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Repetitive Set-based Learning Robust Predictive Control for Lur'e Systems

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Abstract: Robust control of uncertain nonlinear systems subject to constrants often leads to conservatism. Such behaviors can be improved by updating the model of the uncertainty with the data collected during the operation time or by bounding the parameters. This paper proposes an approach to robustly control the discrete-time Lur'e system subject to states and input constraints, where the unknown memoryless nonlinearity is sector-bounded and its Lipschitz constant is assumed to be given. In the first phase of operation, when no data has been collected, a robust MPC controller obtained from solving linear matrix inequalities is used. This formulation is also used to compute the safe region, in which the system can operate safely. After sufficient data has been collected, an upper and lower bound of of the nonlinearity can be constructed by using the Lipschitz constant. A controller based on tube-based MPC is used, which results in less conservatism and provides more flexibility. Data of the nonlinearity can be further updated to reduce uncertainty, and hence, to decrease the size of the tube. Under additional conditions, the controller can safely explore the region outside the safe regions to collect more data of the unknown nonlinearity to improve performance and region of attraction.

Keywords: Lur'e systems, learning, constrained model predictive control, robustness, uncertainty.

1. INTRODUCTION

Nonlinear model predictive control (NMPC) has received great attention over the last decades thanks to its flexibility to efficiently handle systems with constraints, its capability of using preview information, the direct consideration of nonlinear systems, and a coherent implementation of multiple hierarchical layers, see e. g. Rawlings et al. (2017), Findeisen et al. (2007), Lucia et al. (2016)). The fundamental idea of predictive control is to solve at each time step an optimal control problem which predicts the future system behavior based on the current state information. The first part of the resulting optimal trajectory is then applied and the process is repeated at the next step.

In spite of its significant progress, the control of uncertain nonlinear systems using predictive control still faces challenges. Numerous approaches, spanning from worst case and tube based formulations (Rakovic et al., 2012; Mayne et al., 2011), to scenario based methods (Lucia et al., 2013), as well as stochastic model predictive control formulations (Mesbah, 2016) have been studied so far. Besides, there exist several approaches to analyze the robustness of predictive control, see e.g. (Grimm et al., 2004; Findeisen et al., 2007). Most of the existing robust predictive control approaches for MPC often lead to conservatism, which in general can have adverse influence on the performance of the controller by excessively reducing the feasible solution space of the optimization problem. One of possible solution to reduce the conservatism of robust approaches is to use the data collected during the past operation to learn the uncertainty model, resulting in the reduction of a uncertainty bounds.

One of the robust controller for uncertain Lur'e system is proposed in Böhm et al. (2009) in which the solution is obtained by recasting the problem as the search for a linear feedback law, where the feedback matrix can be calculated efficiently due to a reformulation in an LMI form. The derived linear feedback law is applied continuously and recalculated at the following discrete sampling times. This approach was improved by Nguyen et al. (2018), in which the optimization problem in the form of LMIs is modified by adding new data collected during operation. Since the problem is formulated as LMIs, the computation cost is decreased. However, this approach results in a small feasible set for optimal solutions. Additionally, the controller is not flexible for tasks such as path following.

Nevertheless, safety must be ensured during the process of collecting data for learning, especially for critical systems, in which any violation of constraints may damage the system. Several learning methods like reinforcement learning (e.g. Lewis and Vrabie (2009)) or Gaussian process (e.g. Liu et al. (2018) and references therein), are promising but in practice are unlikely to guarantee safety since they do not consider constraints. Furthermore, the free-model approach of reinforcement learning needs a large amount of data to work. Gaussian processes use a stochastic ap-

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proach, resulting a stochastic guarantee, which may not be suitable for critical systems. Besides, it is also crucial that during the process of collecting data, where the model accuracy may be not sufficient, the safety is also guaranteed.

From above discussion, Model Predictive Control combined with set-based approaches shows advantages, since all constraints are well integrated in the formulation of the optimization problem. Also, the first principles and prior knowledge of the system can provide a good-enough initial model for control, avoiding the need for large amount of data for learning. The approach proposed in this paper exploits the prior knowledge, namely the Lipschitz constant and sector bound, as an initial approximate information, but during its operation, more data are collected and are used to refine the model to reduce uncertainty.

Specifically, the initial controller based on solving optimization problem as LMIs is used due to the large uncertainty, which leads to a small feasible set, but it guarantees safety during operation and data collection process. The LMIs formulation also provides a safe robust control invariant set for the system, which can play the role of the terminal set for a second controller. By assuming that all the states of the system are observable and the information is re-constructable and the measure error bound is given, based on the collected data, an upper and lower bound function of the unknown nonlinear function are constructed (see, for example, in Beliakov (2006) or Calliess (2016)). The maximum difference between the upper and lower bound function can be reduced by collecting more data. Accordingly, a nominal nonlinear function can be chosen and the system can be consider as a nominal system with a bound disturbance. With more appropriate data, the bound of disturbance can be reduced. When data are sufficient, a tube-based MPC controller is utilized, which can be less conservative and more flexible. This controller can be used to explore the region outside the safe region.

The paper is structured as follows. Section 2 describes the considered class of Lur'e system and the control problem. Section 3 provides a way to obtain robustly stabilizing linear feedback with in form of LMIs. Section 4 outlines the tube-based MPC scheme and its usage for exploring the region outside the safe region. In Section 5 the results are illustrated with the help of simulation example that considers a flexible link robotic arm. Finally, conclusions are provided in Section 6.

2. PROBLEM SETUP - SAMPLED-DATA MPC OF UNCERTAIN CONTINUOUS TIME LUR'E SYSTEMS

In this section, the description of the nonlinear system is given and the control task is outlined.

We consider nonlinear Lur'e systems, which are given by

$$\begin{aligned} x(k+1) &= Ax(k) + G\gamma(z(k)) + Bu(k), \\ z(k) &= Hx(k), \end{aligned} \tag{1}$$

(for example see Khalil (2002)). Here $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $G \in \mathbf{R}^{n\times 1}$ and $H \in \mathbf{R}^{1\times n}$ are constant matrices. The mapping $\gamma(z) : \mathbf{R} \to \mathbf{R}$ is a nonlinearity satisfying the following assumption:

Assumption 1. (Sector bounded Lur'e system) The nonlinearity $\gamma(z)$ satisfies a sector-bound condition, described in the form

$$(\beta z - \gamma(z))^T \gamma(z) \geq 0, \qquad \forall z \in \mathbf{R}. \tag{2}$$

It should be noted that, this work can also be applied for a more general case, in which $\beta \in [\beta_1, \beta_2]$, as with a simple loop transformation, the nonlinearity can be transformed into (2), cf. e.g. Khalil (2002). Moreover, this approach can be extended to the case where the system has multiple nonlinearities.

Assumption 2. (Lipschitz continuity) The nonlinearity is Lipschitz continuous, i.e. there exists a real number $L \geq 0$ such that, for all z_1 and z_2

$$\| \gamma(z_1) - \gamma(z_2) \| \leq L \| z_1 - z_2 \| \quad \forall z_1, z_2 \in \mathbf{R}, \tag{3}$$

and L is known.

The system (1) is controlled while state and inputs are subject to constraints

$$x(k) \in \mathbf{X}, \quad u(k) \in \mathbf{U}, \quad \forall k \geq 0.$$

Assumption 3. (Input and state constraints) The constraint sets $\mathbf{X}, \mathbf{U}$ are convex polytopes

$$\begin{aligned} \mathbf{X} &= \{x \in \mathbf{R}^n : \tilde{c}_\iota^T x \leq 1, \ \iota = 1, 2, ..., r_x\} \\ \mathbf{U} &= \{u \in \mathbf{R}^m : \tilde{d}_\kappa^T u \leq 1, \ \kappa = 1, 2, ..., r_u\} \end{aligned},$$

in which $\tilde{c}_\iota \in \mathbf{R}^n, \tilde{d}_\kappa \in \mathbf{R}^m$, and r_x, r_u are the number of the state and input constraints, respectively.

The sets $\mathbf{X}$ and $\mathbf{U}$ can be combined in one constraint set

$$\mathbf{C} = \{[x^T \ u^T]^T \in \mathbf{R}^{n+m} : c_i^T x + d_i^T u \leq 1, \ i = 1, ..., r\}, \tag{4}$$

where $r = r_x + r_u$.

In the next section we outline how the optimal control problem can be solved efficiently using an LMI formulation which leads to the linear feedback matrix K_k to apply to the system each time step.

3. LMI-BASED CONTROLLER FOR COLLECTING DATA

In this section, the linear feedback control law $u = K_k x$ is designed to guarantee that the system (1) is robustly asymptotically stabilized while the state and input constraints are taken into account. The infinite horizon problem is solved using LMIs. The approach leads to smaller feasible set but guarantees stability for the system. During this time, data is safely collected and later incorporated in the model, resulting in a better performance. For computational reasons, we focus on finding at each time step a linear control law

$$u = K_k x, \tag{5}$$

which minimizes the infinite horizon cost functional

$$J = \sum_{k}^{\infty} x^T(k) Q x(k) + u^T(k) R u(k). \tag{6}$$

The weight matrices $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{m\times m}$ are positive definite, and (6) is minimized subject to the system dynamics (1) and the constraints (4).

Furthermore, the constraint set (4) can be written as follows

$$\mathbf{C} = \{x \in \mathbf{R}^n : (c_i^T + d_i^T K)x \leq 1, \ i = 1, 2, ..., r\}. \tag{7}$$

The following lemma will be exploited to guarantee the satisfaction of the constraint (7).

Lemma 4. The ellipsoid $\mathcal{E} = \{x \in \mathbf{R}^n : x^T P x \leq \alpha\}$ is contained in the set (7) if and only if

$$(c_i^T + d_i^T K)(\alpha P^{-1})(c_i^T + d_i^T K)^T \leq 1, \quad i = 1, 2, ...r. \quad (8)$$

Proof. See Boyd et al. (1994) or Chen and Ballance (1999).

Lemma 4 implies that if the condition (8) is satisfied, then the $\mathcal{E}$-neighborhood of a state x lies in the constraint set $\mathcal{C}$, hence satisfying the state and input constraints. The condition (8) will be later written in form of an LMI.

Furthermore, the sector-bounding condition (2) can be written in matrix form as follows (see e.g. Boyd et al. (1994)):

$$\begin{bmatrix} x \\ \gamma \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2} H^T \beta^T \\ -\frac{1}{2}\beta H & I \end{bmatrix} \begin{bmatrix} x \\ \gamma \end{bmatrix} \leq 0. \quad (9)$$

Incorporating (8) and (9) guarantees constraint satisfaction and robustness. However, the resulting feedback $u = K_k x$ will be conservative, since no knowledge of the nonlinearity γ is used.

Ensuring a stable Linear Feedback Law

To guarantee stability of the resulting feedback we add a Lyapunov condition. We choose a quadratic Lyapunov function $V(x) = x^T P x$ where $P > 0$. Asymptotic stability of the feedback $u = Kx$ is guaranteed if

$$V(x(k+1)) - V(x(k)) + x(k)^T Q x(k) + u(k)^T R u(k) < 0, \quad (10)$$

From the dynamics (1) and the control law (5), the condition is equivalent to

$$\begin{bmatrix} x \\ \gamma \end{bmatrix}^T \begin{bmatrix} \begin{bmatrix} (A^T + K^T B^T) P (A + BK) \\ -P + Q + K^T R K \end{bmatrix} & \star \\ G^T P (A + BK) & 0 \end{bmatrix} \begin{bmatrix} x \\ \gamma \end{bmatrix} \leq 0. \quad (11)$$

The condition (11) needs to hold for all $[x^T \quad \gamma^T]^T$ that satisfy (9). Applying the so-called S-procedure (e.g. Boyd et al. (1994)), it follows that

$$\begin{bmatrix} x \\ \gamma \end{bmatrix}^T \begin{bmatrix} \begin{bmatrix} (A^T + K^T B^T) P (A + BK) \\ -P + Q + K^T R K \end{bmatrix} & \star \\ G^T P (A + BK) + \tau \beta H & -\tau I \end{bmatrix} \begin{bmatrix} x \\ \gamma \end{bmatrix} \leq 0. \quad (12)$$

Changing the variables to $P = \alpha N_1^{-1}, K = N_2 N_1^{-1}$, where $\alpha > 0$ is a constant number, $0 < N_1 = N_1^T \in \mathbf{R}^{n \times n}$, and applying the Schur complement, we obtain

$$\begin{bmatrix} N_1 & \star & \star & \star & \star & \star \\ S & \alpha\tau I & \alpha G^T & \star & \star & \star \\ M & \alpha G & N_1 & \star & \star & \star \\ 0 & 0 & 0 & I & \star & \star \\ Q^{\frac{1}{2}} N_1 & 0 & 0 & 0 & \alpha I & \star \\ R^{\frac{1}{2}} N_2 & 0 & 0 & 0 & 0 & \alpha I \end{bmatrix} \geq 0. \quad (13)$$

where $M = AN_1 + BN_2, S = -\frac{\tau}{2}\beta H N_1$.

Combining all conditions leads to the following Theorem.

Theorem 5. (Stability of the separate feedback laws) Consider the system (1) satisfying Assumption 1 and 2. Suppose that there exist matrices $0 < N_1 = N_1^T \in \mathbf{R}^{n \times n}$ and $N_2 \in \mathbf{R}^{m \times n}$, and constants $\tau \in \mathbf{R}^+$ and $\alpha \in \mathbf{R}^+$ such that the inequality (13) holds for $M = AN_1 + BN_2, S = -\frac{\tau}{2}\beta H N_1$. Then, for $P = \alpha N_1^{-1}$ and $K = N_2 N_1^{-1}$, the control input $u = Kx$ asymptotically stabilizes the system (1). Moreover, at time k, $x(k)^T P x(k)$ is an upper bound of the infinite horizon cost functional (6).

Proof. By construction, if the inequality (13) is satisfied, then clearly $V(x) = x^T P x$ is a Lyapunov function and consequently the system (1) is asymptotically stable with the control law $u = Kx$. Furthermore, if we take the sum of both side of inequality (10) from t to ∞, we obtain

$$x(k)^T P x(k) > \sum_{k}^{\infty} x^T(k) Q x(k) + u^T(k) R u(k)$$

since $x(k) \to 0$ when $k \to \infty$, which shows that $x(k)^T P x(k)$ is an upper bound of the infinite horizon cost functional (6).

The following algorithm explains how we use LMI based controller.

Algorithm 1. (for controlling and collecting data)
1. At $k \in \mathbf{Z}_+$, measure the state $x(k)$ and construct information of $\gamma(z)$.
2. Solve the following optimization problem for fixed τ:

$$\min_{\alpha_k, N_{1k}, N_{2k}} \alpha_k \quad (14)$$

subject to

$$\begin{bmatrix} 1 & x^T(k) \\ x(k) & N_{1k} \end{bmatrix} > 0, \quad (14a)$$

$$\begin{bmatrix} N_{1k} & \star & \star & \star & \star & \star \\ S & \alpha\tau I & \alpha G^T & \star & \star & \star \\ M & \alpha G & N_{1k} & \star & \star & \star \\ 0 & 0 & 0 & I & \star & \star \\ Q^{\frac{1}{2}} N_{1k} & 0 & 0 & 0 & \alpha I & \star \\ R^{\frac{1}{2}} N_{2k} & 0 & 0 & 0 & 0 & \alpha I \end{bmatrix} \geq 0. \quad (14b)$$

$$\begin{bmatrix} 1 & c_i N_{1k} + d_i N_{2k} \\ (c_i N_{1k} + d_i N_{2k})^T & N_{1k} \end{bmatrix} \geq 0. \quad (14c)$$

3. Compute $K_k = N_{2k} N_{1k}^{-1}$ and apply the input

$$u(k) = K_k x(k), \quad \forall k \geq 0. \quad (15)$$

4. Set $k := k + 1$ and go to step 1.

The following theorem guarantees the feasibility of the method.

Theorem 6. (Repeated feasibility and stability with offline learning) Consider the system (1) satisfying Assumption 1 and 2 controlled by using Algorithm 1. The following properties are guaranteed:
a. If the optimization problem (14) is feasible at the time k, it is also feasible at time $k + 1$.
b. If the optimization problem (14) is feasible initially, the origin is asymptotically stable.

Proof. Hier the sketch of the proof is given. The solution of the optimization problem at the time k also satisfies (14b), (14c) at the time $k + 1$. The constraint (14a) is satisfied because $x^T(k+1) P_k x(k+1) < x^T(k) P_k x(k) < \alpha_k \ \forall k$.
Therefore, all states in stay in the ellipsoid $\mathcal{E}_k = \{x \in \mathbf{R}^n : x^T P_k x \leq \alpha_k\}$. From Lemma 4 and (14c), which is equivalent to $(c_i^T + d_i^T K_k)(\alpha P_k^{-1})(c_i^T + d_i^T K_k)^T \leq 1, \ i =$

$1, 2, ...r.$, it is clear that for each time k, the ellipsoid $\mathcal{E}_k$ is contained in the constraint set (7). Since it holds for every k, it follows that the constraints are satisfied for all $k \geq 0$.

4. TUBE-MPC FOR CONTROLLING AND EXPLORING NEW DATA

In this section, we outline the method to integrate the data we collect of the function $\gamma(z)$ to reduce the conservatism of the controller. Without loss of generality, let us consider the function $\gamma(z)$ with $z > 0$. On the other half of the plane, where $z < 0$, the same conclusions apply. Assume to measure l data points in the form (z_i, γ_i) where $i = 1, 2, \ldots, l$ and $0 < z_1 < z_2 < \cdots < z_l$. Let ε denote the measurement error and assume that the upper bound of this error, denoted as ϵ, is known, which means that $\varepsilon_i < \epsilon$ for all $i = 1, 2, \ldots, l$.

Let $\overline{\gamma}(z)$, $\underline{\gamma}(z) : \mathbf{R} \to \mathbf{R}$ denote the upper and lower bound of the function $\gamma(z)$, which can be constructed by utilizing the data set $(z_i, \gamma_i)_{i=0}^{l}$ (which is denoted as the set $\mathcal{D}$) and the assumption that the Lipschitz constant L of the function $\gamma(z)$ is given. The method used to construct these functions can be found in Beliakov (2006) and Calliess (2016). Let us pick an arbitrary function $\tilde{\gamma}(z)$ as a nominal function such that $\underline{\gamma}(z) \leq \tilde{\gamma}(z) \leq \overline{\gamma}(z)$ for all $z \in \mathbf{R}^p$ and $j \in [0, p]$. From how upper and lower bound function are constructed and how the nominal function is chosen it follows that the difference of $\tilde{\gamma}(z)$ and the real function $\gamma(z)$ is always bounded by

$$\|\tilde{\gamma}(z) - \gamma(z)\| \leq 2Ld + 2\epsilon, \tag{16}$$

where $d = \min_{z_i \in \mathcal{D}} \|z - z_i\|$. Denote $\tilde{d} = \max_{z_i \in \mathcal{D}} \|z_{i+1} - z_i\| \quad \forall i \in [1, 2, 3, ..., l-1]$, obviously $d \leq \tilde{d}$ for all $\|z\| \leq \max_{i \in [0,l]} \|z_i\|$. Therefore, the error between the nominal and the real function is always bounded by

$$\|\tilde{\gamma}(z) - \gamma(z)\| \leq W, \tag{17}$$

where $W = 2L\tilde{d} + 2\epsilon$. Accordingly, by making the set $\mathcal{D}$ denser (by collecting data), the error w is reduced and for a certain data set $\mathcal{D}$, w is known.

Control in the known region with tube MPC

We now use the standard approach of MPC to control the system. When a nominal nonlinear fuction $\tilde{\gamma}(z)$ is chosen, the corresponding nominal system is

$$\begin{aligned} \tilde{x}(k+1) &= A\tilde{x}(k) + G\tilde{\gamma}(\tilde{z}(k)) + B\tilde{u}(k), \\ \tilde{z}(k) &= H\tilde{x}(k), \end{aligned} \tag{18}$$

The cost function can be chosen as

$$F(\tilde{x}, \tilde{u}) := \tilde{x}^T Q \tilde{x} + \tilde{u}^T R \tilde{u}, \tag{19}$$

with the weighting matrix $Q, R \in \mathbf{R}^{n \times n}$ and $Q, R \succeq 0$. The terminal cost is defined as

$$E(x) := \tilde{x}^T P \tilde{x}, \tag{20}$$

where $P \in \mathbf{R}^{n \times n}$ and $P \succeq 0$. P is determined by solving the optimization problem (14). The optimal control problem of the proposed approach is given by

$$\min_{\tilde{u}(k)} \sum_{k}^{N} F(\tilde{x}(k), \tilde{u}(k)) + E(\tilde{x}(k+N)), \tag{21}$$

subject to

$$\tilde{x}(k+1) = A\tilde{x}(k) + G\tilde{\gamma}(H\tilde{x}(k)) + B\tilde{u}(k), \tag{21a}$$

$$\tilde{x}(k) \in \mathbf{X} \ominus \mathcal{R}(k), \tag{21b}$$

$$\tilde{u}(k) \in \mathbf{U} \ominus K_e \mathcal{R}(k), \tag{21c}$$

$$\tilde{x}(k+N) \in \mathbf{E} \ominus \mathcal{R}(k+N). \tag{21d}$$

Hereby, the nominal states have to satisfy the shrinking constraints obtained by subtracting the original constraints with the tubes $\mathcal{R}(k)$. Likewise, the terminal set, which is used to guarantee the recursive feasibility, is also shrunk by the tubes. The construction of the tubes is elaborated in the next section. The terminal set can be chosen as the ellipsoid computed from section 3.

Robust constraint satisfaction and recursive feasibility

The difference between the real states and the nominal states are denoted as

$$e(k) = x(k) - \tilde{x}(k).$$

The input is chosen in the form of

$$u(k) = \tilde{u}(k) + K_e e(k), \tag{22}$$

which has two terms. The term $\tilde{u}(k)$ comes from solving the optimization problem (21), while the term $K_e e(k)$ is used to stabilize the error dynamics. Thus, it can be written as

$$e(k+1) = (A + BK_e)e(k) + Gd(k), \tag{23}$$

where $d(k) = \gamma(z(k)) - \tilde{\gamma}(\tilde{z}(k))$ and can be bounded by

$$\|\gamma(z) - \tilde{\gamma}(\tilde{z})\| \leq \|\gamma(z) - \gamma(\tilde{z})\| + \|\gamma(\tilde{z}) - \tilde{\gamma}(\tilde{z})\|,$$

for each time k. The first term of the right hand side in (4) can be bounded by using the Lipschitz constant in Assumption 2

$$\|\gamma(z) - \gamma(\tilde{z})\| \leq L\|z - \tilde{z}\| \leq L\|H\|\|x - \tilde{x}\| = L\|H\|\|e\|,$$

while the second term is bounded by using (17) as follows

$$\|\gamma(\tilde{z}) - \tilde{\gamma}(\tilde{z})\| \leq W.$$

Therefore, we have

$$\|d(k)\| \leq \tilde{L}\|e(k)\| + W. \tag{24}$$

where $\tilde{L} = L\|H\|e$. To find the suitable K_e and the minimum robust positively invariant set of (23) with respect to the disturbance (24), we can use the method in Löfberg (2003). In short, we need to find an ellipsoid $\mathbf{E}_\Omega$ such that $e_{k+1} \in \mathbf{E}_\Omega$ for all $e_k \in \mathbf{E}_\Omega$ and d that satisfies (24). The problem becomes finding $\Omega \in \mathbf{R}^{n \times n}$, $\Omega \geq 0$ and $\Theta \in \mathbf{R}^{1 \times n}$ such that

$$\tau_1 \tilde{W} + \tau_2 \leq 1, \tag{25}$$

$$\begin{bmatrix} \tau_2 \Omega & \star & \star & \star \\ 0 & \tau_1 I & \star & \star \\ A\Omega + B\Theta & G & \Omega & \star \\ \sqrt{2\tau_1 \tilde{L}} \Omega & 0 & 0 & I \end{bmatrix} \geq 0. \tag{25a}$$

where $\tilde{W} = 2W^2$, τ_1 and τ_2 are constants coming from S-procedure and K_e is computed by

$$K_e = \Theta \Omega^{-1}. \tag{26}$$

The following theorem guarantees the constraints satisfaction and recursive feasibility for the real system.

Theorem 7. If there exists a feasible solution $\kappa_i(x(k_0))$ for the problem and if the states of the nominal model (18) satisfy the constraints (21b), then the states of the real system (1) robustly satisfy the real constraints $(\mathbf{X}, \mathbf{U})$

Proof. This results are direct results from standard tube-based MPC approach, see for example Mayne et al. (2011).

Exploring the unknown region

The ellipsoid safe region $\mathcal{E}$ can be the computed from solving the optimization (14) and the procedure of collecting data for the function $\gamma(z)$ can be done from method discussed above. Also, z_l denotes the largest value of z we have collected in the data set $\mathcal{D}$ and the upper and lower bound $\overline{\gamma}, \underline{\gamma}$ have been constructed for $0 < z < z_l$. In this part, we discuss how to safely explore the region outside of ellipsoid $\mathcal{E}$. Assume that we need to steer the system to the state x_{l+1} corresponding to the value z_{l+1} so that we can measure the value γ_{l+1}, which can help update the the nonlinear functions $\overline{\gamma}(z)$ and $\underline{\gamma}(z)$ to the value of z_{l+1}.

The problem of safe exploration is that, on the one hand, we must ensure that the system can safely be driven from the state x_l to the state x_{l+1}, which means there exists always at least a control input $u \in \mathbf{U}$ that can take the system to the state x_{l+1} such that $x \in \mathbf{X}$. In other words, x_{l+1} must lie inside the reachable set of the system with the initial condition x_l. On the other hand, when the system is at the state x_{l+1}, it should be able to come back to the so-called known set, denoted as set $\mathcal{S}$, which includes the safe set $\mathcal{E}$ and all points that have been explored, which means there exist at least a control input $u \in \mathbf{U}$ that can take the system to the set $\mathcal{S}$ such that $x \in \mathbf{X}$. That is to say, the intersection set of the set $\mathcal{S}$ and the reachable set of the system with the initial condition x_{l+1} is non-empty.

In the case of systems with a bounded uncertainty, the problem is more complicated because even though the state is x_{l+1} is fixed and the nominal system can reach x_{l+1}, the state of the real system lies in a bounded set that surrounds x_{l+1}, denoted as $\mathcal{X}_{l+1}$. If the set $\mathcal{X}_{l+1}$ is computable, then in order to make sure that the system can travel back to the set $\mathcal{S}$, we have to check that condition for all the points in the set $\mathcal{X}_{l+1}$. Additionally, since the problem is formulated as a tube-based MPC problem, the necessary condition is that the set of state constraints $\mathbf{X} \ominus \mathcal{R}(k)$ in (21b) is not empty, in which the set $\mathcal{R}(k)$ depends on the distance $\|x_{l+1} - x_l\|$. In general, this is not a trivial problem for a nonlinear system. However, we can exploit the structure of the nominal system (18) and limit to computations of reachable sets in one step ahead to come up with a simpler approach as follows.

As long as $z_{l+1} > z_l$, the system can obtain new data to construct the bounds for the nonlinear function, so the option for z_{l+1} is not unique. One possible way is to choose x_{l+1} such that $z_{l+1} - z_l = \min_{z_i \in \mathcal{D}} \|z_{i+1} - z_i\| := \underline{d}$, which guarantees the condition that $\mathbf{X} \ominus \mathcal{R}(k)$ in (21b) is not empty. After, z_{l+1} is picked, $\tilde{x}_{l+1}$ must satisfy

$$H\tilde{x}_{l+1} = z_{l+1}. \tag{27}$$

Here note that $\tilde{x}_{l+1}$ is the state that the nominal system is planned to reach, while the real system, due to the disturbance, will arrive at the state $x_{l+1} \in \tilde{x}_{l+1} \oplus \underline{\mathcal{W}}$ where $\underline{\mathcal{W}} = \{w : \|w\| \leq 2L\underline{d} + 2\epsilon\}$. In order to guarantee that the nominal system can reach $\tilde{x}_{l+1}$ in one step, $\tilde{x}_{l+1}$ must belong to the set

$$\tilde{x}_{l+1} \in [(Ax_l + G\tilde{\gamma}(z_l)) \oplus B\mathbf{U}] \cap [\mathbf{X} \ominus \mathcal{W}]. \tag{28}$$

If the optimization problem is feasible, then one x_{l+1} can be chosen. The next step is to check if the system can be driven back to the safe set from all the points in the set $\tilde{x}_{l+1} \oplus \underline{\mathcal{W}}$. Consider

$$x_{l+2} = Ax_{l+1} + Bu + G\tilde{\gamma}(z_{l+1}) + w. \tag{29}$$

where $x_{l+1} \in \tilde{x}_{l+1} \oplus \underline{\mathcal{W}}$ and $z_{l+1} \in [\tilde{z}_{l+1} - d, \tilde{z}_{l+1} + d]$. By finding the minimum $\underline{\Gamma}$ and maximum $\overline{\Gamma}$ of $\tilde{\gamma}(z_{l+1})$ over $[\tilde{z}_{l+1} - d, \tilde{z}_{l+1} + d]$, we have $\tilde{\gamma}(z_{l+1}) \in [\underline{\Gamma}, \overline{\Gamma}]$, denoted as the set Γ. From (4), since it is a linear combination of all polytopes, we only need to check the condition on the boundaries.

5. SIMULATION EXAMPLE

In this section we illustrate the proposed approach with a flexible link robot arm modelwhich is similar the one in Böhm et al. (2008), Böhm et al. (2009). The simulations were run using the python toolbox do-mpc (Lucia et al., 2017). The robot is given in form of (1) with the matrices

$$A = \begin{bmatrix} 1 & 0.05 & 0 & 0 \\ -2.43 & 0.9375 & 2.43 & 0 \\ 0 & 0 & 1 & 0.05 \\ 0.975 & 0 & -0.835 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1.08 \\ 0 \\ 0 \end{bmatrix},$$
$$G^T = [0 \ 0 \ 0 \ -0.1665], \quad H = [0 \ 0 \ 1 \ 0].$$

The considered nonlinear function γ takes the form $\gamma(z) = 0.25(\sin(z) + z)$. The following input and state constraints need to be satisfied

$$u(k) \in [-2, 2], \quad x_1(k), x_3(k) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \quad \forall k \geq 0.$$

The control task is to steer the robotic arm to the origin. The weighting matrices in (6) are chosen as $Q = 0.01\text{diag}(1, 0.1, 1, 0.1), R = 0.01$. First, the system is controlled with LMI-based controller. The parameters are set to $\tau = 1$. We first consider the initial conditions $x_0 = [1.1, 0.2, 0, 0]^T$. The problem is feasible and the trajectory is shown in Fig. 1. However, if $x_0 = [1.2, 0.2, 0, 0]^T$, the problem becomes infeasible. We start from different initial conditions where the problem is feasible to collect more data to build the upper and lower-bound functions. The gain $K_e = [-4.9, -1.19, 2.58, -0.5]$ is computed by (26). The constraints becomes $\tilde{u}(k) \in [-1.26, 1.26]$, $x_1(k), x_3(k) \in [-1.41, 1.41]$. The tube-based approach can find solution from $x_0 = [1.2, 0.2, 0, 0]^T$ as shown in Fig. 2, which shows that it slightly reduces conservatism. This result still can be improved by using a better computational method for computing $\mathcal{R}(k)$ and a better choice of K_e.

6. CONCLUSIONS

In this paper we proposed a set-based robust model predicted control combined with learning for controlling Lur'e systems with unknown sector-bounded nonlinearity. The data collecting phase is conducted by using an LMI-based controller which guarantees stability and constraints. These points are used to construct upper and lower bound functions of the unknown nonlinerity by using the Lipschitz constant, and the error between the real nonlinear function and the nominal one is bounded. The measurement error is also taken into account in the problem formulation.

The proposed approach can be improved as follows. Firstly, the rough initial estimation of Lipschitz constant may be re-estimated and improved over time from the data. Second, a strategy to pick up a suitable K_e that introduces less conservatism to the method can be considered. Third, the exploration strategy can only be applied

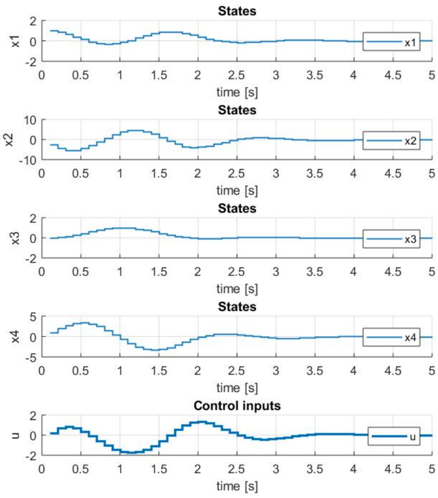

Fig. 1. The flexible link robotic arm controlled by LMI-based controller.

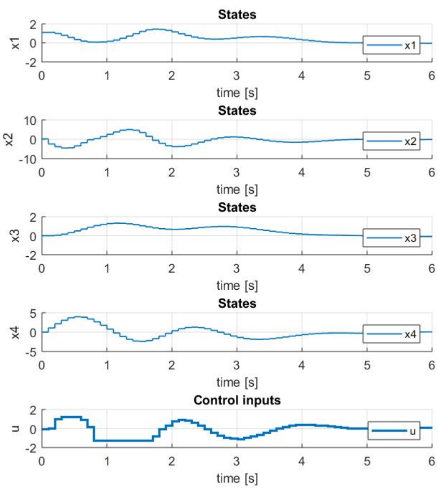

Fig. 2. The flexible link robotic arm controlled by tube-based controller.

for one step ahead computation due to the complexity of the computation of forward and backward reachable sets. To decrease the computational burden one could exploit the structure of Lur'e system and obtain an appropriate approximation of the sets.

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