

An Alternative Setup to Study Stabilization of Networked Control Systems Over Correlated Fading Channels^{*}

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Abstract: This paper studies stability of control systems over a communication network with spatially correlated fading channels. We consider a multiple-input multiple-output linear time-invariant discrete-time closed-loop system in which communications are done through multiple correlated multiplicative channels. Particularly, we provide conditions that relate the stability of the above networked control system (NCS) to the stability of an auxiliary closed-loop system that replaces the multiplicative channels by additive noise channels. This provides a simple framework to analyse NCSs over correlated multiplicative channels since standard linear systems tools can be used to analyse stability as opposed to consider the multiplicative non-linearity explicitly in the analysis. Thereby, we extend existing results in the literature that contain similar conditions for uncorrelated channels only.

Keywords: Networked control systems, fading channels, spatial correlation, stability

1. INTRODUCTION

Networked control systems (NCSs) are control loops in which the communication is done via a network (Baillieul and Antsaklis, 2007), and have received much attention in the last decade given their practical benefits. However, in order to study how the network influences the closed-loop behaviour, the communication constraints that come with the network have to be taken into account. Several constraints exist and have been studied in the literature, such as random delays and packet losses, signal-to-noise (SNR) ratio constraints, quantization errors, among others (Baillieul and Antsaklis (2007); Chen and Qiu (2016); Elia (2005); Maass and Silva (2014)). Here we restrict our attention to NCS over multiple parallel and correlated multiplicative channels.

Fading channels, also known as multiplicative channels, assume there is a time-varying random multiplicative gain that affects the transmitted signal (Goldsmith, 2005), (Elia, 2005). In general, these channels are relatively hard to analyse given the non-linearity imposed by the channel noise multiplication, and thus the standard analytical tools from linear systems cannot be directly applied. However, numerous works can be found in the literature in which stability and performance of NCSs over fading channels are studied, see e.g. (Elia, 2005; Maass and Silva, 2014; Qi et al., 2017; You et al., 2015). In these works, conditions for stability and achievable performance are obtained, which

generally depend on the unstable poles, non-minimum phase zeros, and delays of the plant. However, these results only apply to uncorrelated fading channels.

In this paper, we propose a framework to study the stability of the NCSs over correlated fading channels in terms of the internal stability of an auxiliary system that replaces the multiplicative channel by an additive noise. This framework exploits a second order statistical equivalence between the original NCS and the auxiliary system. A similar statistical equivalence was used in (Maass and Silva, 2014) to obtain optimal performance results on NCSs over uncorrelated fading channels. Basically, this equivalence states that the NCS with analogous transmission over fading channels can be analysed by studying an auxiliary NCS that contains additive noise channels subject to SNR constraints. Such a framework facilitates the study of the original NCS by allowing the use of standard linear control systems.

Additive noise channels have been well studied in the literature (Li et al., 2016; Vargas et al., 2013). Given the network induced noise enters the system additively, it is possible to use a set of linear analytical tools to study its stability and performance. For instance, the authors in (Li et al., 2016; Vargas et al., 2013) characterise stability conditions under SNR constraints on the channels. These works consider that the additive noises are white and uncorrelated with other channels. In (González et al., 2018), stability of a MIMO system is studied when the noises are spatially correlated. This result was later extended

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in (Gonzalez et al., 2019) for the case of time-correlated noises, i.e. coloured noises.

In this work, we extend the statistical equivalence presented in (Maass and Silva, 2014) for the case in which the fading channels are correlated between each other. We obtain conditions such that the first and second order moments of the original NCS over fading channels are equal to the first and second order moments of the auxiliary NCS over additive channels. We use this equivalence to relate the mean square stability of the NCS over fading channels to the internal stability of the linear time-invariant NCS over additive channels. This opens the door to use results like (González et al., 2018), (Gonzalez et al., 2019) to further obtain explicit stability conditions or optimal performance for NCSs over correlated fading channels.

This paper is organised as follows. Section 2 presents the notation and preliminaries. The problem formulation is given in Section 3. A statistical analysis of NCSs over fading and AWN channels is provided in Section 4. The main results of this paper are presented in Section 5. Lastly, we draw conclusions in Section 6.

2. NOTATION AND PRELIMINARIES

We denote the expectation operator, Hadamard product, and spectral radius respectively by $\mathcal{E}\{\cdot\}$, \odot and $\rho(\cdot)$. We say that a linear operator \mathcal{A} is stable if $\rho(\mathcal{A}) < 1$. A positive (semi-definite) definite matrix A is denoted by $A > 0$ ($A \geq 0$). $\text{diag}\{\cdot\}$ denotes a diagonal matrix. Given a vector $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, a matrix $D_z = \text{diag}\{z_1, \dots, z_n\}$, and a square matrix $M \in \mathbb{R}^{n \times n}$, then the following holds (Bernstein, 2005), $D_z M D_z^T = (z z^T) \odot M$. Let x be a discrete-time stochastic process, and $x(k)$ the corresponding random variable at time instant k . For simplicity we define

$$\bar{x}(k) \triangleq x(k) - \mathcal{E}\{x(k)\}.$$

The mean, covariance matrix, second moment, and auto-covariance function of x are denoted respectively by $\mu_x(k)$, $P_x(k)$, $Q_x(k)$, $R_x(k + \tau, k)$, and are defined as follows, $\mu_x(k) \triangleq \mathcal{E}\{x(k)\}$, $Q_x(k) \triangleq \mathcal{E}\{x(k)x(k)^T\}$, $P_x(k) \triangleq \mathcal{E}\{\bar{x}(k)\bar{x}(k)^T\}$, and $R_x(k + \tau, k) \triangleq \mathcal{E}\{\bar{x}(k + \tau)\bar{x}(k)^T\}$. Note that $Q_x(k) = P_x(k) + \mu_x(k)\mu_x(k)^T$. When $R_x(k + \tau, k) = 0$ for all $\tau \in \mathbb{Z}$ except for $\tau = 0$, we say that x is a white process. If, in general, $P_x(k)$ is not diagonal for all k , then we say that x has spatial correlation.

3. SETUP AND PROBLEM FORMULATION

Consider the closed-loop system in Fig. 1.a), where N is a multiple-input multiple-output linear time-invariant discrete-time system, d is a vector that contains external signals, e is a measurement of interest, v corresponds to the channel input, and w is the channel output.

This type of closed-loop systems can be often found in the literature, see e.g. (Maass and Silva, 2014; Zhou et al., 1996), and it is a general way of writing many control architectures such as the standard one- and two-degrees of freedom feedback loops. In such scenarios: the plant, controller, and their interaction are represented inside system N ; inputs, outputs, and perturbations are

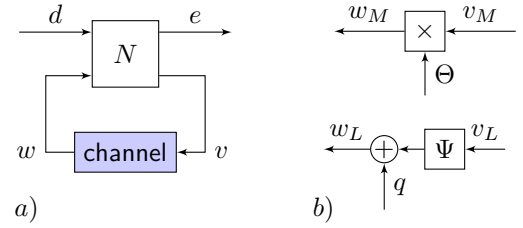


Fig. 1. a) System N closed over a communication channel. b) Top: Multiplicative noise channel. Bottom: Additive noise channel.

all contained in the vector d ; e denotes a user-defined performance signal, e.g. tracking error; and the signals that are being transmitted over the communication network are contained in vector v (see Fig. 1.a)).

System N is described in state-space by

$$\begin{bmatrix} x(k+1) \\ e(k) \\ v(k) \end{bmatrix} = \begin{bmatrix} A & B_d & B_w \\ C_e & D_{de} & D_{we} \\ C_v & D_{dv} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \\ w(k) \end{bmatrix}, \quad (1)$$

where $k \in \mathbb{N}_0$, and $x(k) \in \mathbb{R}^{n_x}$ denotes the state of system N . We assume that the initial state $x(0)$ is a random variable with mean and covariance matrix $P_x(0)$. Note that the 0 element in the above state-space representation is needed to ensure that there is no algebraic loop inside the feedback, since the communication channels we consider have no delays.

Assumption 1. For all $k \in \mathbb{N}_0$, the input vector $d(k) \in \mathbb{R}^{n_d}$ is a stationary white noise sequence with constant mean $\mu_d(k) = \mu_d$ and constant covariance matrix $P_d(k) = P_d$. We also assume that d is not correlated with $x(0)$. ■

Remark 1. As foreshadowed in the introduction, we will use an auxiliary NCS over AWN channels as per Fig. 1.b) to state our results. Depending on the channel being used in the closed-loop architecture of Fig. 1.a), the realisation of the corresponding signals might be different, except signal d which is the same in both cases. Therefore, we incorporate the subindex M to refer to signals in (1) when fading channels are in place, and the subindex L when AWN channels are in place. ■

3.1 Fading channels

Consider the fading channel in the top of Fig. 1.b) with input $v_M = [v_{M1}, \dots, v_{M\ell}]^T$, output $w_M = [w_{M1}, \dots, w_{M\ell}]^T$, and multiplicative noise Θ . We define the multiplicative channel noise matrix as $\Theta(k) \triangleq \text{diag}\{\theta_1(k), \dots, \theta_\ell(k)\}$, with mean $\Upsilon(k) \triangleq \mathcal{E}\{\Theta(k)\} = [\mu_{\theta_1}(k), \dots, \mu_{\theta_\ell}(k)]^T$. The above definitions correspond to the channel having multiple inputs and outputs.

Assumption 2. We assume that each $\theta_i(k)$, $i \in \{1, \dots, \ell\}$, is an i.i.d. process with non-zero mean and correlated with the other channels $\theta_j(k)$ for $i \neq j$. This implies that $\Upsilon(k) \triangleq \Upsilon$ is a constant matrix, and that $P_\theta \triangleq \mathcal{E}\{(\Theta(k) - \Upsilon)(\Theta(k) - \Upsilon)^T\}$ is non-diagonal and constant. Moreover, we assume that every $\theta_i(k)$ (and thus $\Theta(k)$) is independent of $(x_M(0), d(k))$. ■

Definition 1. The signals $w_M(k)$ and $v_M(k)$, are connected over a fading channel if and only if the channel output $w_M(k)$ and the channel input $v_M(k)$ are related through $w_M(k) = \Theta(k)v_M(k)$. ■

3.2 AWN channels

The AWN communication channel is depicted in Fig. 1.b), where $v_L = [v_{L_1}, \dots, v_{L_\ell}]^\top$ and $w_L = [w_{L_1}, \dots, w_{L_\ell}]^\top$ are the input and output of the channel, $q = [q_1, \dots, q_\ell]^\top$ is the additive noise vector, and $\Psi \in \mathbb{R}^{\ell \times \ell}$ is a deterministic and constant matrix acting as a channel gain.

Assumption 3. Each $q_i(k)$, $i \in \{1, \dots, \ell\}$, is a white noise sequence correlated with $q_j(k)$ for $i \neq j$. Furthermore, we assume that $q(k)$ is not correlated with $(x_L(0), d(k))$. ■

Definition 2. The signals $w_L(k)$ and $v_L(k)$ are related through an AWN channel with constant gain if and only if $w_L(k)$ and $v_L(k)$ are related via $w_L(k) = q(k) + \Psi v_L(k)$. ■

4. STATISTICS OF THE NCS

The goal of this section is to provide expressions for the first and second order moments of the state of system N , and also the performance measurement $e_M(k)$, when the system uses the channels models defined above to transmit over the network. The proof of the results presented here are in the appendix.

4.1 Statistics over fading channels

When fading channels are in place, the moments of the plant signals can be written as follows.

Lemma 1. Define the constant matrices $\Delta_1 \triangleq A + B_w \Upsilon C_v$ and $\Delta_2 \triangleq B_d + B_w \Upsilon D_{dv}$, and suppose Assumptions 1 and 2 hold. Then, the mean, covariance matrix, and covariance function of the state x_M of system N are given by

$$\mu_{x_M}(k+1) = \Delta_1 \mu_{x_M}(k) + \Delta_2 \mu_d, \quad (2)$$

$$P_{x_M}(k+1) = \Delta_1 P_{x_M}(k) \Delta_1^\top + \Delta_2 P_d \Delta_2^\top + B_w H(k) B_w^\top, \quad (3)$$

$$R_{x_M}(k+\tau, k) = \Delta_1^\tau P_{x_M}(k), \quad (4)$$

where

$$H(k) = P_\theta \odot \left(C_v P_{x_M}(k) C_v^\top + D_{dv} P_d D_{dv}^\top + (C_v \mu_{x_M}(k) + D_{dv} \mu_d) (C_v \mu_{x_M}(k) + D_{dv} \mu_d)^\top \right). \quad (5)$$

Lemma 1 characterises the first and second moments of the state $x_M(k)$ recursively. We emphasize that the recursion for the covariance matrix involves the Hadamard product between the noise variance and the second moment of the channel input, making this expression non-linear.

Given that system N is linear, the moments of the other signals in the loop can be written as a function of the state moments. We can generalise this assertion via the performance measurement $e_M(k)$.

Lemma 2. Define the constant matrices $\Delta_3 \triangleq C_e + D_{we} \Upsilon C_v$ and $\Delta_4 \triangleq D_{de} + D_{we} \Upsilon D_{dv}$, and suppose Assumptions 1 and 2 hold. Then, the mean, covariance matrix, and covariance function of signal $e_M(k)$ are given by

$$\mu_{e_M}(k) = \Delta_3 \mu_{x_M}(k) + \Delta_4 \mu_d, \quad (6)$$

$$P_{e_M}(k) = \Delta_3 P_{x_M}(k) \Delta_3^\top + \Delta_4 P_d \Delta_4^\top + D_{we} H(k) D_{we}^\top, \quad (7)$$

$$R_{e_M}(k+\tau, k) = \Delta_3 \Delta_1^\tau P_{x_M}(k) \Delta_3^\top + \Delta_3 \Delta_1^{\tau-1} [\Delta_2 P_d \Delta_4^\top + B_w H(k) D_{we}^\top], \quad (8)$$

where $P_{x_M}(k)$ and $H(k)$ are given in Lemma 1.

Lemma 2 characterises the moments of the signal $e_M(k)$ as a function of the state and input perturbation moments. It is important to recall that $e_M(k)$ is an arbitrary performance measurement, and thus this result allows us to compute the moments of any linear combination of the state and input.

4.2 Statistics over AWN channels

On the other hand, when the information is sent through the AWN channels defined in Section 3 the first and second order moments of the plant signals are given below.

Lemma 3. Define constant matrices $\Lambda_1 \triangleq A + B_w \Psi C_v$, $\Lambda_2 \triangleq B_d + B_w \Psi D_{dv}$, $\Lambda_3 \triangleq C_e + D_{we} \Psi C_v$ and $\Lambda_4 \triangleq D_{de} + D_{we} \Psi D_{dv}$, and suppose Assumptions 1 and 3 hold. Then, the mean, covariance matrix, and covariance function of the state x_L of system N are given by

$$\mu_{x_L}(k+1) = \Lambda_1 \mu_{x_L}(k) + \Lambda_2 \mu_d + B_w \mu_q(k), \quad (9)$$

$$P_{x_L}(k+1) = \Lambda_1 P_{x_L}(k) \Lambda_1^\top + \Lambda_2 P_d \Lambda_2^\top + B_w P_q(k) B_w^\top, \quad (10)$$

$$R_{x_L}(k+\tau, k) = \Lambda_1^\tau P_{x_L}(k). \quad (11)$$

Additionally, the mean, covariance matrix, and covariance function of the performance signal e_L , are given by

$$\mu_{e_L}(k) = \Lambda_3 \mu_{x_L}(k) + \Lambda_4 \mu_d + D_{we} \mu_q(k), \quad (12)$$

$$P_{e_L}(k) = \Lambda_3 P_{x_L}(k) \Lambda_3^\top + \Lambda_4 P_d \Lambda_4^\top + D_{we} P_q(k) D_{we}^\top, \quad (13)$$

$$R_{e_L}(k+\tau, k) = \Lambda_3 \Lambda_1^\tau P_{x_L}(k) \Lambda_3^\top + \Lambda_3 \Lambda_1^{\tau-1} [\Lambda_2 P_d \Lambda_4^\top + B_w P_q(k) D_{we}^\top]. \quad (14)$$

Lemma 3 characterises the moments of the state $x_L(k)$ recursively. As opposed to the fading channel case (see (3)), the covariance matrix does satisfy a linearity property with respect to the noise and state covariance. This considerably eases the analysis of this type of systems compared to those where multiplicative channels are in place. Lemma 3 also presents expressions for the moments of the performance measurement as linear combinations of the state, channel noise, and input.

5. MAIN RESULTS

5.1 Statistical equivalence

In this section, we present the statistical equivalence for which we relate the moment recursions in each case of Section 4.

Theorem 1. Suppose Assumptions 1, 2, and 3 hold, and further assume that $\mu_{x_M}(0) = \mu_{x_L}(0)$, $P_{x_M}(0) = P_{x_L}(0)$, and $\Psi = \Upsilon$. If the noise q is such that $\mu_q(k) = 0$ for all $k \in \mathbb{N}_0$, and such that

$$P_q(k) = P_\theta \odot \left(C_v P_{x_L}(k) C_v^\top + D_{dv} P_d D_{dv}^\top + (C_v \mu_{x_L}(k) + D_{dv} \mu_d) (C_v \mu_{x_L}(k) + D_{dv} \mu_d)^\top \right), \quad (15)$$

then, for every time instant $k \in \mathbb{N}_0$, the following holds.

$$\mu_{\alpha_M}(k) = \mu_{\alpha_L}(k),$$

$$P_{\alpha_M}(k) = P_{\alpha_L}(k),$$

$$R_{\alpha_M}(k+\tau, k) = R_{\alpha_L}(k+\tau, k),$$

where $\alpha \in \{x, e\}$.

Proof. Given that $\Psi = \Upsilon$, then by construction $\Lambda_i = \Delta_i$, for $i = \{1, 2, 3, 4\}$. This implies that, when we compare the recursion (2) with (9), it is clear that if $\mu_q(k) = 0$ for all $k \in \mathbb{N}_0$ and $\mu_{x_M}(0) = \mu_{x_L}(0)$, then the state means will evolve similarly. Consequently, the same will happen with the performance signals given in (6) and (12).

For the state covariances, we first compare (3) with (10) and we note that, given $\Lambda_i = \Delta_i$, for $i = \{1, 2, 3, 4\}$, and $P_{x_M}(0) = P_{x_L}(0)$, the state covariance matrices will evolve identically if $P_q(k) = H(k)$, where $H(k)$ is defined in (5). In that case, we have that $P_{x_M}(k) = P_{x_L}(k)$, and thus we can use such equality and replace $P_{x_M}(k)$ in (5) by $P_{x_L}(k)$, leading to (15). If such condition holds, then from (7) and (13) we get $P_{e_M}(k) = P_{e_L}(k)$.

The same above requirements suffice for the auto-covariance functions of the state (4) and (11) to be identical, since this holds if $P_{x_M}(k) = P_{x_L}(k)$ and $\Lambda_1 = \Delta_1$. Similarly, these conditions guarantee that the auto-covariance functions for the performance signal are identical, which is concluded directly by comparing (8) with (14). ■

Theorem 1 provides conditions for which NCSs closed over correlated fading channels have the same second order moments as NCSs closed over correlated AWN channels, for both the state and performance signal. Thereby, this extends the results presented in (Maass and Silva, 2014) to the case in which the fading channels are correlated between each other. This result provides a tool to tackle a broader class of problems related to NCSs over correlated fading channels since it allows to write the original problem as an equivalent problem but using an auxiliary noise variable that satisfies the requirements given in (15) for the equivalence to hold.

To use the equivalence in practice, instead of studying the original NCS over fading channel, one creates the auxiliary system and sets $\mu_{x_L}(0) \leftarrow \mu_{x_M}(0)$, $P_{x_L}(0) \leftarrow P_{x_M}(0)$, $\Psi \leftarrow \Upsilon$, and picks the additive noise such that $\mu_q(k) = 0$ for all $k \in \mathbb{N}_0$ and such that its variance is equal to (15). Then, any analysis involving the second order moments of the original system can be done using the auxiliary setup under the aforementioned constraints.

5.2 Stability

The equivalence result from Theorem 1 also allows us to study stability of NCSs over correlated fading channels by studying the equivalence when $k \rightarrow \infty$, which is what we do in this section. Particularly, we relate the mean square stability of the NCS with fading channels with the internal stability of the NCS with AWN channels.

Theorem 2. Consider the system in Fig. 1, under Assumptions 1, 2, and 3. Then, the NCS of Fig. 1 when fading channels are in place is MSS if and only if

- (i) the LTI system of Fig. 1 when AWN channels are in place is internally stable; and
- (ii) there exists a finite and positive semi-definite choice for the auxiliary noise covariance matrix P_q such that $P_q = P_\theta \odot (P_{v_L} + \mu_{v_L} \mu_{v_L}^\top)$, where $P_{v_L} = C_v P_{x_L} C_v^\top + D_{dv} P_d D_{dv}^\top$ and $\mu_{v_L} = C_v \mu_{x_L} + D_{dv} \mu_d$.

Proof. (\Rightarrow): If the NCS of Fig. 1 with fading channels in place is MSS, then by using Lemma 1 in (Maass and Silva, 2014)¹ together with Lemma 4 (see appendix) and (3), we can conclude that

$$P_{x_M} = \Delta_1 P_{x_M} \Delta_1^\top + \Delta_2 P_d \Delta_2^\top + B_w (P_\theta \odot [C_v P_{x_M} C_v^\top]) B_w^\top + B_w (P_\theta \odot [D_{dv} P_d D_{dv}^\top + \mu_{v_M} \mu_{v_M}^\top]) B_w^\top, \quad (16)$$

admits a unique solution which is also positive semi-definite (see (Styan, 1973)), where $\mu_{v_M} = C_v \mu_{x_M} + D_{dv} \mu_d$. In addition, Lemma 1 in (Maass and Silva, 2014) also implies that there exists $M > 0$ such that

$$M - \Delta_1 M \Delta_1^\top - B_w (P_\theta \odot (C_v M C_v^\top)) B_w^\top > 0. \quad (17)$$

From (17) we conclude that there exists $M > 0$ such that $M - \Delta_1 M \Delta_1^\top > 0$, which in turn implies that $\rho(\Delta_1) < 1$. Consequently, the NCS in Fig. 1 with AWN channels in place is internally stable.

Since Δ_1 is stable, we have that

$$P_{x_L} = \Delta_1 P_{x_L} \Delta_1^\top + \Delta_2 P_d \Delta_2^\top + B_w P_q B_w^\top \quad (18)$$

admits a unique positive semi-definite solution for all positive semi-definite P_d and P_q . Particularly, (18) admits a solution when we choose

$$P_q = P_\theta \odot (C_v P_{x_M} C_v^\top + D_{dv} P_d D_{dv}^\top + \mu_{v_M} \mu_{v_M}^\top),$$

where P_{x_M} satisfies (16). Then, (18) can be written as

$$P_{x_L} = \Delta_1 P_{x_L} \Delta_1^\top + B_w (P_\theta \odot (C_v P_{x_M} C_v^\top)) B_w^\top + \Delta_2 P_d \Delta_2^\top + B_w (P_\theta \odot (D_{dv} P_d D_{dv}^\top + \mu_{v_M} \mu_{v_M}^\top)) B_w^\top. \quad (19)$$

Since both (16) and (19) admit unique solutions, we conclude that $P_{x_M} = P_{x_L}$, and that there exists $P_q > 0$ such that $P_q = P_\theta \odot (P_{v_L} + \mu_{v_L} \mu_{v_L}^\top)$, completing the proof.

(\Leftarrow): If the NCS in Fig. 1 with AWN channels in place is internally stable and there exists a choice for $P_q = P_\theta \odot (P_{v_L} + \mu_{v_L} \mu_{v_L}^\top)$, then $P_q = P_\theta \odot (C_v P_{x_L} C_v^\top + D_{dv} P_d D_{dv}^\top + \mu_{v_L} \mu_{v_L}^\top)$, where P_{x_L} satisfies (18). In addition, there exists $P > 0$ such that

$$P - \Delta_1 P \Delta_1^\top > 0. \quad (20)$$

With Observation 21.6 in (Zhou, 1996), we write (18) as

$$P_{x_L} = \sum_{i=0}^{\infty} \Delta_1^i (\Delta_2 P_d \Delta_2^\top + B_w P_q B_w^\top) \Delta_1^{i\top}. \quad (21)$$

Therefore,

$$P_q = P_\theta \odot \left[C_v \left(\sum_{i=0}^{\infty} \Delta_1^i \Delta_2 P_d \Delta_2^\top \Delta_1^{i\top} + Q \right) C_v^\top + B^+ \right] > P_\theta \odot [C_v Q C_v^\top], \quad (22)$$

where $B^+ \triangleq D_{dv} P_d D_{dv}^\top + \mu_{v_L} \mu_{v_L}^\top$ and

$$Q \triangleq \sum_{i=0}^{\infty} \Delta_1^i B_w P_q B_w^\top \Delta_1^{i\top} \iff Q = \Delta_1 Q \Delta_1^\top + B_w P_q B_w^\top.$$

From (22) we conclude that there exists $\epsilon > 0$ such that

$$P_q > P_\theta \odot [C_v (Q + \epsilon P) C_v^\top],$$

where P satisfies (20). We can thus construct the following inequality

¹ The operator is defined by $\mathcal{A}(P_x) \triangleq \Delta_1 P_x \Delta_1^\top + B_w (P_\theta \odot (C_v P_x C_v^\top)) B_w^\top$

$$\begin{aligned} \mathcal{A}(Q + \epsilon P) &\leq \Delta_1(Q + \epsilon P)\Delta_1^\top + B_w P_q B_w^\top \\ &= Q + \epsilon \Delta_1 P \Delta_1^\top \\ &< Q + \epsilon P. \end{aligned}$$

We thus conclude that there exists $M = Q + \epsilon P > 0$ that satisfies $M - \mathcal{A}(M) > 0$. Therefore, the NCS in Fig. 1 with fading channels in place is MSS, concluding the proof. ■

Theorem 2 shows that the mean square convergence of the moment of the NCS over fading channels is guaranteed when the alternative LTI setup is internally stable, and the stationary constraint for the noise variance is met. This result is obtained exploiting the fact that the equivalence is valid also when $k \rightarrow \infty$.

Remark 2. The result in Theorem 2 is really convenient to further obtain explicit conditions for stability or performance of an NCS over correlated fading channels. For instance, similar results have been used for uncorrelated fading channels in Maass and Silva (2014) to obtain explicit stability and performance conditions in terms of unstable poles/zeros and channels statistics. These type of results have been also popular in uncorrelated channels with packet losses, see e.g. Silva and Pulgar (2013) for stability and performance, and Silva and Solis (2013) for estimation.

6. CONCLUSION

We studied NCSs communicating over spatially correlated fading channels. Particularly, we provided a statistical equivalence that relates the moments of the NCS with fading channels, to an auxiliary NCS that communicates over AWN channels. This equivalence extends the existing results in the literature for uncorrelated fading channels. In addition, we provided necessary and sufficient conditions that relate the MSS stability of the original system to the internal stability of the auxiliary system.

Appendix A. PROOF OF LEMMA 1

From (1), the state of system N can be written as

$$x_M(k+1) = \Delta_1 x_M(k) + \Delta_2 d(k) + B_w \bar{\Theta}(k) v_M(k), \quad (\text{A.1})$$

where $\Delta_1 = A + B_w \Upsilon C_v$ and $\Delta_2 = B_d + B_w \Upsilon D_{dv}$ are constant matrices. Moreover, given the linearity of the expectation operator, we know that

$$\mu_{x_M}(k+1) = \Delta_1 \mu_{x_M}(k) + \Delta_2 \mu_d + B_w \mathcal{E} \{ \bar{\Theta}(k) v_M(k) \}. \quad (\text{A.2})$$

Given that $\Theta(k)$ is independent of $v_M(k)$, we get $\mathcal{E} \{ \bar{\Theta}(k) v_M(k) \} = \mathcal{E} \{ \bar{\Theta}(k) \} \mathcal{E} \{ v_M(k) \} = 0$, which proves expression (2). To prove the expression for P_{x_M} , we use (2) and (A.1) to conclude that

$$\bar{x}_M(k+1) = \Delta_1 \bar{x}_M(k) + \Delta_2 \bar{d}(k) + B_w \bar{\Theta}(k) v_M(k). \quad (\text{A.3})$$

We proceed by using the definition $P_{x_M}(k+1) = \mathcal{E} \{ \bar{x}_M(k+1) \bar{x}_M(k+1)^\top \}$ together with (A.3). This computation generates a number of cross terms which will be zero given Assumptions 1 and 2. In fact, given that $d(k)$ is a white noise process independent of $\Theta(k)$, $\forall k \in \mathbb{N}_0$, and not correlated with the initial condition, we conclude that

$\mathcal{E} \{ \bar{x}_M(k) \bar{d}(k) \} = 0$. Moreover, given Assumption 2, and emphasizing that $\mathcal{E} \{ \bar{\Theta}(k) \} = 0$, we have that

$$\mathcal{E} \{ \bar{\Theta}(k) v_M(k) \bar{x}_M(k)^\top \} = 0, \quad (\text{A.4})$$

$$\mathcal{E} \{ \bar{\Theta}(k) v_M(k) \bar{d}(k)^\top \} = 0. \quad (\text{A.5})$$

Therefore, we conclude that

$$\begin{aligned} P_{x_M}(k+1) &= \Delta_1 P_{x_M}(k) \Delta_1^\top + \Delta_2 P_d \Delta_2^\top \\ &\quad + B_w \mathcal{E} \{ \bar{\Theta}(k) \mathcal{E} \{ v_M(k) v_M(k)^\top \} \bar{\Theta}(k)^\top \} B_w^\top \\ &= \Delta_1 P_{x_M}(k) \Delta_1^\top + \Delta_2 P_d \Delta_2^\top \\ &\quad + B_w (P_\theta(k) \odot [P_{v_M}(k) + \mu_{v_M}(k) \mu_{v_M}(k)^\top]) B_w^\top, \end{aligned}$$

where the mean of $v_M(k)$ is given by

$\mu_{v_M}(k) = \mathcal{E} \{ C_v x_M(k) + D_{dv} d(k) \} = C_v \mu_{x_M}(k) + D_{dv} \mu_d$, and its corresponding covariance matrix given by

$$\begin{aligned} P_{v_M}(k) &= \mathcal{E} \{ \bar{v}_M(k) \bar{v}_M(k)^\top \} \\ &= \mathcal{E} \left\{ (C_v \bar{x}_M(k) + D_{dv} \bar{v}_M(k)) \right. \\ &\quad \left. \times (C_v \bar{x}_M(k) + D_{dv} \bar{v}_M(k))^\top \right\} \\ &= C_v P_{x_M}(k) C_v^\top + D_{dv} P_d D_{dv}^\top. \quad (\text{A.6}) \end{aligned}$$

Then, by defining $H(k) \triangleq P_{v_M}(k) + \mu_{v_M}(k) \mu_{v_M}(k)^\top$ and substituting $P_{v_M}(k)$ and $\mu_{v_M}(k)$ we prove (3) and (5).

On the other hand, by using (A.3) and the definition of the covariance function, we get

$$\begin{aligned} R_{x_M}(k+\tau+1, k) &= \mathcal{E} \{ \bar{x}_M(k+\tau+1) \bar{x}_M(k)^\top \} \\ &= \mathcal{E} \{ [\Delta_1 \bar{x}_M(k+\tau) + \Delta_2 \bar{d}(k+\tau) \\ &\quad + B_w \bar{\Theta}(k+\tau) v_M(k+\tau)] \bar{x}_M(k)^\top \} \\ &= \Delta_1 R_{x_M}(k+\tau, k) \\ &= \Delta_1^{\tau+1} P_{x_M}(k) \quad (\text{A.7}) \end{aligned}$$

where we used the fact that $\mathcal{E} \{ \bar{d}(k+\tau) \bar{x}_M(k)^\top \} = 0$, and that $\mathcal{E} \{ \bar{\Theta}(k+\tau) v_M(k+\tau) \bar{x}_M(k)^\top \} = 0$.

Appendix B. PROOF OF LEMMA 2

We can write $e_M(k)$ as

$$e_M(k) = \Delta_3 x_M(k) + \Delta_4 d(k) + D_{we} \bar{\Theta}(k) v_M(k), \quad (\text{B.1})$$

with $\Delta_3 = C_e + D_{we} \Upsilon C_v$ and $\Delta_4 = D_{de} + D_{we} \Upsilon D_{dv}$. Applying the expectation operator in (B.1), and by noting that $\mathcal{E} \{ \bar{\Theta}(k) v_M(k) \} = 0$, we immediately get (6). In order to compute the covariance matrix $P_{e_M}(k)$, we use (B.1) and (6) to get

$$\bar{e}_M(k) = \Delta_3 \bar{x}_M(k) + \Delta_4 \bar{d}(k) + D_{we} \bar{\Theta}(k) v_M(k). \quad (\text{B.2})$$

It is clear that $\bar{e}_M(k)$ in (B.2) has the same structure of $\bar{x}_M(k+1)$ in (A.3), in which the only difference is given by the multiplying matrices. The above implies that, in order to compute $P_{e_M}(k)$, we can use the same procedure and properties utilised to obtain $P_{x_M}(k+1)$ in the proof of Lemma 1, concluding the proof of (7).

On the other hand, given Assumptions 1 and 2, we have that

$$\begin{aligned} R_{e_M}(k+\tau, k) &= \mathcal{E} \{ \bar{e}_M(k+\tau) \bar{e}_M(k)^\top \} \\ &= \mathcal{E} \left\{ [\Delta_3 \bar{x}_M(k+\tau) + \Delta_4 \bar{d}(k+\tau) \right. \\ &\quad \left. + D_{we} \bar{\Theta}(k+\tau) v_M(k+\tau)] \bar{e}_M(k)^\top \right\} \\ &= \Delta_3 \mathcal{E} \{ \bar{x}_M(k+\tau) \bar{e}_M(k)^\top \}. \quad (\text{B.3}) \end{aligned}$$

Moreover, using (A.3) recursively, we can write

$$\begin{aligned}\bar{x}_M(k+\tau) &= \Delta_1^\tau \bar{x}_M(k) + \sum_{i=1}^{\tau} \Delta_1^{i-1} \Delta_2 \bar{d}(k+\tau-i) \\ &+ \sum_{i=1}^{\tau} \Delta_1^{i-1} B_w \bar{\Theta}(k+\tau-i) v_M(k+\tau-i).\end{aligned}\quad (\text{B.4})$$

We now use (B.4) and (B.2) to expand the product inside the expectation. This procedure contains several cross product terms which will be zero given Assumptions 1 and 2. Therefore, we get

$$\begin{aligned}R_{e_M}(k+\tau, k) &= \Delta_3 \mathcal{E} \{ \bar{x}_M(k+\tau) \bar{e}_M(k)^\top \} \\ &= \Delta_3 \mathcal{E} \{ \Delta_1^\tau \bar{x}_M(k) \bar{x}_M(k)^\top \Delta_3^\top \} \\ &+ \Delta_3 \mathcal{E} \{ \Delta_1^{\tau-1} \Delta_2 \bar{d}(k) \bar{d}(k)^\top \Delta_4^\top \} \\ &+ \Delta_3 \mathcal{E} \{ \Delta_1^{\tau-1} B_w \bar{\Theta}(k) v_M(k) v_M(k)^\top \bar{\Theta}(k)^\top D_{we}^\top \},\end{aligned}$$

from which we prove (8), completing the proof.

Appendix C. PROOF OF LEMMA 3

We re-write the state $x_L(k+1)$ in (1) by including the AWN channel dynamics and get

$$x_L(k+1) = \Lambda_1 x_L(k) + \Lambda_2 d(k) + B_w q(k). \quad (\text{C.1})$$

We then apply the expectation operator to (C.1) and get

$$\mu_{x_L}(k+1) = \Lambda_1 \mu_x(k) + \Lambda_2 \mu_d + B_w \mu_q(k). \quad (\text{C.2})$$

Moreover, from (C.1) and (C.2) we have that

$$\bar{x}_L(k+1) = \Lambda_1 \bar{x}_L(k) + \Lambda_2 \bar{d}(k) + B_w \bar{q}(k), \quad (\text{C.3})$$

Consequently, the covariance matrix can be computed as

$$\begin{aligned}P_{x_L}(k+1) &= \Lambda_1 P_{x_L}(k) \Lambda_1^\top + B_w P_d B_w^\top \\ &+ \Lambda_2 P_q(k) \Lambda_2^\top,\end{aligned}\quad (\text{C.4})$$

where we exploited the fact that, given Assumptions 1 and 3, $\mathcal{E} \{ \bar{x}(k) \bar{q}_L(k)^\top \} = 0$, $\mathcal{E} \{ \bar{x}(k) \bar{d}(k)^\top \} = 0$, and $\mathcal{E} \{ \bar{q}(k) \bar{d}(k)^\top \} = 0$. Lastly, by using the definition of covariance function together with (C.3), we have that

$$R_{x_L}(k+\tau+1, k) = \Lambda_1^\tau P_{x_L}(k), \quad (\text{C.5})$$

where we used the fact that, for all $\tau > 0$, $\mathcal{E} \{ \bar{d}(k+\tau) \bar{x}_L(k)^\top \} = 0$, and $\mathcal{E} \{ \bar{q}(k+\tau) \bar{x}_L(k)^\top \} = 0$.

On the other hand, since the signal e_L is a linear combination of the state, the result for e_L follows directly from the results for the state x_L , and thus the details are omitted for brevity.

Appendix D. TECHNICAL RESULT

Lemma 4. Define the operator $\mathcal{T}(A) \triangleq MAM^T + L(P \odot [NAN^T])L^T$, where A, M, L, P , and N are matrices of appropriate dimensions, and $P \geq 0$. Then, $\mathcal{T}(\cdot)$ is a monotonic linear operator.

Proof. Let $\alpha, \beta \in \mathbb{R}$, then linearity follows from

$$\begin{aligned}\mathcal{T}(\alpha A + \beta B) &= M(\alpha A + \beta B)M^T \\ &+ L(P \odot [N(\alpha A + \beta B)N^T])L^T \\ &= \alpha MAM^T + \beta MBM^T \\ &+ L(P \odot [\alpha NAN^T])L^T + L(P \odot [\beta NBN^T])L^T \\ &= \alpha \mathcal{T}(A) + \beta \mathcal{T}(B),\end{aligned}$$

where we used the fact that $(\alpha A) \odot B = \alpha(A \odot B)$ (Bernstein, 2005). To show monotonicity we let $X \geq Y$

and show that $\mathcal{T}(X) \geq \mathcal{T}(Y)$. Given that $X - Y \geq 0$, by using the fact that the Hadamard product of two positive semi-definite matrices is also positive semi-definite (see (Styan, 1973)), we get $\mathcal{T}(X - Y) \geq 0$. The proof is then complete by linearity of $\mathcal{T}(\cdot)$. ■

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