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A positive real lemma for singular hybrid systems

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Abstract: This paper introduces a positive real lemma for continuous-time singular hybrid systems. The necessary and sufficient condition of stochastic admissibility and strict passivity for the singular hybrid systems is obtained in terms of linear matrix inequalities. The sufficient condition is derived by using mode-dependent Lyapunov function. In this step, two slack variables are inserted to make the proposed condition be necessary and sufficient condition in terms of strict linear matrix inequalities. Then, to prove the necessary condition, the positive real lemma for the hybrid system is proposed. Since the admissible singular system can be reformulated into stable normal system, the positive real lemma for the hybrid system holds. Thus, we give a necessary condition by constructing the solution of the proposed lemma from that of the hybrid systems.

Keywords: Stochastic hybrid systems, stability and stabilization of hybrid systems, linear matrix inequality, positive real lemma, descriptor system

1. INTRODUCTION

In the real world, many physical systems have suddenly changing components caused by system failure or repair. To mimic these phenomena, the hybrid systems which have changes in the operation point have been widely used. Markovian jump systems, in which the operating point follows a Markov process, are one of the popular examples of the hybrid system Ni et al. (2019); Zheng et al. (2019); Zhang et al. (2019b,a). Since the Markov process requires only the current information to determine the next state, Markovian jump systems have been useful systems to represent the chemical process Érdi and Tóth (1989), smart grid Singh et al. (2017) and network control system Wu and Wu (2019). Otherwise, singular systems are composed of both differential and algebraic equations. Due to this property, singular systems can describe the dynamics and static constraint of the state at once. For this reason, it is well known that the singular systems are more general than the normal systems.

Therefore, the singular hybrid systems are one of the popular models nowadays. On its practical aspect, researches on application to many practical models such as DC motor and electrical circuit Wang et al. (2015) is studied. In the theoretical side, the researchers have tried to find the stability and stabilization criterion in terms of linear matrix inequalities (LMIs). For example, the authors of Xia et al. (2009) obtain the equivalent condition for the stochastic admissibility criterion. In the case of the stabilization problem, the necessary and sufficient conditions for state-feedback control Xia et al. (2009) and dynamic output-feedback control Kwon et al. (2017b) are introduced in terms of LMIs.

On the other hand, external disturbance in the system is inevitable in the real world. However, the disturbance degrades the system performance or causes the instability of the system. Therefore, treating the disturbance is one of the important issues in the robust control theory field. Many researches have been investigated the stability analysis for the system with external disturbance. Especially, they have been focused on the relation between the disturbance and system output. For example, many papers regarding stability analysis and stabilization for $\mathcal{H}_\infty$ and passivity-based control/filter have been published Shin and Park (2019); Wu et al. (2019); Kwon et al. (2017a); Park et al. (2018, 2019); Li et al. (2017); Wu et al. (2015).

These studies start from bounded real lemma and positive real lemma, which show $\mathcal{H}_\infty$ and passivity-based stability criterion, respectively. The bounded real lemma is well developed for linear system Boyd et al. (1994), Markovian jump systems in discrete- Seiler and Sengupta (2003) and continuous-time Li and Ugrinovskii (2007) domain, positive system Tanaka and Langbort (2011), singular systems Uezato and Ikeda (1999); Masubuchi (2006), and singular Markovian jump systems Park et al. (2019). Also, the positive real lemma for linear system Boyd et al. (1994), singular system in continuous- and discrete-time domain Zhang et al. (2002) have been published. However, to the best of the authors' knowledge, the positive real

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lemma for the singular Markovian jump system is not studied yet. This is the motivation behind this paper.

This paper introduces the equivalent condition of the positive real lemma for the continuous-time singular hybrid systems in terms of LMIs. To show the sufficient condition, we verify that there exists a mode-dependent Lyapunov function for stochastic admissibility and strict passivity. In the case of the necessary condition, we propose the equivalent condition of positive real lemma for the hybrid system. Then, by using this lemma, we prove that the existence of a feasible solution for the proposed LMIs.

The notations used in this paper are standard. For vector $x \in \mathcal{R}^n$ and matrix $X \in \mathcal{R}^{n\times m}$, $x^T(X^T)$ denotes the transpose of $x(X)$. For symmetric matrices X and Y, $X > (\geq) Y$ means that $X - Y > (\geq) Y$ is positive (semi-) definite. For any square matrix Z, $sym\{Z\} = Z + Z^T$. The notation I_m is an identity matrix with dimension m and the notation I is used for the identity matrix with an appropriate dimension. In the symmetric matrix, the notation $(*)$ represents an ellipsis of the terms which can be induced by symmetry. The notation $\mathcal{E}[\cdot]$ denotes the expectation value.

2. PRELIMINARIES

2.1 Problem statement

Let us consider the following singular hybrid system:

$$E\dot{x}(t) = A(\theta_t)x(t) + B(\theta_t)w(t), \quad (1)$$
$$y(t) = C(\theta_t)x(t) + D(\theta_t)w(t), \quad (2)$$

where $x(t) \in \mathcal{R}^n$, $w(t) \in \mathcal{R}^k$, $y(t) \in \mathcal{R}^m$ is the state, the external disturbance which belongs to the Hilbert space such that $\mathcal{L}_2 \triangleq \{w(t) \mid \int_0^\infty \mathbb{E}\{|w(t)|^2\}dt < \infty\}$, the measured output. The continuous-time Markov process $\{\theta_t \mid \text{for all } t > 0\}$ on the probability space takes values in the finite set $\mathbb{N} \triangleq \{1, 2, \cdots, N\}$. The mode transition probabilities are defined as follows:

$$\Pr(\theta_{t+\delta t} = j|\theta_t = i) = \begin{cases} \pi_{ij}\delta t + o(\delta t) & \text{if } j \neq i, \\ 1 + \pi_{ii}\delta t + o(\delta t) & \text{otherwise,} \end{cases} \quad (3)$$

where $\delta t > 0$, $\lim_{\delta t \to 0}(o(\delta t)/\delta t) = 0$, and π_{ij} is the transition rate from mode i at time t to mode j at time $t + \delta t$. The mode transition rate matrix $\Pi = [\pi_{ij}]_{i,j\in\mathbb{N}}$ belongs to

$$\mathcal{S}_\Pi = \left\{ [\pi_{ij}]_{i,j\in\mathbb{N}} \;\middle|\; \sum_{j=1}^N \pi_{ij} = 0, \pi_{ij} \geq 0 \text{ for } i \neq j \right\}. \quad (4)$$

To simplify the notations, we will use the subscript i for all mode-dependent matrix at $\theta_t = i$, i.e., $A(\theta_t = i) = A_i$. Also, the matrix $E \in \mathcal{R}^{n\times n}$ is supposed to be singular, i.e., $rank(E) = r < n$. To handle the matrix E, we will use the matrices $R \in \mathcal{R}^{(n-r)\times n}, S \in \mathcal{R}^{n\times(n-r)}, E_L \in \mathcal{R}^{n\times r}, E_R \in \mathcal{R}^{r\times n}$, where R and S are of full rank, $RE = 0$, $ES = 0$, and $E_L E_R^T = E$.

In this paper, we will use the following assumption.

Assumption 1. Assume that the pair (A_i, C_i) for all $i \in \mathbb{N}$ is observable.

The main objective of this paper is to show the positive real lemma for singular hybrid systems. The definitions are introduced as follows:

Definition 2. (Xu and Lam (2006))

i) The continuous-time singular hybrid system (1) with $w(t) = 0$ is said to be regular, if for all $i \in \mathbb{N}$, $det(sE - A_i)$ is not identically zero.
ii) The continuous-time singular hybrid system (1) with $w(t) = 0$ is said to be impulse-free, if for all $i \in \mathbb{N}$, $deg(det(sE - A_i)) = rank(E)$.
iii) The continuous-time singular hybrid system (1) with $w(t) = 0$ is said to be stochastically stable, if for all initial mode $r_0 \in \mathbb{N}$ and any initial state x_0, there exists a scalar $M(x_0, r_0)$ such that

$$\mathcal{E}\left\{ \int_0^\infty ||x(t)||^2 dt \mid x_0, r_0 \right\} \leq M(x_0, r_0). \quad (5)$$

iv) The continuous-time singular hybrid system (1) with $w(t) = 0$ is said to be stochastically admissible if it is regular, impulse-free and stochastically stable.

The equivalent condition for the stochastic admissibility of the singular hybrid system is known as the following lemma.

Lemma 3. (Feng and Shi (2017)) The singular hybrid system (1) with $w(t) = 0$ is stochastically admissible if and only if there exist symmetric matrices $P_i \in \mathcal{R}^{n\times n}$ and matrices $X_i \in \mathcal{R}^{(n-r)\times(n-r)}$ such that for all $i \in \mathbb{N}$

$$0 < E_L^T P_i E_L, \quad (6)$$

$$0 > sym\{A_i^T(P_i E + R^T X_i S^T)\} + \sum_{j=1}^N \pi_{ij} E^T P_j E. \quad (7)$$

Also, the stochastically passive system is defined as below.

Definition 4. (Florchinger (1999)) The singular hybrid system (1)-(2) is said to be stochastically passive, under zero initial condition, if for any zero initial condition the inequality

$$\mathcal{E}\left[\int_0^T w^T(t) y(t) dt \right] \geq 0, \quad (8)$$

for any $T > 0$.

2.2 Positive real lemma for the hybrid system

Before considering the singular hybrid system, we deal with the hybrid system such that

$$\dot{x}(t) = A_i x(t) + B_i w(t), \quad (9)$$
$$z(t) = C_i x(t) + D_i w(t). \quad (10)$$

For the above system, the following lemma will be used in this section.

Lemma 5. (Li and Ugrinovskii (2007)) The hybrid system (9) with $w(t) = 0$ is stochastically stable if and only if there exists a unique solution $Z_i > 0$ for all $i \in \mathbb{N}$ and any given $\mathcal{S}_i$ such that

$$sym\{A_i^T Z_i\} + \sum_{j=1}^N \pi_{ij} P_j + \mathcal{S}_i = 0. \quad (11)$$

We also insist the following proposition.

Proposition 6. If the hybrid system (9) with $w(t) = 0$ is stochastically stable, then there exist positive definite matrices Δ_i for all $i \in \mathbb{N}$ such that

$$0 > sym\{A_i^T \Delta_i\} + \sum_{j=1}^{N} \pi_{ij}\Delta_j + \Delta_i B_i (D_i + D_i^T)^{-1} B_i^T \Delta_i, \quad (12)$$

where $D_i + D_i^T > 0$.

Proof. Since the system matrix A_i in (9) is stochastically stable, from Lemma 5, there exist positive definite matrices $\tilde{\Delta}_i$ such that for all $i \in \mathbb{N}$,

$$sym\{A_i^T \tilde{\Delta}_i\} + \sum_{j=1}^{N} \pi_{ij}\tilde{\Delta}_j + \mathcal{S}_i = 0, \quad (13)$$

where $\mathcal{S}_i > 0$ for all $i \in \mathbb{N}$ are given. We can find a sufficiently small scalar $\delta > 0$ such that

$$\delta\tilde{\Delta}_i B_i (D_i + D_i^T)^{-1} B_i^T \tilde{\Delta}_i < \mathcal{S}_i. \quad (14)$$

Setting $\Delta_i = \delta\tilde{\Delta}_i$ completes the proof.

Lemma 7. Hybrid system (9)-(10) is stochastically stable and strictly positive real if and only if there exist matrices $P_i \in \mathcal{R}^{n\times n}$ such that for all $i \in \mathbb{N}_+$

$$0 < P_i, \quad (15)$$

$$0 > \begin{bmatrix} sym\{A_i^T P_i\} + \sum_{j=1}^{N} \pi_{ij} P_j & (*) \\ B_i^T P_i - C_i & -D_i^T - D_i \end{bmatrix}. \quad (16)$$

Proof. (Sufficient) Let us select mode-dependent Lyapunov function candidate $V_i = x^T(t)P_i x(t)$, where $P_i > 0$ holds by (15). The weak infinitesimal operator acting on V_i is known as follows:

$$\nabla V_i = 2x^T(t)P_i(A_i x(t) + B_i w(t)) + \sum_{j=1}^{N} \pi_{ij} P_j. \quad (17)$$

Then we can obtain the following relation:

$$\nabla V_i - y^T(t)w(t) - w^T(t)y(t) \quad (18)$$

$$= \xi^T(t) \begin{bmatrix} sym\{A_i^T P_i\} + \sum_{j=1}^{N} \pi_{ij} P_j & (*) \\ B_i^T P_i - C_i & -D_i^T - D_i \end{bmatrix} \xi(t), \quad (19)$$

where $\xi(t) = \begin{bmatrix} x^T(t) & w^T(t) \end{bmatrix}^T$. From the condition (16),

$$0 > \nabla V_i - y^T(t)w(t) - w^T(t)y(t). \quad (20)$$

The above condition concludes to the following two state:

a) At $w(t) = 0$, the condition (19) becomes

$$0 > sym\{A_i^T P_i\} + \sum_{j=1}^{N} \pi_{ij} P_j, \quad (21)$$

which means the stochastic stability of the hybrid system from Lemma 5.

b) At $w(t) \neq 0$, the condition (20) guarantees the following relation:

$$0 < V_i(T) - V_i(0) < 2\mathcal{E}\left[\int_0^T w^T(t)y(t)dt\right], \quad (22)$$

which means strictly passive of the hybrid system.

Thus, the proposed conditions (15)-(16) guarantee the stochastic stability and positive realness of the hybrid system.

(Necessary) Since the hybrid system is stochastically stable and positive real, i.e., $D_i + D_i^T > 0$, there exist a positive definite solution $\bar{P}_i$ for the following algebraic Riccati equation (ARE):

$$sym\{A_i^T \bar{P}_i\} + \sum_{j=1}^{N} \pi_{ij}\bar{P}_j + (\bar{P}_i B_i - C_i^T)(D_i + D_i^T)^{-1}(B_i^T \bar{P}_i - C_i) = 0. \quad (23)$$

It means that the system matrix $A_i - B_i(D_i + D_i^T)^{-1}C_i + B_i(D_i + D_i^T)^{-1}B_i^T\bar{P}_i$ is stochastically stable, i.e., there exist positive definite matrix $\tilde{P}_i$ such that

$$0 > sym\{A_i^T \tilde{P}_i\} + \sum_{j=1}^{N} \pi_{ij}\tilde{P}_j + \tilde{P}_i B_i (D_i + D_i^T)^{-1} B_i^T \tilde{P}_i \quad (24)$$

from the proposition 6. By adding the conditions (23) and (24), we can obtain the following relation:

$$0 > sym\{A_i^T(\tilde{P}_i + \bar{P}_i)\} + \sum_{j=1}^{N} \pi_{ij}(\tilde{P}_j + \bar{P}_j) + ((\tilde{P}_i + \bar{P}_i)B_i - C_i^T)(D_i + D_i^T)^{-1}((\tilde{P}_i + \bar{P}_i)B_i - C_i^T)^T. \quad (25)$$

Letting $P_i = \bar{P}_i + \tilde{P}_i$, and applying Schur complement to (25) concludes to (16). This completes the proof.

3. MAIN RESULTS

Based on the positive real lemma for hybrid system in Lemma 7, we will derive the positive real lemma for the singular hybrid system.

Theorem 8. The singular hybrid system (1)-(2) is stochastically admissible and strictly passive if and only if there exist symmetric matrices $P_i \in \mathcal{R}^{n\times n}$, non-singular matrices $X_i \in \mathcal{R}^{(n-r)\times(n-r)}$, and matrices $Y_i \in \mathcal{R}^{(n-r)\times k}$ such that for all $i \in \mathbb{N}$

$$0 < E_L^T P_i E_L, \quad (26)$$

$$0 > \begin{bmatrix} sym\{A_i^T \Omega_i\} + \sum_{j=1}^{N} \pi_{ij} E^T P_j E & (*) \\ B_i^T \Omega_i + Y_i^T R A_i - C_i & sym\{B_i^T R^T Y_i - D_i\} \end{bmatrix} \quad (27)$$

where $\Omega_i \triangleq P_i E + R^T X_i S^T$.

Proof. (Sufficient) Let us choose mode-dependent Lyapunov function candidate $V_i = x^T(t)E^T P_i E x(t)$, where $E^T P_i E = E_R\{E_L^T P_i E_L\}E_R^T \geq 0$ holds by (26). The weak infinitesimal operator ∇ of the Markov process acting on V_i is calculated such that

$$\nabla V_i = 2x^T(t)E^T P_i(A_i x(t) + B_i w(t)) + \sum_{j=1}^{N} \pi_{ij} E^T P_j E. \quad (28)$$

Note that the relation $RE = 0$ holds the condition $RE\dot{x} = R(A_i x(t) + B_i w(t)) = 0$. Therefore, we can add the free-variable X_i and Y_i into the relation (28):

$$\nabla V_i = 2x^T(t)E^TP_i(A_ix(t)+B_iw(t)) + \sum_{j=1}^{N}\pi_{ij}E^TP_jE + 2(x^T(t)SX_i^T + w^T(t)Y_i^T)R(A_ix(t)+B_iw(t)) = \xi^T(t)\begin{bmatrix} (1,1) & (*) \\ B_i^T\Omega_i + Y_i^TRA_i & sym\{B_i^TR^TY_i\}\end{bmatrix}\xi(t), \tag{29}$$

where $(1,1) = sym\{A_i^T\Omega_i\} + \sum_{j=1}^{N}\pi_{ij}P_j$ and $\xi(t) = \begin{bmatrix} x^T(t) & w^T(t)\end{bmatrix}^T$. Next, to consider the passivity property (8), let us consider the following inequality:

$$\nabla V_i - w^T(t)y(t) - y^T(t)w(t) = \xi^T(t)\begin{bmatrix} sym\{A_i^T\Omega_i\} + \sum_{j=1}^{N}\pi_{ij}E^TP_jE & (*) \\ B_i^T\Omega_i + Y_i^TRA_i - C_i & (2,2)\end{bmatrix}\xi(t), \tag{30}$$

where $(2,2) = sym\{B_i^TR^TY_i - D_i\}$. From the condition (27), it is clear that $\nabla V_i < 2w^T(t)y(t)$ which concludes that

$$0 < V_i(T) - V_i(0) < \mathcal{E}\left[\int_0^T w^T(t)y(t)dt\right]. \tag{31}$$

Since the conditions (26)-(27) hold (15)-(16), the singular hybrid system (1) with $w(t) = 0$ is stochastically admissible and strictly passive.

(Necessary) By singular value decomposition, singular matrix E can be represented as follows:

$$E = U\begin{bmatrix}\Sigma & 0 \\ 0 & 0\end{bmatrix}V^T, \tag{32}$$

where $\Sigma \in \mathcal{R}^{r\times r}$ is diagonal matrix. Let us use $\bar{U} = U\begin{bmatrix}\Sigma & 0 \\ 0 & I\end{bmatrix}$. Then we can use the following relation:

$$E = \bar{U}\begin{bmatrix} I_r & 0 \\ 0 & 0\end{bmatrix}V^T,\ A_i = \bar{U}\begin{bmatrix} A_{1i} & A_{2i} \\ A_{3i} & A_{4i}\end{bmatrix}V^T. \tag{33}$$

Regular and impulse-free of the singular hybrid system means that $A_{4i} \in \mathcal{R}^{(n-r)\times(n-r)}$ is non-singular. Therefore we can use the the following relation:

$$E = U_i\begin{bmatrix} I_r & 0 \\ 0 & 0\end{bmatrix}V^T,\ A_i = U_i\begin{bmatrix} \bar{A}_i & 0 \\ A_{3i} & A_{4i}\end{bmatrix}V^T, \tag{34}$$

where the system $\bar{A}_i = A_{1i} - A_{2i}A_{4i}^TA_{3i}$ is stochastically stable and $U_i = \begin{bmatrix} I & A_{2i}A_{4i}^{-1} \\ 0 & I\end{bmatrix}$. Then the equivalent system of the singular hybrid system (1)-(2) can be represented as follows:

$$\begin{bmatrix} I_r & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}_1(t) \\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}\bar{A}_i & 0 \\ A_{3i} & A_{4i}\end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t)\end{bmatrix} + \begin{bmatrix} B_{1i} \\ B_{2i}\end{bmatrix}w(t), \tag{35}$$

$$y(t) = \begin{bmatrix} C_{1i} & C_{2i}\end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t)\end{bmatrix} + D_iw(t), \tag{36}$$

where

$$V^Tx(t) = \begin{bmatrix} x_1(t) \\ x_2(t)\end{bmatrix}, x_1(t) \in \mathcal{R}^r, B_i = U_i\begin{bmatrix} B_{1i} \\ B_{2i}\end{bmatrix},$$
$$C_i = \begin{bmatrix} C_{1i} & C_{2i}\end{bmatrix}V^T, B_{1i} \in \mathcal{R}^{r\times k}, C_{1i} \in \mathcal{R}^{m\times r}.$$

To facilitate the equation, we will use the equivalent system (35)-(36) in this proof. Since this proof start from the stochastic admissibility of the system (35)-(36), it is clear the the following hybrid system is stochastic stable.

$$\dot{x}_1(t) = \bar{A}_ix_1(t) + B_{1i}w(t), \tag{37}$$
$$y(t) = \bar{C}_ix_1(t) + \bar{D}_iw(t), \tag{38}$$

where $\bar{C}_i = C_{1i} - C_{2i}A_{4i}^{-1}A_{3i}$ and $\bar{D}_i = D_i - C_{2i}A_{4i}^{-1}B_{2i}$. From Lemma 7, the stochastic stability and strictly passivity of the hybrid system (37)-(38) means the existence of positive definite matrix $\bar{P}_i \in \mathcal{R}^{r\times r}$ such that

$$0 > \Theta_i \triangleq \begin{bmatrix} sym\{\bar{A}_i^T\bar{P}_i\} + \sum_{j=1}^{N}\pi_{ij}\bar{P}_j & (*) \\ B_{1i}^T\bar{P}_i - \bar{C}_i & -\bar{D}_i^T - \bar{D}_i\end{bmatrix}. \tag{39}$$

To obtain P_i, X_i and Y_i which satisfy the proposed condition (26)-(27), let us construct the following block matrix by using $\bar{P}_i$ in (39):

$$P_i = \begin{bmatrix}\bar{P}_i & P_{2i} \\ P_{2i}^T & 0\end{bmatrix}, \tag{40}$$

where $P_{2i} \in \mathcal{R}^{r\times(n-r)}$ is to be determined. The proposed conditions (26)-(27) for the system (35)-(36) are represented as follows:

$$0 < \bar{P}_i, \tag{41}$$

$$0 > \begin{bmatrix}\mathcal{L}_{1i} & (*) & (*) \\ \mathcal{L}_{2i} & sym\{A_{4i}^T\bar{X}_i\} & (*) \\ \mathcal{L}_{3i} & \mathcal{L}_{4i} & sym\{B_{2i}^T\bar{Y}_i - D_i\}\end{bmatrix}, \tag{42}$$

$$\mathcal{L}_{1i} = sym\{\bar{A}_i\bar{P}_i + A_{3i}^TP_{2i}\} + \sum_{j=1}^{N}\pi_{ij}\bar{P}_j, \tag{43}$$

$$\mathcal{L}_{2i} = A_{4i}^TP_{2i} + \bar{X}_i^TA_{3i}, \tag{44}$$
$$\mathcal{L}_{3i} = B_{1i}^T\bar{P}_i + B_{2i}^TP_{2i}^T + \bar{Y}_i^TA_{3i} - C_{1i}, \tag{45}$$
$$\mathcal{L}_{4i} = B_{2i}^T\bar{X}_i + \bar{Y}_i^TA_{4i} - C_{2i}, \tag{46}$$

where

$$E_L = \begin{bmatrix} I_r \\ 0\end{bmatrix}, R = \mathcal{M}\begin{bmatrix} 0 & I_{n-r}\end{bmatrix}, S = \begin{bmatrix} 0 \\ I_{n-r}\end{bmatrix}\mathcal{N}, \tag{47}$$
$$\bar{X}_i = \mathcal{M}^TX_i\mathcal{N}^T, \bar{Y}_i = \mathcal{M}^TY_i \tag{48}$$

and $\mathcal{M}, \mathcal{N}$ are arbitrary invertible matrices. Let us suppose $\bar{X}_i = -1/2A_{4i}^{-T}\mathcal{Q}$, where $\mathcal{Q}$ is arbitrary positive definite matrix. Then, we can apply the Schur complement to the inequality (42) and obtain the following result:

$$0 > \begin{bmatrix} sym\{\bar{P}_iA_i + \sum_{j=1}^{N}\pi_{ij}\bar{P}_j\} & (*) \\ B_{1i}^T\bar{P}_i - \bar{C}_i & -sym\{D_i\}\end{bmatrix} + \begin{bmatrix}\mathcal{L}_{2i}^T \\ \mathcal{L}_{4i}\end{bmatrix}\mathcal{Q}^{-1}\begin{bmatrix}\mathcal{L}_{2i}^T \\ \mathcal{L}_{4i}\end{bmatrix}^T \tag{49}$$

$$= \Theta_i + \begin{bmatrix}\mathcal{T}_{1i} \\ \mathcal{T}_{2i}\end{bmatrix}\mathcal{Q}^{-1}\begin{bmatrix}\mathcal{T}_{1i} \\ \mathcal{T}_{2i}\end{bmatrix}^T, \tag{50}$$

where

$$\mathcal{T}_{1i} = P_{2i}^TA_{4i} + 1/2A_{3i}^TA_{4i}^{-T}\mathcal{Q}, \tag{51}$$
$$\mathcal{T}_{2i} = \bar{Y}_i^TA_{4i} - C_{2i} + 1/2B_{2i}^TA_{4i}^{-T}\mathcal{Q}. \tag{52}$$

The detailed proof for this step is shown in Appendix. Here, we can set the variables such that

$$P_{2i} = -1/2A_{4i}^{-T}\mathcal{Q}A_{4i}^{-1}A_{3i}, \tag{53}$$
$$\bar{Y}_i = A_{4i}^{-T}C_{2i}^T - 1/2A_{4i}^{-T}\mathcal{Q}A_{4i}^{-1}B_{2i}. \tag{54}$$

Then the proposed condition (49) concludes to $0 > \Theta_i$ because the variables (53)-(54) make $\mathcal{T}_{1i} = 0$ and $\mathcal{T}_{2i} = 0$. Therefore, it is clear that there exist feasible solutions P_i, X_i and Y_i which satisfy LMIs (26)-(27). This complete the proof.

Remark 9. The necessary proof starts by representing the singular hybrid system as the equivalent normal hybrid system (37)-(38). Note that there exists $P_i > 0$ which satisfies $0 > \Theta_i$, from Lemma 7. Therefore, if there exist matrices X_i and Y_i which makes the condition (16) be equal to $0 > \Theta_i$, they can be solution of the condition (15)-(16). By obtaining the relation (50), we can always find the solutions.

4. CONCLUSION

This paper proposed the necessary and sufficient condition for the stochastic admissibility and strictly positive realness of the singular hybrid systems. First, the sufficient condition was derived by using a mode-dependent Lyapunov function. Next, for the necessary condition, the authors introduced the positive real lemma for the hybrid system in terms of LMIs. Based on the lemma and solution, we proved that there always exists the feasible solution of the positive real lemma for the singular hybrid systems. Our future work is extending the result to discrete-time singular hybrid systems.

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Appendix A. PROOF FOR THE EQUATION (50)

To show the equation (50), let us start from (49):

$$\begin{bmatrix} (1,1)_A & (*) \\ B_{1i}^T\bar{P}_i - \bar{C}_i & -sym\{D_i\} \end{bmatrix} + \begin{bmatrix} \mathcal{L}_{2i}^T \\ \mathcal{L}_{4i} \end{bmatrix} \mathcal{Q}^{-1} \begin{bmatrix} \mathcal{L}_{2i}^T \\ \mathcal{L}_{4i} \end{bmatrix}^T \quad \text{(A.1)}$$

$$= \Theta_i + \begin{bmatrix} \mathcal{Z}_{1i} & \mathcal{Z}_{2i} \\ \mathcal{Z}_{2i}^T & \mathcal{Z}_{3i} \end{bmatrix}, \quad \text{(A.2)}$$

where

$$(1,1)_A = sym\{\bar{P}_iA_i + \sum_{j=1}^{N}\pi_{ij}\bar{P}_j\}, \quad \text{(A.3)}$$

$$\begin{aligned} \mathcal{Z}_{1i} &= sym\{P_{2i}^TA_{3i}\} \\ &\quad + (P_{2i}^TA_{4i} + A_{3i}^T\bar{X}_i)^T\mathcal{Q}^{-1}(P_{2i}^TA_{4i} + A_{3i}^T\bar{X}_i), \end{aligned} \quad \text{(A.4)}$$

$$\begin{aligned} \mathcal{Z}_{2i} &= (P_{2i}^TA_{4i} + A_{3i}^T\bar{X}_i)^T\mathcal{Q}^{-1}(\bar{X}_i^TB_{2i} + A_{4i}^T\bar{Y}_i - C_{2i}^T) \\ &\quad + P_{2i}^TB_{2i} + A_{3i}^T\bar{Y}_i - A_{3i}^TA_{4i}^{-T}C_{2i}^T, \end{aligned} \quad \text{(A.5)}$$

$$\begin{aligned} \mathcal{Z}_{3i} &= -sym\{C_{2i}A_{4i}^{-1}B_{2i} + \bar{Y}_i^TB_{2i}\} \\ &\quad + \{(B_{2i}^T\bar{X}_i + \bar{Y}_i^TA_{4i} - C_{2i}) \\ &\quad \times\mathcal{Q}^{-1}(B_{2i}^T\bar{X}_i + \bar{Y}_i^TA_{4i} - C_{2i})^{-1}\}. \end{aligned} \quad \text{(A.6)}$$

The above terms can be reformulated as follows:

$$\begin{aligned} \mathcal{Z}_{1i} &= sym\{P_{2i}^TA_{4i}(A_{4i}^{-1} + \mathcal{Q}^{-1}\bar{X}_i^T)A_{3i}\} \\ &\quad + P_{2i}^TA_{4i}\mathcal{Q}^{-1}A_{4i}^TP_{2i} + A_{3i}^T\bar{X}_i\mathcal{Q}^{-1}\bar{X}_i^TA_{3i}. \end{aligned} \quad \text{(A.7)}$$

Note that the equation $A_{4i}^T\bar{X}_i + \bar{X}_i^TA_{4i} = -\mathcal{Q}$ holds the relation

$$A_{4i}^{-1} + \mathcal{Q}^{-1}\bar{X}_i^T = -\mathcal{Q}^{-1}A_{4i}^T\bar{X}_iA_{4i}^{-1}. \quad \text{(A.8)}$$

Thus,

$$\begin{aligned} \mathcal{Z}_{1i} &= -sym\{P_{2i}^TA_{4i}\mathcal{Q}^{-1}A_{4i}^T\bar{X}_iA_{4i}^{-1}A_{3i}\} \\ &\quad + P_{2i}^TA_{4i}\mathcal{Q}^{-1}A_{4i}^TP_{2i} + A_{3i}\bar{X}_i\mathcal{Q}^{-1}\bar{X}_i^TA_{3i} \quad \text{(A.9)} \\ &= \{(P_{2i}^TA_{4i} - A_{3i}^TA_{4i}^{-T}\bar{X}_i^TA_{4i})\mathcal{Q}^{-1} \\ &\quad \times(P_{2i}^TA_{4i} - A_{3i}^TA_{4i}^{-T}\bar{X}_i^TA_{4i})^T\} \\ &\quad + \{A_{3i}^TA_{4i}^{-T}(A_{4i}^T\bar{X}_i\mathcal{Q}^{-1}\bar{X}_i^TA_{4i} - \bar{X}_i^TA_{4i}\mathcal{Q}^{-1}A_{4i}^T\bar{X}_i) \\ &\quad \times A_{4i}^{-1}A_{3i}\}. \end{aligned} \quad \text{(A.10)}$$

The term $\mathcal{Z}_{2i}$ is also calculated as follows:

$$\begin{aligned} \mathcal{Z}_{2i} &= P_{2i}^TA_{4i}\mathcal{Q}^{-1}\bar{X}_i^TB_{2i} + P_{2i}^TA_{4i}\mathcal{Q}^{-1}(A_{4i}^T\bar{Y}_i - C_{2i}^T) \\ &\quad + A_{3i}^T\bar{X}_i\mathcal{Q}^{-1}(A_{4i}^T\bar{Y}_i - C_{2i}^T) + A_{3i}^T\bar{X}_i\mathcal{Q}^{-1}\bar{X}_i^TB_{2i} \\ &\quad + P_{2i}^TB_{2i} + A_{3i}^TA_{4i}^{-T}(A_{4i}^T\bar{Y}_i - C_{2i}^T) \quad \text{(A.11)} \\ &= P_{2i}^TA_{4i}(A_{4i}^{-1} + \mathcal{Q}^{-1}\bar{X}_i^T)B_{2i} + A_{3i}^T\bar{X}_i\mathcal{Q}^{-1}\bar{X}_i^TB_{2i} \\ &\quad + A_{3i}^T(A_{4i}^{-T} + \bar{X}_i\mathcal{Q}^{-1})(A_{4i}^T\bar{Y}_i - C_{2i}^T) \\ &\quad + P_{2i}^TA_{4i}\mathcal{Q}^{-1}(A_{4i}^T\bar{Y}_i - C_{2i}^T). \end{aligned} \quad \text{(A.12)}$$

By substituting the condition (A.8) into above equation, we can obtain the following relation:

$$\begin{aligned} \mathcal{Z}_{2i} &= P_{2i}^TA_{4i}\mathcal{Q}^{-1}(A_{4i}^T\bar{Y}_i - C_{2i}^T) + A_{3i}^T\bar{X}_i\mathcal{Q}^{-1}\bar{X}_i^TB_{2i} \\ &\quad - A_{3i}^TA_{4i}^{-T}\bar{X}_i^TA_{4i}\mathcal{Q}^{-1}(A_{4i}^T\bar{Y}_i - C_{2i}^T) \\ &\quad - P_{2i}A_{4i}\mathcal{Q}^{-1}A_{4i}^T\bar{X}_iA_{4i}^{-1}B_{2i} \quad \text{(A.13)} \\ &= \{(P_{2i}^TA_{4i} - A_{3i}^TA_{4i}^{-T}\bar{X}_i^TA_{4i})\mathcal{Q}^{-1} \\ &\quad \times(A_{4i}^T\bar{Y}_i - C_{2i} - A_{4i}^T\bar{X}_iA_{4i}^{-1}B_{2i})\} \\ &\quad + \{A_{3i}^TA_{4i}^{-T}(A_{4i}^T\bar{X}_i\mathcal{Q}^{-1}\bar{X}_i^TA_{4i} - \bar{X}_i^TA_{4i}\mathcal{Q}^{-1}A_{4i}^T\bar{X}_i) \\ &\quad \times A_{4i}^{-1}B_{2i}\}. \end{aligned} \quad \text{(A.14)}$$

The term $\mathcal{Z}_{3i}$ also reformulated as follows:

$$\begin{aligned} \mathcal{Z}_{3i} &= sym\{B_{2i}^T(\bar{X}_i\mathcal{Q}^{-1} + A_{4i}^{-T})(A_{4i}^{-T}\bar{Y}_i - C_{2i}^T)\} \\ &\quad + B_{2i}^T\bar{X}_i\mathcal{Q}^{-1}\bar{X}_i^TB_{2i} \\ &\quad + (A_{4i}^T\bar{Y}_i - C_{2i}^T)^T\mathcal{Q}^{-1}(A_{4i}^T\bar{Y}_i - C_{2i}^T) \quad \text{(A.15)} \\ &= -sym\{B_{2i}^TA_{4i}^{-T}\bar{X}_i^TA_{4i}\mathcal{Q}^{-1}(A_{4i}^{-T}\bar{Y}_i - C_{2i}^T)\} \\ &\quad + B_{2i}^T\bar{X}_i\mathcal{Q}^{-1}\bar{X}_i^TB_{2i} \\ &\quad + (A_{4i}^T\bar{Y}_i - C_{2i}^T)^T\mathcal{Q}^{-1}(A_{4i}^T\bar{Y}_i - C_{2i}^T) \quad \text{(A.16)} \\ &= \{(\bar{Y}_i^TA_{4i} - C_{2i} - B_{2i}^TA_{4i}^{-T}\bar{X}_i^TA_{4i})\mathcal{Q}^{-1} \\ &\quad \times(\bar{Y}_i^TA_{4i} - C_{2i} - B_{2i}^TA_{4i}^{-T}\bar{X}_i^TA_{4i})\} \\ &\quad + \{B_{2i}^TA_{4i}^{-T}(A_{4i}^T\bar{X}_i\mathcal{Q}^{-1}\bar{X}_iA_{4i} \\ &\quad -\bar{X}_i^TA_{4i}\mathcal{Q}^{-1}A_{4i}^T\bar{X}_i)A_{4i}^{-1}B_{2i}\} \end{aligned} \quad \text{(A.17)}$$

Remark that $A_{4i}^T\bar{X}_i\mathcal{Q}^{-1}\bar{X}_i^TA_{4i} - \bar{X}_i^TA_{4i}\mathcal{Q}^{-1}A_{4i}^T\bar{X}_i = 0$ from Lemma 10. Therefore, the term $\begin{bmatrix} \mathcal{Z}_{1i} & \mathcal{Z}_{2i} \\ \mathcal{Z}_{2i}^T & \mathcal{Z}_{3i} \end{bmatrix}$ conclude to

$$\begin{bmatrix} \mathcal{T}_{1i} \\ \mathcal{T}_{2i} \end{bmatrix} \mathcal{Q}^{-1} \begin{bmatrix} \mathcal{T}_{1i} \\ \mathcal{T}_{2i} \end{bmatrix}^T \quad \text{(A.18)}$$

in (50) by substituting $\bar{X}_i = -1/2A_{4i}^{-T}\mathcal{Q}$. This completes the proof.

Lemma 10. For invertible matrix A, the following relation is true:

$$A(A + A^T)^{-1}A^T = A^T(A + A^T)^{-1}A. \quad \text{(A.19)}$$

Proof.

$$\begin{aligned} A(A + A^T)^{-1}A^T &= \{A^{-T}(A + A^T)A^{-1}\}^{-1} && \text{(A.20)} \\ &= \{A^{-T} + A^{-1}\}^{-1} && \text{(A.21)} \\ &= \{A^{-1}(A + A^T)A^{-T}\}^{-1} && \text{(A.22)} \\ &= A^T(A + A^T)^{-1}A. && \text{(A.23)} \end{aligned}$$