Y}\mathbb{Q}^{-1}\mathcal{Y}^T)|\} \tag{36}$$

where

$$\mathbb{Q} = \begin{pmatrix} 2R_h & 2A_h & 0 & \cdots & \cdots & 0 \\ 2A_h^T & 2R_h & 2A_h & \ddots & & \vdots \\ 0 & 2A_h^T & 2R_h & 2A_h & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & 2A_h^T & 2R_h & 2A_h \\ 0 & \cdots & \cdots & 0 & 2A_h^T & 2R_h \end{pmatrix} \in \mathcal{M}_{(n_0-2)n}(\mathbb{R}) \tag{37}$$

and

$$\mathcal{Y} = \begin{pmatrix} 0 & 0\ 0 \cdots 0 \\ 2A_h^T & 0\ 0 \cdots 0 \end{pmatrix} \in \mathcal{M}_{2n,(n_0-2)n}\mathbb{R}. \tag{38}$$

We can associate a classical observer with the system (35):

$$\mathbb{M}_h^\star \dot{\tilde{\mathbb{F}}}_h^\star = (\mathbb{A}_h^\star + \Lambda^\star)\tilde{\mathbb{F}}_h^\star + \mathbb{K}^\star(\mathbb{Y}_h^\star - \mathbb{C}_h^\star\tilde{\mathbb{F}}_h^\star) \tag{39}$$

where $\tilde{\mathbb{F}}_h^\star$ represents the estimate of $\mathbb{F}_h^\star$ and $\mathbb{K}^\star$ must be constructed in such a way that the estimation error $\mathfrak{e}_r^\star = \mathbb{F}_h^\star - \tilde{\mathbb{F}}_h^\star$ converges asymptotically to zero for any initial condition $\tilde{\mathbb{F}}_{0h}^\star$ et $\mathbb{F}_{0h}^\star$.

Theorem 4. *Let the gain matrix $\mathbb{K}^\star$ of appropriate dimension such that the LMI is verified*

$$(\mathbb{A}_h^\star + \Lambda^\star)^T + (\mathbb{A}_h^\star + \Lambda^\star) - \mathbb{C}^{\star T}\mathbb{K}^{\star T} - \mathbb{K}^\star\mathbb{C}^\star \prec 0 \tag{40}$$

then, the observer gain defined by:

$$\mathbb{K} = [\mathbb{K}^\star\ 0] \tag{41}$$

satisfies the LMI (32).

Proof. Let

$$\Theta = (\mathbb{A}_h - \mathbb{K}\mathbb{C}_h)^T + (\mathbb{A}_h - \mathbb{K}\mathbb{C}_h) = \begin{pmatrix} \mathcal{Z} & \mathcal{Y} \\ \mathcal{Y}^T & \mathbb{Q} \end{pmatrix}$$

where

$$\mathcal{Z} = (\mathbb{A}_h^\star - \mathbb{K}^\star\mathbb{C}_h^\star)^T + (\mathbb{A}_h^\star - \mathbb{K}^\star\mathbb{C}_h^\star)$$

and $\mathbb{Q}$ is given by (37). En By applying the lemma of Schur complement, $\Theta \prec 0$ is equivalent to

$$\mathbb{Q} < 0 \text{ et } \mathbb{W} = \mathcal{Z} - \mathcal{Y}\mathbb{Q}^{-1}\mathcal{Y}^T. \tag{42}$$

Let's show that $\mathbb{W} < 0$

$$\mathbb{W} = (\mathbb{A}_h^\star - \mathbb{K}^\star\mathbb{C}_h^\star)^T + (\mathbb{A}_h^\star - \mathbb{K}^\star\mathbb{C}_h^\star) - \mathcal{Y}\mathbb{Q}^{-1}\mathcal{Y}^T$$
$$= (\mathbb{A}_h^\star + \Lambda^\star - \mathbb{K}^\star\mathbb{C}_h^\star)^T + (\mathbb{A}_h^\star + \Lambda^\star - \mathbb{K}^\star\mathbb{C}_h^\star) - 2\chi\mathbb{I} - \mathcal{Y}\mathbb{Q}^{-1}\mathcal{Y}^T.$$

(36) implies $-2\chi\mathbb{I} - \mathcal{Y}\mathbb{Q}^{-1}\mathcal{Y}^T < 0$, then $\Theta \prec 0$. So by the proposition 3 we conclude that (41) verifie the LMI (32). □

5.3 Observer synthesis H_∞

This section extends the results of the observer based synthesis to the case of systems with noise in the dynamics and output of the system. The system is given by:

$$\begin{cases} \mathbb{M}_h\dot{\mathbb{F}}_h = \mathbb{A}_h\mathbb{F}_h + \mathbf{G}_h(\mathbb{F}_h) + \mathcal{W}_1 w \\ Y_h = \mathbb{C}_h\mathbb{F}_h + \mathcal{W}_2 w \end{cases} \tag{43}$$

where $\mathcal{W}_1$ and $\mathcal{W}_2$ are constant matrices of appopriate size and $w \in L^2(\mathbb{R}^n)$ is the bounded disturbance vector, where the matrix $\mathcal{W}_1$

$$\mathcal{W}_1 = [\mathcal{W}_1^\star\ 0]^T,\ \ \textit{where } \mathcal{W}_1^\star \in \mathcal{M}_{2n,n}(\mathbb{R}). \tag{44}$$

Let the following observer:

$$\mathbb{M}_h\dot{\tilde{\mathbb{F}}}_h = \mathbb{A}_h\tilde{\mathbb{F}}_h + \mathbf{G}_h(\tilde{\mathbb{F}}_h) + \mathcal{W}_2 w + \mathbb{K}(Y_h - \mathbb{C}_h\tilde{\mathbb{F}}_h). \tag{45}$$

The dynamics of the state observer error $\mathfrak{e}_r = \mathbb{F}_h - \tilde{\mathbb{F}}_h$ is given by

$$\mathbb{M}_h\mathfrak{e}_r = (\mathbb{A}_h + \Lambda(\eta) - \mathbb{K}\mathbb{C}_h)\mathfrak{e}_r + (\mathcal{W}_1 - \mathbb{K}\mathcal{W}_2)w.$$

The robust H_∞ observer design problem is to determine the matrix $\mathbb{K}$ such that the estimation error converges asymptotically to zero, i.e,

$$\lim_{t\to t}\mathfrak{e}_r = 0\ \ \textit{with } w(t) = 0 \tag{46}$$

$$\|\mathfrak{e}_r(t)\|_{L^2} \leqslant \lambda\|w(t)\|_{L^2}\ \textit{with } w(t) \neq 0 \textit{ and } \mathfrak{e}_r(0) = 0 \tag{47}$$

with $\lambda > 0$ a prescribed scalar disturbance attenuation level. However, to satisfy (46) and (47), it is sufficient to find a Lyapunov function V so that

$$\dot{V} + \mathfrak{e}^T\mathfrak{e} - \lambda w^T w < 0. \tag{48}$$

Then it will remain to demonstrate the implication of (48) to (46) and (47).

- For $w = 0$, if (48) is verified then $\dot{V} < 0$. Thus, from the Lyapunov theory, we deduce that the estimation error converges asymptotically to zero, which implies (46).
- If $w \neq 0$ and $\mathfrak{e}_r(0) = 0$, then (48) implies that

$$V(\mathfrak{e}_r(t)) + \int_0^t \mathfrak{e}_r^T(s)\mathfrak{e}_r(s)ds - \lambda^2\int_0^t w^T(s)w(s)ds < 0. \tag{49}$$

We know that $V(\mathfrak{e}_r) \geqslant 0$, for all $t \geqslant 0$, then for $t \to \infty$ we obtain

$$\int_0^\infty \mathfrak{e}_r^T(s)\mathfrak{e}_r(s)ds \leqslant \lambda^2\int_0^\infty w^T(s)w(s)ds, \tag{50}$$

hence (47).

Theorem 5. *Let $\lambda > 0$, the H_∞ observer design problem corresponding to the system* (43) *and the observer* (45) *is solvable if there exists a gain matrix $\mathbb{K}$ of appropriate dimensions such the following LMI is feasible:*

$$Block-Diag\,(\Gamma(\eta_1,\lambda), \Gamma(\eta_2,\lambda), \ldots, \Gamma(\eta_{2^N},\lambda)) \prec 0, \tag{51}$$

$\eta_i \in \varpi_N,\ \forall i \in 1,\ldots,2^N,$ *with*

$$\Gamma(\eta,\lambda) = \begin{bmatrix} (\mathbb{A}_h + \Lambda(\eta) - \mathbb{K}\mathbb{C})^T + \mathbb{A}_h + \Lambda(\eta) - \mathbb{K}\mathbb{C} + \mathbb{I} & \mathcal{W}_1 - \mathbb{K}\mathcal{W}_2 \\ (\mathcal{W}_1 - \mathbb{K}\mathcal{W}_2)^T & -\lambda^2\mathbb{I} \end{bmatrix} \tag{52}$$

Proof. Let us consider the following Lyapunov function:

$$V(\mathfrak{e}_r = \mathfrak{e}^T\mathbb{M}_h\mathfrak{e}_r,$$

hence,

$$\dot{V}(\mathfrak{e}_r(t)) + \mathfrak{e}_r^T \mathfrak{e}_r - \lambda^2 w^T(t) w(t) = $$
$$= \begin{bmatrix} \mathfrak{e}_r \\ w(t) \end{bmatrix}^T \begin{bmatrix} \mathfrak{M}(\mathbb{K},\eta) & (W_1 - \mathbb{K}W_2) \\ (W_1 - \mathbb{K}W_2)^T & -\lambda^2 \mathbb{I} \end{bmatrix} \begin{bmatrix} \mathfrak{e}_r \\ w(t) \end{bmatrix} \quad (53)$$

where

$$\mathfrak{M}(\mathbb{K},\eta) = (\mathbb{A}_h + \Lambda(\eta) - \mathbb{K}\mathbb{C})^T + \mathbb{A}_h + \Lambda(\eta) - \mathbb{K}\mathbb{C} + \mathbb{I}.$$

Then,

$$\dot{V}(\mathfrak{e}_r(t)) + \mathfrak{e}_r^T \mathfrak{e}_r - \lambda^2 w^T(t) w(t) = \begin{bmatrix} \mathfrak{e}_r \\ w(t) \end{bmatrix}^T \Gamma(\eta,\lambda) \begin{bmatrix} \mathfrak{e}_r \\ w(t) \end{bmatrix},$$

where $\Gamma(\eta,\lambda)$ is given by (52), which is identical to (51). Consequently, we deduce that under the condition (51), the state observer error converges asymptotically to zero. □

Proposition 6. *Let $\lambda > 0$. If there exists an output-injection gain $\mathbb{K}$ of appropriate dimension such that the following LMI condition holds:*

$$\begin{bmatrix} \mathbb{A}_h^T + \mathbb{C}^T\mathbb{K}^T + \mathbb{A}_h - \mathbb{K}\mathbb{C} + \mathbb{I} & \mathcal{W}_1 - \mathbb{K}\mathcal{W}_2 \\ (\mathcal{W}_1 - \mathbb{K}\mathcal{W}_2)^T & -\lambda^2\mathbb{I} \end{bmatrix} \quad (54)$$

then the observer gain matrix $\mathbb{K}$ verifies (51).

Proof. Let $\lambda > 0$. If $\mathbb{K}$ satisfies

$$\begin{bmatrix} \mathbb{A}_h^T + \mathbb{C}^T\mathbb{K}^T + \mathbb{A}_h - \mathbb{K}\mathbb{C} + \mathbb{I} & \mathcal{W}_1 - \mathbb{K}\mathcal{W}_2 \\ (\mathcal{W}_1 - \mathbb{K}\mathcal{W}_2)^T & -\lambda^2\mathbb{I} \end{bmatrix} \prec 0,$$

we can conclude using the Schur complement lemma and the structure of Jacobian mathix Λ that $\mathbb{K}$ verifies (51). □

$$\varrho = \max |Sp(\mathcal{Y}\tilde{\mathbb{Q}}^{-1}\mathcal{Y}^T)|, \quad (55)$$

where $\tilde{\mathbb{Q}} = \mathbb{Q} + \mathbb{I}$.

Theorem 7. *Let $\lambda > 0$, the H_∞ observer design problem corresponding to the system* (43) *and the observer* (45) *is solvable if there exists a gain matrix $\mathbb{K}$ of appropriate dimensions such the following LMI is feasible:*

$$\begin{bmatrix} (\mathbb{A}_h^\star + \Lambda^\star)^T + \mathbb{A}_h^\star + \Lambda^\star(\eta) - \mathbb{C}_h^\star\mathbb{K}^{\star T} - \mathbb{K}^\star\mathbb{C}_h^\star + \mathbb{I} & \mathcal{W}_1^\star - \mathbb{K}^\star\mathcal{W}_2 \\ (\mathcal{W}_1^\star - \mathbb{K}^\star\mathcal{W}_2)^T & -\lambda^2\mathbb{I} \end{bmatrix} \prec 0, \quad (56)$$

where $\Lambda^\star = \varrho\mathbb{I} \in \mathcal{M}_{2n}(\mathbb{R})$, then the observer gain

$$\mathbb{K}^\star = [\mathbb{K}^\star \ 0]^T \quad (57)$$

satisfies (54).

6. CONCLUSION

To validate the theory that we have just done, we develop numerical codes. This in order to compare the LMI obtained by the original and reduced systems. We will calculate these LMIs with Toolboox Yalmip. We will also develop controls for the Vlasov-Poisson system.

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