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Mixed Fractional Order Adaptive Control: Theory and Applications ⋆

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Abstract: In this paper we study the adaptive control problem of integer order plants using fractional order adaptive laws in the controller. The study is based on a general methodology developed recently to establish boundeness and asymptotic behavior of solutions to multi-order systems (set of differential equations with different derivation orders) having multiple time-varying delays. Also it is based on recent results for fractional order systems under the perspective of the so called "Error Models". The method relies on vector Lyapunov-like functions and on comparison arguments. Boundedness and convergence of the solutions are theoretically analyzed and applications to fractional adaptive schemes are presented towards the end of the paper, including numerical simulations to verify the analytical results.

Keywords: Fractional order systems; Adaptive control; Stability; Multi-order fractional systems; Boundedness and convergence.

1. INTRODUCTION

In the last decades there has been numerous efforts towards improving the control of systems with some degree of uncertainty, including unknown parameters, external perturbations and unmodeled dynamics, among others. To this extent, adaptive control has been one of the several techniques extensively studied to cope for the aforementioned uncertainties. In an historical perspective this discipline appeared in the late fifties Aseltine et al. (1958); with some landmarks published in 1980, where the linear case was completely solved. Narendra et al. (1980a), Narendra et al. (1980b), Goodwin et al. (1980), Morse (1980). Next, the robust adaptive control problem was solved by different methods and techniques Narendra et al. (1989). Lately the interest has shifted to developing robust adaptive control techniques for nonlinear systems Astolfi et al. (2008), Krstic et al. (2006). On the other hand fractional order operators (FOO) theory burst into the calculus discipline in the late nineties providing a new viewpoint in the areas of system identification and control although the concept has its origins in a letter to L'Hopital written by Leibniz in 1695 it was only in the nineties when this idea was seriously studied in the control area from theoretical as well as from application viewpoints Kilbas et al. (2006), Baleanu et al. (2012).

In this paper we study the adaptive control problem for integer order (IO) plants using fractional order (FO) controllers where the derivation order of each adaptive law is allowed to be different to each other. We start summarizing some new results recently published in the control literature on the case of multi-order systems (MOS) (understood as systems described by FO differential equations (FODE) with different derivation orders for each variable) and having multiple time-varying delays (MTVD), as well as some previous results for fractional order systems (FOS) under the perspective of the so called Error Models (EM) Narendra et al. (1989). The contributions of this paper are organized as follows. In Section 2, we present stability results for MOS with MTVD using Lyapunov-like functions (VLLF) to establish asymptotic stability for mixed FO systems. Next we summarize some stability results for mixed order systems from an Error Model perspective. Finally, we provide illustrative examples and simulations of MOS in the adaptive control area.

2. BASIC CONCEPTS AND DEFINITIONS

This section presents some basic concepts and definitions of FO calculus as well as some proprieties of FOO that will be used along the paper. $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the set of reals and nonnegative reals numbers, respectively. For $x \in \mathbb{R}^n$, we use the norm $\|x\|_1 := \sum_{i=1}^{n} \|x_i\|$ and $\|\cdot\|$ is the Euclidean norm . $\mathcal{C}\big([-\tau, 0], \mathbb{R}^n\big)$ denotes the set of continuous real-valued functions on $[-\tau, 0]$ endowed with the infinite norm $\|\phi\|_\infty = \sup_{t\in[-\tau,0]} \|\phi(t)\|$. For $\alpha \in \mathbb{R}$, $[\alpha]$ denotes the integer part of α. For $x \in \mathbb{R}^n$, $x \succeq 0$ means $x_i \geq 0$ for any $i \in \{1, \ldots, n\}$. For $A \in \mathbb{R}^{n\times m}$, A^T denotes its transpose. $A \in \mathbb{R}^{n\times n}$ is Metzler if the off-diagonal elements are nonnegative. $A \in \mathbb{R}^{n\times n}$ is Hurwitz if all its eigenvalues have negative real part. $A \in \mathbb{R}^{n\times m}$ is nonnegative if all its entries are nonnegative. I_n denotes

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the identity matrix in $\mathbb{R}^{n\times n}$. A class $\mathcal{K}$ function is an strictly increasing continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\gamma(0) = 0$. The Riemann-Liouville fractional integral (RLFI) is one of the main concepts of fractional calculus (FC). For a measurable function $f : [a, b] \to \mathbb{R}$ s.t. $\int_a^b |f(s)|ds < \infty$, the RLFI of order $\alpha \in \mathbb{R}$ is given in Definition 1.

Definition 1. Kilbas et al. (2006). The RLFI of order $\alpha \in \mathbb{R}_{\geq 0}$ of a function $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is defined as

$$I_{t_0}^{\alpha} f(t) = 1/\Gamma(\alpha) \int_{t_0}^{t} f(\tau)/(t-\tau)^{1-\alpha} d\tau, \quad t > t_0, \tag{1}$$

where $\Gamma(\alpha)$ is the Gamma function Kilbas et al. (2006).

Although there exist several definitions for fractional derivatives (FD) of order $\alpha > 0$ of a function, in this study we will use the Caputo definition (CFD), which is most frequently used in engineering problems given as follows:

Definition 2. Kilbas et al. (2006). Let $\alpha \geq 0$ and $n = [\alpha]$. The CFD of order $\alpha \in \mathbb{R}_{\geq 0}$ of a function $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is

$${}^{C}D_{t_0}^{\alpha} f(t) = 1/\Gamma(n-\alpha) \int_{t_0}^{t} f^{(n)}(\tau)/(t-\tau)^{\alpha-n+1} d\tau, \tag{2}$$

$f^{(n)} \in L_1[t_0, t]$, space of Lebesque integrable functions.

The following lemma, reported in Duarte et al. (2015), will be useful in proving boundedness of FODE.

Lemma 1. (Duarte et al. (2015)). Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable functions. Then, for all $t > t_0$, the following relationship holds

$$\frac{1}{2} {}^{C}D_{t_0}^{\alpha} \left\{ x^T\, P\, x \right\}(t) \leq x^T(t)\, P\, {}^{C}D_{t_0}^{\alpha} x(t), \tag{3}$$

where $\alpha \in (0, 1]$ and $P \in \mathbb{R}^{n\times n}$ is a constant, square, symmetric and positive definite matrix.

In the particular case of scalar functions ($x(t) \in \mathbb{R}$) equation (3) takes the form given in Aguila et al. (2014)

$$\frac{1}{2} {}^{C}D_{t_0}^{\alpha} x^2(t) \leq x(t) {}^{C}D_{t_0}^{\alpha} x(t). \tag{4}$$

The following Lemma, proposed in Aguila et al. (2016), is used in establishing convergence of certain type of FODE

Lemma 2. (Aguila et al. (2016)). Let $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ be a bounded nonnegative function. If there exists some $\alpha \in (0, 1]$ such that

$$\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{x(\tau)}{(t-\tau)^{1-\alpha}} d\tau < M, \forall t \geq t_0, with M \in (0, \infty), \tag{5}$$

then

$$\lim_{t\to\infty} [t^{\alpha-\varepsilon}(\int_{t_0}^{t} x(\tau)d\tau)/t] = 0, \qquad \forall \varepsilon > 0 \tag{6}$$

3. MAIN PREVIOUS RESULTS

3.1 Analysis of multi order systems (MOS) with multiple time-varying delays (MTVD)

In Gallegos et al. (2020)we studied positive solutions for the following class of systems

$$D_{0^+}^{\alpha} x(t) = Ax(t) + \sum_{j=1}^{l} A_{d_j} x(t - \tau_j(t)) + Bu(t) \tag{7}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $D_{0^+}^{\alpha} x = (D_{0^+}^{\alpha_1} x_1, \ldots, D_{0^+}^{\alpha_n} x_n)^T$ for $0 < \alpha_i \leq 1$ and all $t > 0$. A, A_{d_j}, B are matrices of suited dimensions. This notation includes delayed inputs such as $\bar{u}(t-\tau(t))$ by redefining $u(t) = \bar{u}(t-\tau(t))$. The only requirement on the delay functions is that $0 \leq \tau_j(t) \leq \tau_j$ for all $t \geq -\tau$ where $\tau := \max_j \tau_j < \infty$. System (7) with initial function $\phi(t)$ is called *positive* (*negative*, respectively) if $\phi(s) \succeq 0$, $u(t) \succeq 0$ ($\phi(s) \preceq 0$, $u(t) \preceq 0$) for all $s \in [-\tau, 0]$, $t \geq 0$, implies $x(t) \succeq 0$ ($x(t) \preceq 0$) for all $t \geq 0$ **?**. We will impose the following assumption.

Assumption 1. For system (7), A is Metzler and A_{d_j}, B are nonnegative for $j = 1, \ldots, l$.

Boundedness and convergence for system (7) is as follows.

Theorem 1. Gallegos et al. (2020). Consider system (7) such that Assumption 1 is satisfied, $\phi \succeq 0$ and $A + \sum_{i=1}^{l} A_{d_i}$ is Hurwitz.

- If $u \equiv 0$, then the trivial solution is asymptotically stable.
- If u is bounded, x is also bounded and if u is bounded and converges to zero, then x also converges to zero.

Let us consider now the following MOS with MTVD defined as:

$$D_{0^+}^{\beta} x(t) = f(x, x(t-\tau_1(t)), \ldots, x(t-\tau_l(t)), t) \tag{8}$$

$x(t) \in \mathbb{R}^n$ for all $t \geq 0$, $D_{0^+}^{\beta} x = (D_{0^+}^{\beta_1} x_1(t), \ldots, D_{0^+}^{\beta_n} x_n(t))^T$ for $0 < \beta_i \leq 1$ and $i = 1, \ldots, n$. f is a smooth enough function guaranteeing continuous solutions for any $t \in [0, \infty)$ and such that $f(0, 0, \ldots, 0, t) = 0$ for any $t \in \mathbb{R}$. The delayed functions satisfy $0 \leq \tau_j(t) \leq \tau_j$ for all $t \geq 0$, where $\tau := \max \tau_j < \infty$. The continuous initial function ϕ is specified on $[-\tau, 0]$. For a given continuous initial function ϕ, the corresponding solution of (8) is denoted by $x(t; \phi)$, or simply $x(t)$ for any $t \geq -\tau$, where $x(t) = \phi(t)$ for any $t \in [-\tau, 0]$. with ϕ now not necessarily positive.

Definition 3. The trivial solution $x \equiv 0$ of (8) is said to be *stable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\phi \in \mathcal{C}\big([-\tau, 0], \mathbb{R}^n\big)$, $\|\phi\|_\infty < \delta$ implies $\|x(t; \phi)\| < \epsilon$ for any $t > 0$; and *asymptotically stable* if, in addition, $\lim_{t\to 0} \|x(t; \phi)\| = 0$ for any ϕ such that $\|\phi\|_\infty < \bar{\delta}$ for some $\bar{\delta} > 0$.

The stability of system (8) is obtained as follows.

Theorem 2. [Gallegos et al. (2020)]. Consider that for system (8) there exists a vector function $V : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

(i) $\gamma_1(\|x\|) \leq \|V(x)\|_1 \leq \gamma_2(\|x\|)$ for some class-$\mathcal{K}$ functions γ_1 and γ_2.

(ii)The function $V(t) := V(x(t))$, for any solution $x(\cdot)$ of (8), is such that $V(t) \succeq 0$ for any $t \geq -\tau$ and there exist a Metzler matrix A, nonnegative matrices A_{d_i} for $i = 1, \ldots, l$ with $A + \sum_{i=1}^{l} A_{d_i}$ a Hurwitz matrix such that $\forall t \geq 0$

$$D_{0^+}^{\alpha} V(t) \preceq AV(t) + \sum_{j=1}^{l} A_{d_j} V(t-\tau_j(t)) + C(t), \quad (9)$$

where $C(t) \succeq 0$ and $D^{\alpha}V$ is the vector of components $D^{\alpha_i}V_i$ with $\alpha_i \in (0,1]$ for $i = 1,\ldots,m$. Then the trivial solution of (8) is asymptotically stable when $C \equiv 0$. In addition, the solutions of (8) remain bounded or converge to zero if C does it.

3.2 Analysis of certain classes of FODE

In what follows, we establish the bounded and stability of three important kind of FODE.

a) FODE of Class 1: One parametrization appearing very often in several important adaptive control problems is

$$\begin{aligned} &y(t) = k_p \eta^T(t)\, u(t) + \xi(t)\, u_1(t) \\ &{}^{C}D_{t_0}^{\alpha} \eta(t) = -\gamma sgn(k_p)\, y(t)\, u(t) \quad \alpha \in (0,1] \\ &{}^{C}D_{t_0}^{\alpha} \xi(t) = -\gamma_1 y(t)\, u_1(t) \qquad \alpha \in (0,1] \end{aligned} \quad (10)$$

where $k_p \in \mathbb{R}$ is an unknown constant with known sign, $\gamma, \gamma_1 \in \mathbb{R}^+$, are positive known constants, $y(t) : \mathbb{R}^+ \to \mathbb{R}$ is measurable, $u(t) : \mathbb{R}^+ \to \mathbb{R}^n$ and and $u_1(t) : \mathbb{R}^+ \to \mathbb{R}$ are assumed to be known and bounded, $\eta(t) : \mathbb{R}^+ \to \mathbb{R}^n$ and $\xi(t) : \mathbb{R}^+ \to \mathbb{R}$ are unknown signals to be adjusted. Boundedness of $\eta(t), \xi(t), y(t)$ and convergence to zero of the mean value of $\|y(t)\|^2$ are proved in Lemma 5 of Aguila et al. (2016).

b) FODE of Class 2: In this case the system is defined by

$$\begin{aligned} &{}^{C}D_{t_0}^{\alpha} y(t) = Ay(t) + k_p\, b\, \eta^T(t) u(t) \\ &{}^{C}D_{t_0}^{\alpha} \eta(t) = -\gamma sgn(k_p y^T(t) P\, b\, u(t) \quad \alpha \in (0,1] \end{aligned} \quad (11)$$

where $A \in \mathbb{R}^{n\times n}$ is an asymptotically stable matrix, $b \in \mathbb{R}^n$, $\eta(t) : \mathbb{R}^+ \to \mathbb{R}^m$, $y(t) : \mathbb{R}^+ \to \mathbb{R}^n$, $u(t) : \mathbb{R}^+ \to \mathbb{R}^m$ assumed to be known and bounded, $P \in \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix satisfying the equation $A^TP + PA = -Q < 0$ (with $Q \in \mathbb{R}^{n\times n}$ positive definite), $k_p \in \mathbb{R}$ is an unknown constant, whose sign is known, and $\gamma \in \mathbb{R}^+$. Boundedness of $\eta(t), y(t)$,and convergence to zero of the mean value of $\|y(t)\|^2$ was studied in Lemma 6 from Aguila et al. (2016).

FODE of Class 3: The structure of FODE of Class 3 is

$$\begin{aligned} &{}^{C}D_{t_0}^{\alpha} y(t) = Ay(t) + b\eta^T(t)\, u(t) \\ &y_1(t) = k_p\, c^T y(t) \\ &{}^{C}D_{t_0}^{\alpha} \eta(t) = -\gamma sgn(k_p)\, y_1(t)\, u(t), \quad \alpha \in (0,1] \end{aligned} \quad (12)$$

where $A \in \mathbb{R}^{n\times n}$ is an asymptotically stable matrix, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $\eta(t) : \mathbb{R}^+ \to \mathbb{R}^m$, $y(t) : \mathbb{R}^+ \to \mathbb{R}^n$ is measurable, $u(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is assumed to be known and bounded, $y_1(t) : \mathbb{R}^+ \to \mathbb{R}$, $k_p \in \mathbb{R}$ is an unknown constant with known sign and $\gamma \in \mathbb{R}^+$. Besides, positive definite matrices $P = P^T \in \mathbb{R}^{n\times n}$ and $Q = Q^T \in \mathbb{R}^{n\times n}$ exist such that

$$A^TP + PA = -Q \qquad \text{and} \qquad Pb = c \quad (13)$$

Boundedness of $\eta(t), y(t)$, as well as the convergence to zero of the mean value of $\|y(t)\|^2$ was analyzed in Lemma 7 from Aguila et al. (2016).

3.3 Analysis of fractional order Error Models 2 and 3

In what follows, we establish the boundedness and stability of two type of fractional order error models (FOEM) important in the adaptive control area Aguila et al. (2019).

a) Analysis of FOEM 2: This is given by

$${}^{C}D^{\beta} e(t) = Ae(t) + k_p\, b\, \phi^T(t)\, \omega(t), \quad e(t_0) = e_0, \quad (14)$$

where $A \in \mathbb{R}^{n\times n}$ is a stable matrix i.e. there exist positive definite symmetric matrices $P, Q \in \mathbb{R}^{n\times n}$ such that $A^TP + PA = -Q$. $e(t) : \mathbb{R}^+ \to \mathbb{R}^n$ is the output error and it is assumed that the whole vector $e(t) \in \mathbb{R}^n$ is accessible. k_p is an unknown constant whose sign is assumed to be known, $b \in \mathbb{R}^n, \phi(t)\, \theta(t) - \theta^*(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is the parameter error, $\omega(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is a vector of known signals and $\beta \in (0,1]$. It was proved in Aguila et al. (2019) that adaptive laws defined in (15) can be used to estimate the unknown parameters $\theta(t)$ keeping bounded all the signals of the resultant adaptive system and the mean value of $\|e(t)\|^2$ converges asymptotically to zero.

$$\begin{aligned} &{}^{C}D^{\beta}\phi(t) = {}^{C}D^{\beta}\theta(t) = -\gamma\, \text{sgn}(k_p) e^T(t) P\, b\, \omega(t), \\ &\phi(t_0) = \phi_0. \end{aligned} \quad (15)$$

$\gamma \in \mathbb{R}^+$ is the adaptive gain. For the case when the order of the adaptive laws are $\alpha \neq \beta$

$$\begin{aligned} &{}^{C}D^{\alpha}\phi(t) = {}^{C}D^{\alpha}\theta(t) = -\gamma\, \text{sgn}(k_p) e^T(t) Pb\omega(t), \\ &\phi(t_0) = \phi_0, \end{aligned} \quad (16)$$

with $\alpha < \beta$, it can be concluded that control error $e(t)$ and parameter error $\phi(t)$ remain bounded. Moreover, if $\omega(t)$ is bounded, then ${}^{C}D^{\beta}e(t)$ and ${}^{C}D^{\alpha}\phi(t)$ remain bounded. Finally, it can also be concluded that the mean value of the squared norm of the output error is $o(t^{\varepsilon-\alpha})$, $\forall \varepsilon > 0$ (Aguila et al. (2019)).

b) Analysis of FOEM 3: This FOEM3 arises when the vector $e(t) : \mathbb{R}^+ \to \mathbb{R}^n$ is not accessible and only one of its variables, $e_1(t) \in \mathbb{R}$, is measurable. It has the form

$$\begin{aligned} &{}^{C}D^{\beta} e(t) = Ae(t) + b\, \phi^T(t)\, \omega(t), \quad && e(t_0) = e_0 \\ &e_1(t) = k_p\, h^T e(t), && e_1(t_0) = e_{1_0}, \end{aligned} \quad (17)$$

where $A \in \mathbb{R}^{n\times n}$ is stable and the triplet $\{A, b, h\}$ satisfies the Kalman-Yakubovich-Popov lemma (Narendra et al. (1989)). k_p is unknown with known sign, $b, h \in \mathbb{R}^n$, $\phi(t) = \theta(t) - \theta^*(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is the parameter error, $\omega(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is a vector of available signals and $\beta \in (0,1]$. Assuming the general case when the orders of the adaptive laws is α,

$$\begin{aligned} &{}^{C}D^{\alpha}\phi(t) = {}^{C}D^{\alpha}\theta(t) = -\gamma\, \text{sgn}(k_p)\, e_1(t)\, \omega(t), \\ &\phi(t_0) = \phi_0, \end{aligned} \quad (18)$$

with $\alpha \in (0,1]$, it was proved in Aguila et al. (2019) that assuming $e(t), \phi(t)$ are differentiable and uniformly continuous functions, then it holds that the parameter error $\phi(t)$, the state error $e(t)$ and the output error $e_1(t)$ remain bounded. If moreover, $\omega(t)$ is bounded, then ${}^{C}D^{\alpha}\phi(t)$ and ${}^{C}D^{\beta}e(t)$ remain bounded and the mean value of the squared norm of $e(t)$ is $o(t^{\varepsilon-\alpha})$, $\forall \varepsilon > 0$.

4. APPLICATIONS AND SIMULATION EXAMPLES

This Section presents the analysis and simulation of some common applications in FO Model Reference Adaptive Control (FOMRAC).

4.1 First order plants with relative degree one. (Case 1)

Let us consider the integer first order plant (either stable or unstable) to be controlled and the model reference described by the following IODE with relative degree one

$$\dot{y}_p(t) + a_p^0 y_p(t) = b_p^0 u(t); \quad y_p(0) = y_{po}, \\ \dot{y}_m(t) + a_m^0 y_m(t) = b_m^0 r(t); \; y_m(0) = y_{m0}, \tag{19}$$

with $y_p(t), y_m(t), u(t), r(t) \in \mathbb{R}$. $r(t)$ is an arbitrary continuous-time known reference signal. The control goal is that the output of the plant asymptotically follows the output of the reference model, that is, the control error $e(t) = y_p(t) - y_m(t) = 0$ satisfies $\lim_{t\to\infty} e(t) = 0$. The controller used in this case has the same form as in the IO case (Narendra et al. (1989))

$$u(t) = k(t)r(t) + \theta(t)y_p(t), \tag{20}$$

where $k(t), \theta(t) \in \mathbb{R}$ are two adjustable parameters ruled by the adaptive laws defined as

$$^C D^{\alpha_1}\theta(t) = -\gamma_1 \mathrm{sgn}(k_p)e(t)y_p(t); \; \theta(t_0) = \theta_0, \\ ^C D^{\alpha_2}k(t) = -\gamma_2 \mathrm{sgn}(k_p)e(t)r(t); \quad k(t_0) = k_0. \tag{21}$$

$k_p = b_p^0$ and unknown. γ_1 and γ_2 are arbitrary, constant and positive adaptive gains. The solution for the particular case when $\alpha_1 = \alpha_2 = 1$ (the IO case) is very well known and be found in Narendra et al. (1989). From equations (19) to (21) we obtain the following set of equation describing the overall adaptive system with mixed order derivatives

$$\dot{e}(t) = -\lambda e(t) + k_p\phi_1(t)y_p(t) + k_p\phi_2(t)r(t); \; e(0) = e_0, \\ ^C D^{\alpha_1}\phi_1(t) = -\gamma_1 \mathrm{sgn}(k_p)e(t)y_p(t); \qquad \phi_1(t_0) = \phi_{10}, \\ ^C D^{\alpha_2}\phi_2(t) = -\gamma_2 \mathrm{sgn}(k_p)e(t)r(t); \qquad \phi_2(t_0) = \phi_{20}. \tag{22}$$

where $\lambda = a_m^0 > 0$, $\phi_1(t) = \theta(t) - \theta^*$ and $\phi_2(t) = k(t) - k^*$. θ^* and k^* are those constant, ideal (but unknown) values for $\theta(t)$ and $k(t)$, respectively, in (20) such that applying this particular control into (19) the resultant system (plant plus ideal controller) exactly matches the reference model. In this particular case $\theta^* = (a_m^0 - a_p^0)/b_p^0$ and $k^* = b_m^0/b_p^0$ Narendra et al. (1989). Now we can state the following Lemma.

Lemma 1. Let us consider the error equation together with the adaptive laws given by (22), then all the the signals of the adaptive system remain bounded and the control error asymptotically converges to zero.

Proof 1. Le us consider the following vector Lyapunov functions (VLP) for system (22); $V_1 = 1/2e^2$, $V_2 = 1/2\phi_1^2$ and $V_3 = 1/2\phi_2^2$. Taking the first, α_1 and α_2 derivatives, respectively, of V_1, V_2 and V_3 Lyapunov functions and using relationships (22), we get $\dot{V}_1 = -\lambda e^2 + k_p e\phi_1 y_p + k_p e\phi_2 r$, $^C D^{\alpha_1} V_2 \le \phi_1(t)^C D^{\alpha_1}\phi_1(t) \le -\gamma_1 \mathrm{sgn}(k_p)\phi_1(t)e(t)y_p(t)$ and $^C D^{\alpha_2} V_3 \le \phi_2(t)^C D^{\alpha_2}\phi_2 \le -\gamma_2 \mathrm{sgn}(k_p)\phi_2(t)e(t)r(t)$. Defining $\psi_1 = k_p e y_p \phi_1$ and $\psi_2 = k_p e r \phi_2$ we can write $|\psi_1| \le CV_1V_2$ and $|\psi_2| \le CV_1V_3$. Considering that $|\phi_1 e y_p| \le C_1V_1V_2$ and $|\phi_2 e r| \le C_2V_1V_3$ we obtain the following inequality (Gallegos et al. (2020)):

$$\begin{bmatrix} \dot{V}_1 \\ ^C D^{\alpha_1} V_2 \\ ^C D^{\alpha_2} V_3 \end{bmatrix} \le \begin{bmatrix} -\lambda & C_1 & C_2 \\ C_1 & -1 & 0 \\ C_2 & 0 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 0 \\ F_1 \\ F_2 \end{bmatrix} \tag{23}$$

where F_1 and F_2 are meant to take into consideration in (22) parameter variations and/or external perturbations decaying to zero as t goes to ∞. The matrix on the right side of equation (23) is Metzler and according to Xu et al. (2016) there exist a Matrix A such that Λ is Hurwitz. Then using Theorem 2 from Gallegos et al. (2020) and since F_1 and F_2 go to zero, the convergence to zero of V_1, V_2 and V_3 follows.

Table 1. FOMRAC implementation for Case 1.

Plant	$y_p(t) = [1/(s-2)]u(t)$;	$y_p(0) = 2$
Reference model	$y_m(t) = [2/(s+6)]r(t)$;	$y_m(0) = 0$
Initial conditions	$\theta(0) = -3$, $k(0) = 0$;	$\gamma_1 \ldots \gamma_4 \in [0.1, 10.0]$
Simulation time	$T = 10s$.	

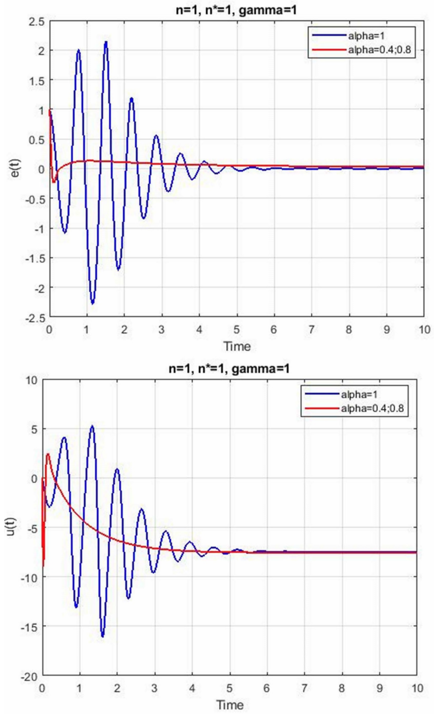

Fig. 1. Control error for first order n*=1 case, using different orders α and reference signal $r(t) = 1$.

Simulation results for this Case 1 are shown in Figure 1 using the numerical values shown in Table 1, unity adaptive gains ($\gamma_i = 1$) and particular values for the fractional orders ($\alpha_i = 0.4; 0.8$).

4.2 Second order plant with relative degree one. (Case 2)

Let us consider now the IO second order plant of relative degree one (either stable or unstable) and a given reference model defined as follows:

$$\ddot{y}_p(t) + a_p^1\dot{y}_p(t) + a_p^0 y_p(t) = b_p^1\dot{u}(t) + b_p^0 u(t); \\ y_p(0), \dot{y}_p(0), u(0), \\ \ddot{y}_m(t) + a_m^1\dot{y}_m(t) + a_m^0 y_m(t) = b_m^1\dot{r}(t) + b_m^0 r(t); \\ y_m(0), \dot{y}_m(0), r(0), \tag{24}$$

where y_p corresponds to the plant output and u is the control signal to be designed. The plant parameters are unknown but the sign of the high frequency gain is assumed to be known. On the other hand the parameters and signals of the reference model are completely known. The control goal is that the control error, defined as $e(t) = y_p(t) - y_m(t)$, satisfies $\lim_{t\to\infty} e(t) = 0$. Following the same ideas as in IOMRAC the controller in this case has the form Narendra et al. (1989)

$$u(t) = \theta^T(t)\omega(t), \\ \theta^T(t) = [k(t)\; \theta_1(t)\; \theta_0(t)\; \theta_2(t)], \\ \omega^T(t) = [r(t)\; \omega_1(t)\; y_p(t)\; \omega_2(t)], \\ \dot{\omega}_1(t) = -\lambda\omega_1(t) + lu(t), \\ \dot{\omega}_2(t) = -\lambda\omega_2(t) + ly_p(t), \tag{25}$$

$\lambda > 0$ is any real constant, $l \neq 0$ and $k(t), \theta_1(t), \theta_0(t), \theta_2(t)$ are the adjustable controller parameters ruled by

$$\begin{aligned} {}^C D^{\alpha_1} k(t) &= -\gamma_1 e(t) r(t); \quad k(0) = k_0, \\ {}^C D^{\alpha_2} \theta_1(t) &= -\gamma_2 e(t) \omega_1(t); \; \theta_1(0) = \theta_{10}, \\ {}^C D^{\alpha_3} \theta_0(t) &= -\gamma_3 e(t) y_p(t); \; \theta_0(0) = \theta_{00}, \\ {}^C D^{\alpha_4} \theta_2(t) &= -\gamma_4 e(t) \omega_2(t); \; \theta_2(0) = \theta_{20}. \end{aligned} \tag{26}$$

with γ_1 to γ_4 positive adaptive gains. From equations (24), and (26) we obtain the following set of equation describing the overall adaptive system with mixed order derivatives

$$\begin{aligned} \dot{e}(t) &= -\lambda e(t) + k_p \phi^T(t) \omega(t); && e(0) = e_0, \\ {}^C D^{\alpha_1} \phi_1(t) &= -\gamma_1 \mathrm{sgn}(k_p) e(t) r(t); && \phi_1(t_0) = \phi_{10}, \\ {}^C D^{\alpha_2} \phi_2(t) &= -\gamma_2 \mathrm{sgn}(k_p) e(t); && \omega_1(t); \phi_2(t_0) = \phi_{20}, \\ {}^C D^{\alpha_3} \phi_3(t) &= -\gamma_3 \mathrm{sgn}(k_p) e(t) y_p(t); && \phi_3(t_0) = \phi_{30}, \\ {}^C D^{\alpha_4} \phi_4(t) &= -\gamma_4 \mathrm{sgn}(k_p) e(t) \omega_2(t); && \phi_4(t_0) = \phi_{40}, \end{aligned} \tag{27}$$

where $\phi(t) = \theta(t) - \theta^* \in \mathbb{R}^4$ and $\theta^* \in \mathbb{R}^4$ are those constant, ideal (but unknown) values for $\theta(t)$ in (25) (defining the ideal controller) such that applying this particular control into (24) the resultant system (plant plus ideal controller) exactly matches the reference model. Now we can state the following Lemma.

Lemma 2. Let us consider the system (24) together with the adaptive laws given by (26), then all the the signals of the adaptive system remain bounded and the control error asymptotically converges to zero.

The proof follows along the same line as in *Lemma 1.*

Proof 2. Le us consider the following vector Lyapunov functions (VLP) for system (27); $V_1 = 1/2e^2$, $V_2 = 1/2\phi_1^2$, $V_3 = 1/2\phi_2^2$, $V_4 = 1/2\phi_3^2$ and $V_5 = 1/2\phi_4^2$.Taking the first, α_1, α_2, α_3 and α_4 derivatives, respectively, of V_1, V_2, V_3, V_4 and V_5 Lyapunov functions and using relationships (27), we get $\dot{V}_1 = -\lambda e^2 + e\phi_1 r + e\phi_2\omega_1 + e\phi_3 y_p + e\phi_4\omega_2$; ${}^C D^{\alpha_1} V_2 \leq \phi_1(t) {}^C D^{\alpha_1} \phi_1(t) \leq -\gamma_1 \mathrm{sgn}(k_p) e(t) r(t)$; ${}^C D^{\alpha_2} V_3 \leq \phi_2(t) {}^C D^{\alpha_2} \phi_2(t) \leq -\gamma_2 \mathrm{sgn}(k_p) e(t) \omega_1)$; ${}^C D^{\alpha_3} V_4 \leq \phi_3(t) {}^C D^{\alpha_3} \phi_3(t) \leq -\gamma_3 \mathrm{sgn}(k_p) e(t) y_p(t)$ and ${}^C D^{\alpha_4} V_5 \leq {}^C D^{\alpha_4} \phi_4 \leq -\gamma_4 \mathrm{sgn}(k_p) e(t) \omega_2(t)$. Considering that $|\phi_1 e r| \leq C_1 V_1 V_2$, $|\phi_2 \omega_1| \leq C_2 V_1 V_3$, $|\phi_3 e y_p| \leq C_3 V_1 V_4$, and $|\phi_4 e \omega_2| \leq C_4 V_1 V_5$ we obtain the following relationship

$$\begin{bmatrix} \dot{V}_1 \\ {}^C D^{\alpha_1} V_2 \\ {}^C D^{\alpha_2} V_3 \\ {}^C D^{\alpha_3} V_4 \\ {}^C D^{\alpha_4} V_5 \end{bmatrix} \leq \begin{bmatrix} -\lambda & C_1 & C_2 & C_3 & C_4 \\ C_1 & -1 & 0 & 0 & 0 \\ C_2 & 0 & -1 & 0 & 0 \\ C_3 & 0 & 0 & -1 & 0 \\ C_4 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} + \begin{bmatrix} 0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}. \tag{28}$$

where F_1 to F_2 are functions decaying to zero as t goes to ∞, meant to take into consideration parameter variations and/or external perturbation in (27). The matrix on the right side of equation (28) is Metzler and according to Xu et al. (2016) there exist a Matrix A such that Λ is Hurwitz. Then using Theorem 2 from Gallegos et al. (2020) and since F_1 to F_5 go to zero, the convergence to zero of V_1, to V_5 follows.

The simulation results for Case 2 are shown in Figure 2 corresponding to the numerical values shown in Table 2, unity adaptive gains γ_i and particular values for the fractional orders ($\alpha_i = 0.9; 0.8; 0.3; 0.4$). From this and other simulations performed, but not shown here for the sake of space, it was observed that the control error

Table 2. Numerical values for Case 2.

Plant	$y_p(t) = [(s+2)/(s^2 - s - 2)]u(t)$
	$y_p(0) = 0, \dot{y}_p(0) = 0, u(0) = 0$
Reference	$y_m(t) = [(s+4)/(s^2+5s+6)]r(t)$
model	$y_m(0) = 0, \dot{y}_m(0) = 0, r(0) = 0$
Control	$u(t) = k(t)r(t) + \theta_1(t)\omega_1(t) +$
signal	$+\theta_0(t)y_p(t) + \theta_2(t)\omega_2(t)$
Auxiliary	$\omega_1(t) = [1/(s+4)]u(t)$
filters	$\omega_2(t) = [1/(s+4)]\ y_p(t)$
Adaptive	${}^C D^{\alpha_1} k(t) = -\gamma_1 e(t) r(t)$
laws	$k(0) = 0, \gamma_1 > 0$
	${}^C D^{\alpha_2} \theta_1(t) = -\gamma_2 e(t) \omega_1(t)$
	$\theta_1(0) = 1, \gamma_2 > 0$
	${}^C D^{\alpha_3} \theta_0(t) = -\gamma_1 e(t) y_p(t)$
	$\theta_0(0) = -3, \gamma_2 > 0$
	${}^C D^{\alpha_4} \theta_2(t) = -\gamma_4 e(t) \omega_2(t)$
	$\theta_2(0) = 10, \gamma_4 > 0$
Simulation	
time	$T = 5s$

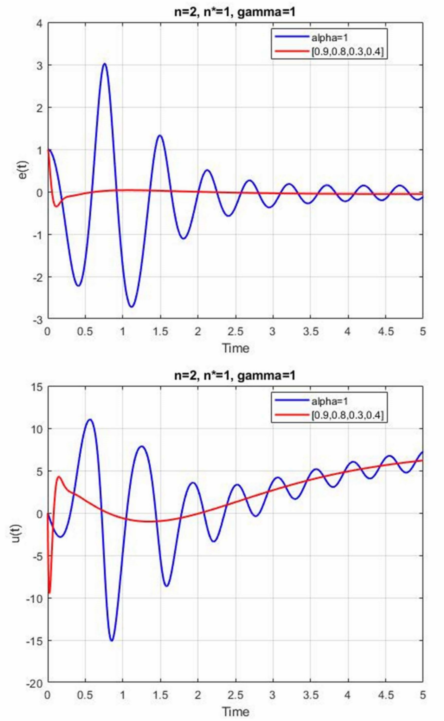

Fig. 2. Control error for second order n*=1 case, using different orders α and reference signal $r(t) = 1$.

$e(t)$ converges to zero for every $\alpha_i \in (0, 1)$ and any positive value of the adaptive gains γ_i. However the speed of convergence as well as the transient behavior depend on the particular values chosen.These facts also apply to the behavior of the control input $u(t)$ (control effort). An analysis of the these aspect was done through performances indices sucha as of the integral of the squared error ISE, the integral of the squared input ISI and the sum of both indexes $J = ISE + ISI$.

4.3 Second order plants with relative degree two.

Let us consider a general IO second order plant with relative degree two and the corresponding reference model of the same characteristics, defined as follows:

$$\begin{aligned} \ddot{y}_p(t) + a_p^1 + \dot{y}_p(t) + a_p^0 y_p(t) &= b_p^0 u(t); y_p(0); \dot{y}_p(0) \\ \ddot{y}_m(t) + a_m^1 \dot{y}_m(t) + a_m^0 y_m(t) &= b_m^0 r(t); y_m(0); \dot{y}_m(0) \end{aligned} \tag{29}$$

The controller in this case has the form Narendra et al. (1989)

$$\begin{aligned} &u(t) = \theta^T(t)\omega(t) - \gamma sgn(k_p)e(t)\bar{\omega}^T(t)\bar{\omega}(t);\\ &\theta^T(t) = [k(t)\ \theta_1(t)\ \theta_0(t)\ \theta_2(t)];\\ &\omega^T(t) = [r(t)\ \omega_1(t)\ y_p(t)\ \omega_2(t)];\\ &\bar{\omega}^T(t) = 1/(s+a)[r(t)\ \omega_1(t)\ y_p(t)\ \omega_2(t)];\\ &\dot{\omega}_1(t) = -\lambda\omega_1(t) + lu(t);\\ &\dot{\omega}_2(t) = -\lambda\omega_2(t) + ly_p(t); \end{aligned} \tag{30}$$

with $\lambda > 0$ and $a > 0$. The adaptive laws have the following form

$$\begin{aligned} &{}^C D^{\alpha_1} k(t) = -\gamma_1 e(t)\bar{r}(t); && k(0) = 0, \gamma_1 > 0,\\ &{}^C D^{\alpha_2} \theta_1(t) = -\gamma_2 e(t)\bar{\omega}_1(t); && \theta_1(0) = 1, \gamma_2 > 0,\\ &{}^C D^{\alpha_3} \theta_0(t) = -\gamma_3 e(t)\bar{y}_p(t); && \theta_0(0) = -3, \gamma_3 > 0,\\ &{}^C D^{\alpha_4} \theta_2(t) = -\gamma_4 e(t)\bar{\omega}_2(t) : && \theta_2(0) = 10, \gamma_4 > 0. \end{aligned} \tag{31}$$

From equations (24), and (26) we obtain the following set of equation describing the overall adaptive system with mixed order derivatives

$$\begin{aligned} &\dot{e}(t) = -\lambda e(t) + k_p\phi^T(t)\omega(t); \quad e(t_0) = e_0,\\ &{}^C D^{\alpha_1}\phi_1(t) = -\gamma_1 \mathrm{sgn}(k_p)e(t)r(t); && \phi_1(t_0) = \phi_{10},\\ &{}^C D^{\alpha_2}\phi_2(t) = -\gamma_2 \mathrm{sgn}(k_p)e(t)\omega_1(t); && \phi_2(t_0) = \phi_{20},\\ &{}^C D^{\alpha_3}\phi_3(t) = -\gamma_1 \mathrm{sgn}(k_p)e(t)y_p(t); && \phi_3(t_0) = \phi_{30},\\ &{}^C D^{\alpha_4}\phi_4(t) = -\gamma_1 \mathrm{sgn}(k_p)e(t)\omega_2(t); && \phi_4(t_0) = \phi_{40}, \end{aligned} \tag{32}$$

where $\phi(t) = \theta(t) - \theta^* \in \mathbb{R}^4$ and $\theta^* \in \mathbb{R}^4$ are those constant ideal (but unknown) values for $\theta(t)$ in (20) such that applying this particular control into (24) the resultant system (plant plus ideal controller) exactly matches the reference model. Now we can state the following Lemma.

Lemma 3. Let us consider the error equation(29) together with the adaptive laws given by (31), then all the the signals of the adaptive system remain bounded and the control error asymptotically converges to zero.

The proof follows along the same line as in *Lemma 1* and *Lemma 2* and therefore will be omitted. For the sake of space the simulation results will be also omitted

5. CONCLUSION

The additional degrees of freedom provided by including FO adaptive laws when controlling IO plants, can lead two improvements in the overall system behavior measured through ISE and ISI indexes.

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REFERENCES

Aguila-Camacho N., Gallegos J. and Duarte-Mermoud M.A. (2019). Analysis of fractional order error models in adaptive systems: Mixed order cases. *Fract. Calc. Appl. Anal.*, 22:1113-1132.

Aguila-Camacho N. and Duarte-Mermoud M.A. (2016). Boundedness of the solutions for certain classes of fractional differential equations with application to adaptive systems. *ISA Transactions*, 60:82–88.

Aguila-Camacho N., Duarte-Mermoud M.A. and Gallegos J. (2014). Lyapunov Functions for Fractional Order Systems. *Commun. Nonlinear. Sci. Numer. Simul.*, 19: 2951–2957.

Aseltine J.A., Mancini AR. and Sartur C.W. (1958). A Survey of Adaptive Control Systems. *IRE TAC*, 3:102–108.

Astolfi A., Karagiannis D. and Ortega R. (2008). Nonlinear and Adaptive Control with Applications, Springer Verlag, New Jersey.

Baleanu D., Tenreiro Machado J.A. and Luo A.C.J. (Eds) (2012). Fractional Dynamics and Control. Springer Verlag, New Jersey.

Drenick, R.F. and Shahbender, R.A. (1957). Adaptive servomechanisms. *AIEE Transactions*, 76:286–292.

Duarte-Mermoud M.A., Aguila-Camacho N., Gallegos J. and Castro-Linares R. (2015). Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems. *Commun.Nonlinear. Sci. Numer. Simul.*, 22:650–659.

Gallegos J., Aguila-Camacho N. and Duarte-Mermoud M.A. (2020). Vector Lyapunov-like functions for multi-order fractional systems with multiple time-varying delays. *Commun. Nonlinear. Sci. Numer. Simul.*, 83: In Press.

Goodwin, G., Ramadge, P. and Caines, P. (1980). Discrete-time multivariable adaptive control. *IEEE Trans. Autom. Control*, 3:449–456.

Kilbas, A., Srivastava, H. and Trujillo, J. (2006). Theory and Applications of Fractional Differential Equations. Elsevier, New York.

Krstic M., Kanellakopoulos I. and Kokotovic P. (2006). Nonlinear and Adaptive Control Design. Wiley-Interscience, New York.

Morse, A. (1980). Global stability of parameter-adaptive control systems. *IEEE Trans. Autom. Control*, 3:433–439.

Narendra K.S. and Annaswamy A.M. (1989). Stable Adaptive Systems. Prentice Hall Inc. , New Jersey.

Narendra, K. Lin, Y.H. and Valavan, L. (1980a). Stable adaptive controller design, Part II: Proof of stability. *IEEE Trans. Autom. Control*,3:440–448.

Narendra, K. and Lin, Y.H. (1980b). Stable discrete adaptive control. *IEEE Trans. Autom. Control*, 3:456–461.

Shen, J. and Lam, J. (2016). A Stability and Performance Analysis for Positive Fractional-Order Systems With Time-Varying Delays. *IEEE Trans. Autom. Control*, 61: 2676–2681.

Xu, D., Wang, X., Hong, Y., Jiang, Z. and Xu, S. (2016). Output feedback stabilization and estimation of the region of attraction for nonlinear systems: A vector control Lyapunov function perspective. *IEEE Trans. Autom. Control*, 61:4034–4040.