Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Analysis of Integral Input-to-State Stable time-delay systems in cascade ⋆

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Abstract: We consider the cascade interconnection of two nonlinear time-delay systems, each of which being integral input-to-state stable (iISS). We provide an explicit growth rate condition on the dissipation rate of the driving system and the input rate of the driven system under which the overall cascade is globally asymptotically stable in the absence of inputs (0-GAS), and its solutions are bounded in response to any input with a suitable bounded-energy assumption.

Keywords: Time-delay systems, integral input-to-state stability, Lyapunov-Krasovskii functional, cascade interconnections.

1. INTRODUCTION

The *input-to-state stability* (ISS) notion has been introduced three decades ago in Sontag (1989) to analyze dynamical systems affected by external disturbances. In few words, the ISS property imposes that, in the absence of disturbances, the dynamical system evolves properly (namely, its equilibrium is globally asymptotically stable: 0-GAS) and that, in the presence of perturbations, this nominal behavior is preserved up to a steady-state error proportional to the magnitude of the applied disturbance. Although the ISS framework was first introduced in a finite-dimensional context, it has more recently been extended to time-delay systems (TDS), see for instance Pepe and Jiang (2006a); Karafyllis and Jiang (2011); Mironchenko and Wirth (2017).

An important and well-known relaxation of ISS is *integral input-to-state stability* (iISS). It was originally introduced in Sontag (1998) for finite-dimensional systems and extended to TDS in Pepe and Jiang (2006a). Instead of measuring the impact of the input *magnitude* on the steady-state behavior of the system, it rather focuses on its *energy*.

Beyond ensuring some robustness to exogenous disturbances, ISS and iISS constitute powerful tools to study the stability of interconnected systems. For feedback interconnected systems, small-gain arguments can be invoked both for simple loops and networks of interconnected systems (Jiang et al., 1996; Dashkovskiy et al., 2010; Ito, 2006). In the simpler case of cascade interconnections, in which the influence of one subsystem on the other is unidirectional, ISS is known to be naturally preserved (Sontag and Teel, 1995; Chaillet et al., 2014). On the contrary, cascades of iISS subsystems may not be iISS (Arcak et al., 2002): additional growth-rate requirements are needed (Angeli and Astolfi, 2005; Chaillet and Angeli, 2008; Ito, 2010). See also Panteley and Loría (2001) and its TDS extension based on Razumikhin approach in Sedova (2008).

So far, the vast majority of the existing stability results about cascades of iISS systems are restricted to finite-dimensional systems. Cascades can be seen as a particular case of feedback interconnection, thus allowing the use iISS small-gain results for time-delay systems such as Ito et al. (2010). However, this approach often leads to overly conservative stability conditions. In this paper, we provide explicit growth-rate conditions under which the cascade of two iISS systems preserves interesting stability and robustness features, namely global asymptotic stability in the absence of exogenous inputs (0-GAS) and bounded solutions in response to a class of inputs with bounded energy, which constitute the two main practical features of the iISS property. These growth rate conditions are based on the Lyapunov-Krasovskii functionals (LKF) associated to the two individual subsystems, and require that the dissipation rate of the driving subsystem has greater growth near zero than the input rate of the driven subsystem.

An appealing feature of the proposed results is that they rely on LKF that dissipate in a *point-wise* manner, meaning solely in terms of the current value of the solution's norm. This differs from more stringent LKF conditions for iISS (Pepe and Jiang, 2006a; Lin and Wang, 2018), which require a dissipation rate that involves the whole LKF itself. See (Chaillet et al., 2017; Kankanamalage et al., 2017) for more discussion on this aspect.

After recalling background notions in Section 2, we state our main result in Section 3 with an immediate corollary for input-free cascades. Proofs are provided in Section 4 and an illustrative example is given in Section 5. We conclude in Section 6 with possible future work directions.

⋆ Gökhan Göksu is supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK) 2214-A (Application No: 1059B141801351).

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2. PRELIMINARIES

2.1 Notation

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{PD}$ if it is continuous, and satisfies $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$. $\alpha \in \mathcal{K}$ if $\alpha \in \mathcal{PD}$ and it is increasing. $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded. $\alpha \in \mathcal{L}$ if it is continuous non-increasing and tends to zero as its argument tends to infinity. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each $t \in \mathbb{R}_{\geq 0}$ and $\beta(s, \cdot) \in \mathcal{L}$ for each $s \in \mathbb{R}_{\geq 0}$. Given $q_1, q_2 \in \mathcal{PD}$, we say that q_1 *has greater growth* than q_2 in a neighborhood of zero (and we write $q_2(s) = \mathcal{O}_{s\to 0^+}(q_1(s))$) if there exists a constant $k \geq 0$ such that $\limsup_{s\to 0^+} q_2(s)/q_1(s) \leq k$. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given $\delta \geq 0$, $\mathcal{C}$ denotes the set of all continuous functions $\varphi : [-\delta; 0] \to \mathbb{R}$. Given any $\phi \in \mathcal{C}^n$, $\|\phi\| := \sup_{\tau\in[-\delta,0]} |\phi(\tau)|$. We denote by $\mathcal{U}$ the set of all measurable essentially bounded signals $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$.

2.2 Definitions

Consider the nonlinear TDS

$$\dot{x}(t) = f(x_t, u(t)), \tag{1}$$

where $u \in \mathcal{U}^m$ is the input, $x(t) \in \mathbb{R}^n$ is the current value of the solution and $x_t \in \mathcal{C}^n$ is the state history defined with the maximum delay $\delta \geq 0$ as

$$x_t(s) := x(t + s), \qquad \forall s \in [-\delta, 0]. \tag{2}$$

$f : \mathcal{C}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be Lipschitz on bounded sets and to satisfy $f(0, 0) = 0$.

We recall the definition of iISS, originally introduced in a delay-free context (Sontag, 1998).

Definition 1. (iISS, Pepe and Jiang (2006a)). The system (1) is said to be *integral input-to-state stable* (iISS) if there exists $\beta \in \mathcal{KL}$ and $\nu, \sigma \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{C}^n$ and all $u \in \mathcal{U}^m$, its solution satisfies

$$|x(t)| \leq \beta(\|x_0\|, t) + \nu\left(\int_0^t \sigma(|u(s)|)ds\right), \quad \forall t \geq 0. \tag{3}$$

A first consequence of this property is forward completeness: since $t \mapsto \int_0^t \sigma(|u(s)|)ds$ is bounded on any bounded time interval, (3) ensures that no finite escape-time can occur which, in view of (Hale, 1977, Theorem 3.2, p. 43), ensures that solutions of (1) exists at all positive times.

Another consequence of iISS is that, in the absence of inputs, the system is globally asymptotically stable.

Definition 2. (0-GAS). The TDS (1) is said to be *globally asymptotically stable in the absence of inputs* (0-GAS) if there exists $\beta \in \mathcal{KL}$ such that, for all $x_0 \in \mathcal{C}^n$, the solution of the input-free system $\dot{x}(t) = f(x_t, 0)$ satisfies

$$|x(t)| \leq \beta(\|x_0\|, t), \qquad \forall t \geq 0.$$

Remark 1. The concept defined above is usually called 0-UGAS in infinite-dimensional ISS literature (see for instance the survey Mironchenko and Prieur (2019)), whereas 0-GAS is often defined as the combination of stability and global attractivity. For finite-dimensional systems both properties are equivalent, but this is not the case for infinite-dimensional systems. Nevertheless, we decide to stick to the 0-GAS acronym (without explicitly stressing that convergence is uniform in the initial state) for the sake of homogeneity with the finite-dimensional terminology.

iISS actually goes beyond the internal stability property of 0-GAS, by ensuring some robustness properties known as *bounded energy-bounded state* and *bounded-energy converging state.*

Definition 3. (BEBS, BECS). The TDS (1) is said to have the *bounded energy-bounded state* (BEBS) property if there exists $\zeta \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{C}^n$ and all $u \in \mathcal{U}^m$, its solution satisfies

$$\int_0^\infty \zeta(|u(s)|)ds < \infty \quad \Rightarrow \quad \sup_{t\geq 0} |x(t)| < \infty. \tag{4}$$

It is said to have the *bounded energy-converging state* (BECS) property if there exists $\zeta \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{C}^n$ and all $u \in \mathcal{U}^m$, its solution satisfies

$$\int_0^\infty \zeta(|u(s)|)ds < \infty \quad \Rightarrow \quad \lim_{t\to\infty} |x(t)| = 0. \tag{5}$$

The strength of iISS lies in the fact that it not only guarantees 0-GAS, but also BEBS and BECS.

Proposition 1. (iISS $\Rightarrow$ 0-GAS, BEBS, BECS). If the TDS (1) is iISS, then it is BEBS and BECS. In particular, if (3) holds, then (4) and (5) hold with $\zeta = \sigma$.

A similar result was proven in (Sontag, 1998, Proposition 6) for finite-dimensional systems. The proof of Proposition 1 follows along the same lines and is therefore omitted.

Remark 2. For finite-dimensional systems, iISS is actually equivalent to 0-GAS combined with the weaker requirement:

$$\int_0^\infty \zeta(|u(s)|)ds < \infty \quad \Rightarrow \quad \liminf_{t\to\infty} |x(t)| < \infty,$$

as proved in Angeli et al. (2004). We are not aware of any extension of this fact to TDS.

2.3 Sufficient condition for iISS

Another strength of iISS is that it can be established using a Lyapunov-Krasovskii functional. We call a *Lyapunov-Krasovskii functional (LKF) candidate* any functional $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, Lipschitz on bounded sets, for which there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{C}^n. \tag{6}$$

Its *upper-right Dini derivative* along the solutions of (1) is then defined for all $t \geq 0$ as

$$D^+V(x_t, u(t)) := \limsup_{h\to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h}. \tag{7}$$

Proposition 2. (iISS LKF, Lin and Wang (2018)). The TDS (1) is iISS if and only if there exists a LKF candidate $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{PD}$ and $\gamma \in \mathcal{K}_\infty$, such that, given any $x_0 \in \mathcal{C}^n$ and any $u \in \mathcal{U}^m$, the following holds for all $t \geq 0$:

$$D^+V(x_t, u(t)) \leq -\alpha(V(x_t)) + \gamma(|u(t)|). \tag{8}$$

This result is reminiscent of the iISS characterization obtained for finite-dimensional systems (Angeli et al., 2000). In what follows, we will actually allow for a weaker dissipation rate which proves very handy in practice, as illustrated through an example in Section 5.

3. CASCADE INTERCONNECTION

Consider two nonlinear TDS in cascade:

$$\dot{x}_1(t) = f_1(x_{1t}, x_2(t-\delta_1)) \tag{9a}$$
$$\dot{x}_2(t) = f_2(x_{2t}, u(t)), \tag{9b}$$

where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$ are the current values of the state, $x_{1t} \in \mathcal{C}^{n_1}$, $x_{2t} \in \mathcal{C}^{n_2}$ are the state histories, and $u \in \mathcal{U}^m$ is the input. The functions $f_1 : \mathcal{C}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ and $f_2 : \mathcal{C}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_2}$ are assumed to be Lipschitz on bounded sets and satisfying $f_1(0,0) = 0$ and $f_2(0,0) = 0$. The interconnection between the two systems is assumed to be through a discrete delay $\delta_1 \in [0, \delta]$.

The main purpose of this study is to investigate under which condition some stability and robustness properties of the cascade (9) can be ensured based on the assumption that each of the subsystems (9a) and (9b) is iISS.

Our main result, proved in Section 4.2, is the following.

Theorem 1. Assume that there exist two LKF candidates $V_i : \mathcal{C}^{n_i} \to \mathbb{R}_{\geq 0}$ and $\eta_i \in \mathcal{K}_\infty$, $i \in \{1,2\}$, such that, along any solution of $\dot{x}_1(t) = f_1(x_{1t}, u_1(t))$ with $u_1 \in \mathcal{U}^{n_2}$,

$$D^+V_1(x_{1t}, u_1(t)) \leq -\frac{\alpha_1(|x_1(t)|)}{1+\eta_1(V_1(x_{1t}))} + \gamma_1(|u_1(t)|) \tag{10}$$

and, along any solution of (9b),

$$D^+V_2(x_{2t}, u(t)) \leq -\frac{\alpha_2(|x_2(t)|)}{1+\eta_2(V_2(x_{2t}))} + \gamma_2(|u(t)|). \tag{11}$$

for all $t \geq 0$. Assume further that (9b) is iISS [1] and

$$\gamma_1(s) = \mathcal{O}_{s\to 0^+}(\alpha_2(s)). \tag{12}$$

Then the cascade (9) is 0-GAS and satisfies the BEBS property.

Note that the dissipation rates required on the LKFs are less conservative than the one in Proposition 2. Indeed, for a LKF candidate V, (6) holds with some $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$. For $\alpha \in \mathcal{PD}$, (Angeli et al., 2000, Lemma IV.1) ensures the existence of $\sigma \in \mathcal{K}_\infty$ and $\ell \in \mathcal{L}$ such that $\alpha(s) \geq \sigma(s)\ell(s) = \ell(0)\sigma(s)\frac{\ell(s)}{\ell(0)}$. Pick any $\eta \in \mathcal{K}_\infty$ such that $\eta(s) \geq \frac{\ell(0)}{\ell(s)} - 1$ (such a function exists as the right-hand side of this expression is continuous, zero at zero, and nondecreasing). Then we have that $\alpha(V(x_t)) \geq \frac{\ell(0)\sigma(V(x_t))}{1+\eta(V(x_t))} \geq \frac{\ell(0)\sigma\circ\underline{\alpha}^{-1}(|x(t)|)}{1+\eta(V(x_t))}$. Hence, starting from a dissipation rate $\alpha(V(x_t))$ as in Proposition 2, we recover a dissipation rate as in (10) and (11).

The growth rate condition (12) requires that the increasing part α_2 of the driving subsystem's dissipation rate has greater growth than the input rate γ_1 of the driven subsystem around the origin. This condition is reminiscent of the one obtained in Chaillet and Angeli (2008) for finite-dimensional systems. Nonetheless, it is worth stressing that the main result in Chaillet and Angeli (2008) goes beyond Theorem 1 by ensuring iISS of the overall cascade whereas, here, only 0-GAS and BEBS (key features of iISS) were obtained. This is due to the fact that, as recalled in Remark 2, the result presented in Angeli et al. (2004), which allows to recover iISS from 0-GAS plus (a relaxed version of) BEBS has not yet been extended to TDS.

It is worth mentioning that the small-gain results for iISS TDS provided in Ito et al. (2010) can also be used to study cascade interconnections. However, the resulting requirement turns out to involve the upper and lower bounds on V_1 and V_2, thus leading to a more conservative stability condition. More crucially, that result imposes that the dissipation rates α_i in (8) for the driving and driven subsystems are of class $\mathcal{K}$ (rather than $\mathcal{PD}$), meaning that both subsystems are required to have an ISS-like behavior for small inputs. In particular, the results in Ito et al. (2010) cannot be used for the example of Section 5.

A straightforward consequence of Theorem 1 is that, under the above growth-rate condition, the cascade composed of an iISS subsystem driven by a GAS one is itself GAS. To state this, consider the following input-free cascade:

$$\dot{x}_1(t) = f_1(x_{1t}, x_2(t-\delta_1)) \tag{13a}$$
$$\dot{x}_2(t) = f_2(x_{2t}), \tag{13b}$$

under similar regularity assumptions as the ones on (9). Then we have the following.

Corollary 1. Assume that there exist two LKF candidates $V_1 : \mathcal{C}^{n_1} \to \mathbb{R}_{\geq 0}$ and $V_2 : \mathcal{C}^{n_2} \to \mathbb{R}_{\geq 0}$, $\underline{\alpha}_i, \overline{\alpha}_i, \eta_i \in \mathcal{K}_\infty$, $\alpha_i \in \mathcal{K}$, $i \in \{1,2\}$, and $\gamma_1 \in \mathcal{K}_\infty$ such that, along any solution of $\dot{x}_1(t) = f_1(x_{1t}, u_1(t))$ with $u_1 \in \mathcal{U}^{n_2}$,

$$D^+V_1(x_{1t}, u_1(t)) \leq -\frac{\alpha_1(|x_1(t)|)}{1+\eta_1(V_1(x_{1t}))} + \gamma_1(|u_1(t)|),$$

and, along any solution of (13b),

$$D^+V_2(x_{2t}) \leq -\frac{\alpha_2(|x_2(t)|)}{1+\eta_1(V_2(x_{2t}))},$$

for all $t \geq 0$. Assume further that

$$\gamma_1(s) = \mathcal{O}_{s\to 0^+}(\alpha_2(s)).$$

Then the cascade (13) is globally asymptotically stable.

This result complements existing delay-free results (Panteley and Loría, 2001; Arcak et al., 2002; Angeli and Astolfi, 2005; Chaillet and Angeli, 2008) as well as the Razumikhin approach developed in Sedova (2008) for TDS.

The proof of Theorem 1 relies on the following change of dissipation rate result, proved in Section 4.1.

Lemma 2. Let $f : \mathcal{C}^n \to \mathbb{R}^n$ be Lipschitz on bounded sets and $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ be a LKF candidate satisfying, along any solution of the TDS $\dot{x}(t) = f(x_t)$,

$$D^+V(x_t) \leq -\frac{\alpha(|x(t)|)}{1+\eta(V(x_t))}, \tag{14}$$

for some $\alpha \in \mathcal{PD}$ and $\eta \in \mathcal{K}_\infty$. Let $\tilde{\alpha}$ be any $\mathcal{PD}$ function satisfying

$$\tilde{\alpha}(s) = \mathcal{O}_{s\to 0^+}(\alpha(s)). \tag{15}$$

Then there exists a continuously differentiable function $\rho \in \mathcal{K}_\infty$ such that the functional $\tilde{V} := \rho \circ V$ satisfies

$$D^+\tilde{V}(x_t) \leq -\tilde{\alpha}(|x(t)|).$$

[1] In Chaillet and Pepe (2018b), it was claimed that (11) implies that the driving subsystem is iISS, but we have recently found a flaw in the proof of that result that we were not able to correct yet. For safety, we therefore assume explicitly that (9b) is iISS, although this is probably redundant with (11).

4. PROOFS

4.1 Proof of Lemma 2

First recall that, since V is a LKF candidate, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{C}^n$,

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|). \tag{16}$$

We will show that there exists a continuous non-decreasing function $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $q(s) > 0$ for all $s > 0$ such that the requested function ρ can be written as $\rho(s) = \int_0^s q(r)dr$ for all $s \geq 0$. Note that, with this choice, ρ is a continuously differentiable $\mathcal{K}_\infty$ function and the function $\tilde{V} = \rho \circ V$ is then Lipschitz on bounded sets. Moreover, in view of (16), $\tilde{V}$ is a LKF candidate. Furthermore, its Dini derivative along the solutions of $\dot{x}(t) = f(x_t)$ reads, in view of (14) and (Ito et al., 2010, Lemma 7),

$$D^+\tilde{V}(x_t) \leq -q(V(x_t))\frac{\alpha(|x(t)|)}{1+\eta(V(x_t))}.$$

So we need to find a continuous non-decreasing function q satisfying $q(s) > 0$ for all $s > 0$ such that

$$q(V(x_t))\frac{\alpha(|x(t)|)}{1+\eta(V(x_t))} \geq \tilde{\alpha}(|x(t)|). \tag{17}$$

Let $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined as

$$\mu(s) := \sup_{r\in[0,s]} \frac{\tilde{\alpha}(r)}{\alpha(r)}, \quad \forall s \geq 0. \tag{18}$$

The function $s \mapsto \tilde{\alpha}(s)/\alpha(s)$ is continuous on $(0, +\infty)$. In addition, condition (15) ensures that it is bounded on any interval of the form $[0, a]$, $a \geq 0$. μ being clearly non-decreasing, it follows that it admits a limit at zero. Hence, σ is continuous and non-decreasing on $\mathbb{R}_{\geq 0}$. We claim that a possible choice of q to fulfill (17) is then

$$q(s) := \mu \circ \underline{\alpha}^{-1}(s)\big(1 + \eta(s)\big), \quad \forall s \geq 0.$$

This function q is indeed a continuous non-decreasing function, positive out of zero. With this choice, and invoking (16) and (18), we have that

$$\begin{aligned} q(V(x_t))\frac{\alpha(|x(t)|)}{1+\eta(V(x_t))} &\geq \sigma \circ \underline{\alpha}^{-1}(V(x_t))\alpha(|x(t)|) \\ &\geq \sigma(|x(t)|)\alpha(|x(t)|) \geq \tilde{\alpha}(|x(t)|), \end{aligned}$$

thus establishing (17).

4.2 Proof of Theorem 1

i) Forward completeness

We start by showing that solutions of the cascade (9) exist at all positive times. This is rather straightforward for the driving subsystem as it is assumed iISS (hence, forward complete). This in turn implies that $x_2(\cdot)$ is continuous on $\mathbb{R}_{\geq -\delta}$ as the solution of a forward complete TDS. (10) then implies that, over the maximal domain of existence of x_{1t}, $D^+V_1(x_{1t}, x_2(t-\delta_1)) \leq \gamma_1(|x_2(t-\delta_1)|)$. This ensures that x_{1t} is bounded over any finite time interval, thus impeding finite escape times. Forward completeness follows (see the remark after (3)).

ii) 0-GAS

We next show that the cascade (9) is 0-GAS, meaning that the input-free system

$$\dot{x}_1(t) = f_1(x_{1t}, x_2(t-\delta_1)) \tag{19a}$$
$$\dot{x}_2(t) = f_2(x_{2t}, 0) \tag{19b}$$

is globally asymptotically stable. Note that, in view of (11), the LKF V_2 satisfies, along the solutions of (19b),

$$D^+V_2(x_{2t}) \leq -\frac{\alpha_2(|x_2(t)|)}{1+\eta_2(V_2(x_{2t}))}.$$

Observe that the growth rate condition (12) ensures that

$$2\gamma_1(s) = \mathcal{O}_{s\to 0^+}(\alpha_2(s)).$$

It follows from Lemma 2 that there exists a continuously differentiable function $\rho \in \mathcal{K}_\infty$ such that the functional

$$\tilde{V}_2 := \rho \circ V_2 \tag{20}$$

satisfies

$$D^+\tilde{V}_2(x_{2t}) \leq -2\gamma_1(|x_2(t)|). \tag{21}$$

We next modify the LKF $\tilde{V}_2$ in such a way that its dissipation rate $2\gamma_1$ involves not only the current value of the driving state $|x_2(t)|$ but also its delayed value $|x_2(t-\delta_1)|$. To that aim, consider the LKF defined as

$$\mathcal{V}_2(\phi_2) := \tilde{V}_2(\phi_2) + \int_{-\delta_1}^0 \gamma_1(|\phi_2(\tau)|)d\tau, \quad \forall \phi_2 \in \mathcal{C}^{n_2}.$$

Since V_2 is a LKF candidate, it satisfies

$$\underline{\alpha}_2(|\phi_2(0)|) \leq V_2(\phi_2) \leq \overline{\alpha}_2(\|\phi_2\|), \quad \forall \phi_2 \in \mathcal{C}^{n_2}. \tag{22}$$

for some $\underline{\alpha}_2, \overline{\alpha}_2 \in \mathcal{K}_\infty$. It follows from (20) that

$$\underline{\tilde{\alpha}}_2(|\phi_2(0)|) \leq \mathcal{V}_2(\phi_2) \leq \overline{\tilde{\alpha}}_2(\|\phi_2\|), \tag{23}$$

where $\underline{\tilde{\alpha}}_2 := \rho \circ \underline{\alpha}_2 \in \mathcal{K}_\infty$ and $\overline{\tilde{\alpha}}_2 := \rho \circ \overline{\alpha}_2 + \delta_1\gamma_1 \in \mathcal{K}_\infty$. Moreover, along the solutions of (19), it holds that

$$\mathcal{V}_2(x_{2t}) := \tilde{V}_2(x_{2t}) + \int_{t-\delta_1}^t \gamma_1(|x_2(\tau)|)d\tau.$$

In view of (21), its Dini derivative therefore reads

$$\begin{aligned} D^+\mathcal{V}_2(x_{2t}) &\leq D^+\tilde{V}_2(x_{2t}) + \gamma_1(|x_2(t)|) - \gamma_1(|x_2(t-\delta_1)|) \\ &\leq -\gamma_1(|x_2(t)|) - \gamma_1(|x_2(t-\delta_1)|). \end{aligned} \tag{24}$$

Furthermore (10) ensures that

$$D^+V_1\big(x_{1t}, x_2(t-\delta_1)\big) \leq -\frac{\alpha_1(|x_1(t)|)}{1+\eta_1(V_1(x_{1t}))} + \gamma_1(|x_2(t-\delta_1)|).$$

Summing this with (24), we get that

$$\begin{aligned} D^+\mathcal{V}(x_t) &\leq -\frac{\alpha_1(|x_1(t)|)}{1+\eta_1(V_1(x_{1t}))} - \gamma_1(|x_2(t)|) \\ &\leq -\frac{\alpha_1(|x_1(t)|) + \gamma_1(|x_2(t)|)}{1+\eta_1(\mathcal{V}(x_t))}, \end{aligned}$$

where $\mathcal{V}(x_t) := V_1(x_{1t}) + \mathcal{V}_2(x_{2t})$. The function $z = (z_1^T, z_2^T)^T \mapsto \alpha_1(|z_1|) + \alpha_2(|z_2|)$ being continuous, positive definite and non-vanishing as $|z| \to \infty$, there exists $\alpha \in \mathcal{K}$ such that $\alpha_1(|z_1|) + \alpha_2(|z_2|) \geq \alpha(|z|)$: see (Khalil, 2002, Lemma 4.3). Therefore

$$D^+\mathcal{V}(x_t) \leq -\frac{\alpha(|x(t)|)}{1+\eta_1(\mathcal{V}(x_t))}. \tag{25}$$

In addition, V_1 being a LKF candidate, there exist $\underline{\alpha}_1, \overline{\alpha}_1 \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_1(|\phi_1(0)|) \leq V_1(\phi_1) \leq \overline{\alpha}_1(\|\phi_1\|), \quad \forall \phi_1 \in \mathcal{C}^{n_1}. \tag{26}$$

Combining (26) and (23), it holds that

$$\underline{\alpha}_1(|\phi_1(0)|) + \underline{\tilde{\alpha}}_2(|\phi_2(0)|) \leq \mathcal{V}(\phi) \leq \overline{\alpha}_1(\|\phi\|) + \overline{\tilde{\alpha}}_2(\|\phi\|).$$

The function $z = (z_1^T, z_2^T)^T \mapsto \underline{\alpha}_1(|z_1|) + \underline{\tilde{\alpha}}_2(|z_2|)$ being continuous, positive definite and radially unbounded, (Khalil, 2002, Lemma 4.3) ensures the existence of $\underline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}_1(|z_1|) + \tilde{\alpha}_2(|z_2|) \geq \underline{\alpha}(|z|)$. Letting $\overline{\alpha} := \overline{\alpha}_1 + \tilde{\overline{\alpha}}_2 \in \mathcal{K}_\infty$, it follows that

$$\underline{\alpha}(|\phi(0)|) \leq \mathcal{V}(\phi) \leq \overline{\alpha}(\|\phi\|). \tag{27}$$

GAS of (19) then follows from (25) and (27) by invoking (Chaillet and Pepe, 2018b, Proposition 1).

iii) BEBS

We finally proceed to establishing the BEBS property. Since the driving subsystem is iISS by assumption, there exists $\beta_2 \in \mathcal{KL}$, ν_2 and $\sigma_2 \in \mathcal{K}_\infty$ such that, given any $u \in \mathcal{U}^m$ and any $x_{20} \in \mathcal{C}^{n_2}$,

$$|x_2(t)| \leq \beta_2(\|x_{20}\|, t) + \nu_2\left(\int_0^t \sigma_2(|u(s)|)ds\right), \quad \forall t \geq 0. \tag{28}$$

Assume that the following bounded energy holds:

$$\int_0^\infty \max\{\gamma_2(|u(\tau)|), \sigma_2(|u(\tau)|)\}d\tau \leq c \tag{29}$$

for some $c \geq 0$. Proposition 1 then ensures that $\lim_{t\to\infty} |x_2(t)| = 0$. Note that this ensures in particular the existence of a finite time $T \geq 0$ (possibly depending on x_{20} and u) such that

$$\|x_{2t}\| \leq 1, \quad \forall t \geq T, \tag{30}$$

which, in view of (22), guarantees that

$$V_2(x_{2t}) \leq \overline{\alpha}_2(1), \quad \forall t \geq T. \tag{31}$$

Consequently, by integrating the dissipation inequality (11) of V_2, we have that, for all $t \geq T$,

$$\begin{aligned} V_2(x_{2t}) - V_2(x_{20}) &\leq -\int_0^t \frac{\alpha_2(|x_2(\tau)|)}{1+\eta_2(V_2(x_{2\tau}))}d\tau \\ &\quad + \int_0^t \gamma_2(|u(\tau)|)d\tau \\ &\leq -\int_T^\infty \frac{\alpha_2(|x_2(\tau)|)}{\bar{\eta}_2}d\tau + \int_0^\infty \gamma_2(|u(\tau)|)d\tau, \end{aligned}$$

where $\bar{\eta}_2 := 1 + \eta_2 \circ \overline{\alpha}_2(1)$. Hence, from (22) and (29),

$$\int_T^\infty \alpha_2(|x_2(\tau)|)d\tau \leq (\overline{\alpha}_2(\|x_{20}\|) + c)\,\bar{\eta}_2. \tag{32}$$

From the growth rate condition (12) and the continuity of γ_1 and α_2, there exists $k > 0$ such that $\gamma_1(s) \leq k\alpha_2(s)$ for all $s \in [0, 1]$. Since (30) implies that $|x_2(t)| \leq 1$ for all $t \geq T$, it follows that

$$\begin{aligned} \int_{-\delta_1}^\infty \gamma_1(|x_2(\tau)|)d\tau &= \int_{-\delta_1}^T \gamma_1(|x_2(\tau)|)d\tau + \int_T^\infty \gamma_1(|x_2(\tau)|)d\tau \\ &\leq \int_{-\delta_1}^T \gamma_1(|x_2(\tau)|)d\tau + \int_T^\infty k\alpha_2(|x_2(\tau)|)d\tau. \end{aligned}$$

Letting $\tilde{x}(\|x_0\|) := k\bar{\eta}_2\,(\overline{\alpha}_2(\|x_{20}\|) + c)$, we get from (32)

$$\int_{-\delta_1}^\infty \gamma_1(|x_2(\tau)|)d\tau \leq \int_{-\delta_1}^T \gamma_1(|x_2(\tau)|)d\tau + \tilde{c}(\|x_0\|). \tag{33}$$

Similarly, by integrating the dissipation inequality (10) with $u_1(t) = x_2(t - \delta_1)$ and using (26), we have that

$$\begin{aligned} \underline{\alpha}_1(|x_1(t)|) &\leq \overline{\alpha}_1(\|x_{10}\|) - \int_0^t \frac{\alpha_1(|x_1(\tau)|)}{1+\eta_1(V_1(x_{1t}))}d\tau \\ &\quad + \int_0^t \gamma_1(|x_2(\tau - \delta_1)|)d\tau \\ &\leq \overline{\alpha}_1(\|x_{10}\|) + \int_{-\delta_1}^{t-\delta_1} \gamma_1(|x_2(\tau)|)d\tau. \end{aligned}$$

In particular, it holds from (33) that

$$\underline{\alpha}_1(|x_1(t)|) \leq \overline{\alpha}_1(\|x_{10}\|) + \int_0^T \gamma_1(|x_2(\tau)|)d\tau + \tilde{c}(\|x_0\|).$$

We conclude from this and (30) that, under the bounded energy assumption (29), the solutions of (9) are bounded, meaning that the cascade owns the BEBS property.

5. ILLUSTRATIVE EXAMPLE

We illustrate the applicability of our result through an academic example. Consider the following cascade TDS involving both discrete and distributed delays:

$$\dot{x}_1(t) = -\operatorname{sat}(x_1(t)) + \frac{1}{4}\operatorname{sat}(x_1(t-1)) + x_1(t)x_2(t-2)^2 \tag{34a}$$

$$\dot{x}_2(t) = -\frac{3}{2}x_2(t) + x_2(t-1) + u(t)\int_{t-1}^t x_2(\tau)d\tau, \tag{34b}$$

where $\operatorname{sat}(s) := \operatorname{sign}(s)\min\{|s|, 1\}$ for all $s \in \mathbb{R}$. This system is in the form (9) with $n_1 = n_2 = 1$, $m = 1$, and $\delta_1 = \delta = 2$. Consider the LKF candidates defined as

$$V_1(\phi_1) := \ln\left(1+\phi_1(0)^2 + \frac{1}{2}\int_{-1}^0 \phi_1(\tau)\operatorname{sat}(\phi_1(\tau))d\tau\right) \tag{35a}$$

$$V_2(\phi_2) := \ln\left(1 + \phi_2(0)^2 + \int_{-1}^0 \phi_2(\tau)^2 d\tau\right) \tag{35b}$$

for all $\phi_1, \phi_2 \in \mathcal{C}$. The derivative of V_2 along the solutions of the driving subsystem (34b) reads

$$\begin{aligned} D^+V_2(x_{2t}, u(t)) &= -\frac{1}{1 + x_2(t)^2 + \int_{-1}^0 x_2(t+\tau)^2 d\tau} \\ &\times \Bigg[2x_2(t)\Big(-\frac{3}{2}x_2(t) + x_2(t-1) + u(t)\int_{t-1}^t x_2(\tau)d\tau\Big) \\ &\quad + x_2(t)^2 - x_2(t-1)^2\Bigg]. \end{aligned}$$

Observing that $x_2(t)x_2(t-1) \leq \frac{1}{2}(x_2(t)^2 + x_2(t-1)^2)$ and

$$\begin{aligned} \left|x_2(t)\int_{t-1}^t x_2(\tau)d\tau\right| &\leq \frac{1}{2}\left(x_2(t)^2 + \left(\int_{t-1}^t x_2(\tau)d\tau\right)^2\right) \\ &\leq \frac{1}{2}\left(x_2(t)^2 + \int_{t-1}^t x_2(\tau)^2 d\tau\right) \end{aligned}$$

and defining $\eta_2(s) := e^s - 1$, we get that

$$D^+V_2(x_{2t}, u(t)) \leq -\frac{x_2(t)^2}{1 + \eta_2(V_2(x_{2t}))} + |u(t)|.$$

On the other hand, the derivative of V_1 along the solutions of the driven subsystem (34a) reads

$$\begin{aligned} D^+V_1(x_{1t}, x_{2t}) &= \frac{1}{1 + x_1(t)^2 + \frac{1}{2}\int_{t-1}^t x_1(\tau)\operatorname{sat}(x_1(\tau))d\tau} \\ &\times \Bigg[2x_1(t)\Big(-\operatorname{sat}(x_1(t)) + \frac{1}{4}\operatorname{sat}(x_1(t-1)) + x_1(t)x_2(t-2)^2\Big) \\ &\quad + \frac{1}{2}\big(x_1(t)^2 - x_1(t-1)^2\big)\Bigg]. \end{aligned}$$

Observe that $x_1(t)\operatorname{sat}(x_1(t-1)) \leq x_1(t)\operatorname{sat}(x_1(t)) + x_1(t-1)\operatorname{sat}(x_1(t-1))$. Defining $\eta_1(s) := e^s - 1$, we obtain that

$$D^+V_1(x_{1t}, x_{2t}) \leq -\frac{x_1(t)\mathrm{sat}(x_1(t))}{1+\eta_1(V_1(x_{1t}))} + 2x_2(t-2)^2.$$

In other words, the assumptions of Theorem 1 are fulfilled with $\alpha_1(s) = \mathrm{sat}(s)s$, $\alpha_2(s) = s^2$, $\eta_1(s) = \eta_2(s) = e^s - 1$, $\gamma_1(s) = 2s^2$ and $\gamma_2(s) = s$. With these rates, condition (12) is fulfilled, so we conclude that the cascade (35) is 0-GAS and owns the BEBS property.

6. CONCLUSION

We have provided conditions under which the cascade of two iISS TDS is 0-GAS and has the BEBS property. Like in the finite-dimensional case, these conditions take the form of growth restrictions on the input rate of the driven subsystem and the dissipation rate of the driving one. An academic example illustrates the applicability of the result.

A limitation of our main result lies in the way the two subsystems are interconnected: although a pure delay is allowed in our setup, we were not able to extend the result to a more generic interconnection (meaning for a driven subsystem of the form $\dot{x}_1(t) = f_1(x_{1t}, x_{2t})$). Another significant limitation is that we were not able to guarantee that the overall cascade is itself iISS. This is due to the fact that, contrarily to the finite-dimensional case, it has not yet been proved that 0-GAS combined with the BEBS property is enough to guarantee iISS for time-delay systems (see Remark 2). Future work will aim at solving these issues and at allowing the input u to impact directly the driven subsystem as well.

ACKNOWLEDGEMENTS

The authors are thankful to Prof. Willie Wong and Prof. Yacine Chitour for fruitful discussions on technical aspects.

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