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Optimal Regulators for Nonlinear Systems with Incompatible State and Input Cost Functions ⋆

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Abstract: When a desired state is different from the state at which input cost becomes zero, a naive application of the optimal control methodology may lead an ill-posed problem. In this study, we propose a new method where the input cost is slightly changed so that the optimal control problem is well defined. The method can realize an energy-efficient control, which considers the actual energy consumption. We also confirm the effectiveness of the proposed method via a case study of an example.

Keywords: optimal control, input offset, energy-saving control, stable-manifold method, non-quadratic cost functions.

1. INTRODUCTION

Optimal regulators are widely used to stabilize various dynamical systems. In particular, Linear-Quadratic (LQ) optimal regulators method as well as Linear-Quadratic-Integral (LQI) control method is the most standard technique for the stabilization of systems described by linear state equations. Numerical solutions of Hamilton-Jacobi equations for the optimal regulators of low-dimensional nonlinear systems can be also obtained easily by stable-manifold methods (e.g. Yamashita and Shima (1998); Sakamoto and van der Schaft (2008)).

It is often explained that an energy-conservative control can be realized by regarding the input cost term in the optimal criterion as physical energy consumption. However, in most actual control systems, matrices in the performance criterion are used solely for adjusting parameters, and controller design is focused only on the excellent characteristics of the optimal regulator — asymptotic stabilization, sector margin, and robustness. In these cases, the optimality itself is neglected. On the other hand, in recent years, the demand for energy saving control has increased, and it is a natural requirement to realize energy saving by using a widely used optimal regulator. Control technology contributes to energy conservation via transient-response improvement and selection of redundant actuators. This study focuses on the former aspect. Transient response is not related to energy consumption at a steady point but considering that much energy in motor and engine operation is consumed during acceleration, improvement of transient response seems to have great significance for energy saving.

Unfortunately, in some cases, physical energy-consumption functions cannot be adopted as the input-cost terms in the optimal control. This problem occurs for the systems that consume energy even at their steady point and momentarily allow smaller inputs than the steady inputs. Such systems include most chemical plant systems, mechanical systems moving at steady state, and mechanical system canceling gravity forces at the steady points. If the input cost at the steady state is not zero, the time integral of the cost function diverges and therefore the optimal control problem becomes ill-posed.

To resolve the ill-posedness, the economic model predictive control (economic MPC) approach (e.g. Ellis et al. (2014)) or usage of discount rate may be adopted, but these methods have some problems on the asymptotical stabilization to a prespecified point. The introduction of the discount rate makes the asymptotic stability of the closed-loop systems uncertain. To ensure the asymptotic stability under the classical MPC methods, we must solve a static optimal problem offline and add an adequate terminate state cost function or a terminal-state constraint. Some new economics MPC methods resolves these problem of the stability by investigating turnpike properties (Grüne (2013)) or using dissipativity (Müller et al. (2015)). Moreover, when an ill-posed infinite-horizon optimal control problem is converted to an MPC problem or an optimal control problem with discount rate, the state of the closed-loop system does not converge to the minimizing point of the state-cost function. Instead, it converges to a compromised point between the input and state costs. Artstein and Leizarowitz (1985) also proposed the concept of overtaking criterion to solve this ill-posedness. For the linear systems, the overtaking optimal control coincides with the limit of finite-horizon optimal control when the length of the control interval goes to infinity. However, the steady state of the closed-loop system of the overtaking optimal control is also a compromised point between the input and state costs.

In some economic problem settings, both the state and input cost terms have the same dimension, which repre-

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Table 1. Comparison of the characteristics between the methods for resolving ill-posed optimal control problems.

	Economic MPC	Discount Rate	Overtaking Criterion	Proposed Method
Guaranteed stability	✓ (guaranteed for some methods)		✓	✓
Applicable to nonlinear systems	✓	✓		✓
Online computational cost	Large	Small	Small	Small
Keeping original cost functional	?	?	✓	?
$x(+\infty)$	Compromised point	Compromised point	Compromised point	Minimizer of the state cost

sents the money per a unit time. For such problems, the convergence to the compromised point between the state and input costs is a preferred characteristic, where the static optimization for finding the best operating point is performed simultaneously. However, in many cases, the designer of the control system wants to assign the final steady state to a specified point and makes the minimizing point of the state cost the specified steady state. The economic MPC method, discount-rate method, and the overtaking optimal control method cannot fulfill this requirement.

In this work, we propose a new method to recover the well-posedness of optimal control having incompatible state and input cost functions. Our method slightly modifies the input-cost term so that the modified optimal control problem becomes well-posed. The closed-loop system of the proposed method is globally asymptotically stable, and the final steady state becomes the minimizing point of the state-cost function. The feedback gain becomes asymmetric even when the controlled object is a linear system, which realizes energy-efficient control. The modified input cost approaches to a positive-definite function, which is not quadratic function, when the state is near the steady point, and it gets closer to the original energy-consumption function minus the steady input as the state goes away from the steady state. The proposed modification is an almost minimal change to ensure the asymptotic stability and the well-posedness of the optimal-control problem.

A comparison among the MPC method, discount-rate method, overtaking criterion, and the proposed method is summarized in Table 1.

2. PROBLEM STATEMENT

We consider a nonlinear system

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{u} \tag{1}$$

as an controlled object, where $\tilde{x}$ ($\in \mathbb{R}^n$) denotes the state vector, $\tilde{u}$ ($\in \mathbb{R}$) the input variable, and $\tilde{f}(\tilde{x})$ and $\tilde{g}(\tilde{x})$ are smooth vector fields. We mainly consider the single-input cases, but in the latter section we will extend our method to systems with multiple inputs.

It is implicitly assumed that the input variable $\tilde{u}$ reflects a physical quantity, and that the energy consumption becomes zero when $\tilde{u} = 0$. For simplicity, we assume that the energy-consumption function can be expressed by a quadratic function of $\tilde{u}$ as $r\tilde{u}^2$. Let $\tilde{x}_0$ ($\neq 0$) denote a specified steady state, and $\tilde{u}_0$ be the nonzero steady input which corresponds to $\tilde{x}_0$. Therefore, the pair $(\tilde{x}_0, \tilde{u}_0)$ must satisfy the following relation:

$$\tilde{f}(\tilde{x}_0) + \tilde{g}(\tilde{x}_0)\tilde{u}_0 = 0. \tag{2}$$

We cosider a performance criterion

$$J_{\text{orig}} = \int_0^\infty L_0(\tilde{x} - \tilde{x}_0) + r\tilde{u}^2 \, dt, \tag{3}$$

where the function $L_0(\cdot)$ for the state cost is positive definite with respect to $\tilde{x} - \tilde{x}_0$. The integrand in the above equation needs to be asymptotic to zero as t goes to infinity, for the integrability. However, since $\tilde{u}_0$ is a nonzero constant, there exists no steady state where the state and input costs become zero simultaneously, and hence the optimal control problem with the cost functional (3) becomes ill-posed.

To resolve this ill-posedness for linear systems, Artstein and Leizarowitz (1985) proposed the overtaking criterion, and obtained an overtaking optimal control law. For linear systems, an overtaking optimal control can be obtained as the limit of the optimal solution for a finite-horizon optimal control problem as the horizon length goes to infinity. Specifically, for a linear system with an ill-posed performance criterion

$$\dot{x} = Ax + Bu,$$
$$J_{\text{lin}} = \int_0^\infty (x - x_0)^\top Q(x - x_0) + r^\top Ru \, dt,$$

the overtaking optimal control law [1] can be obtained by

$$u = -R^{-1}B^\top(Px - g),$$

where P denote a positive-definite solution of a Riccati equation

$$PA + A^\top P - PBR^{-1}B^\top P + Q = 0,$$

and the offset term is defined as

$$g = (A - BR^{-1}B^\top P)^{-\top}Qx_0.$$

The control law consist of a fixed gain feedback with an offset. However, the overtaking optimal solution does not satisfy $x(t) \to x_0$ as $t \to \infty$. Instead, the final steady state becomes a compromised point between the state and input cost terms. In the overtaking criterion, the restriction (2) is not necessary and the steady state does not coincides with $P^{-1}g$ generally. Therefore, specifying the steady state to some point is not easy in general.

In actual control problems, it is usually required that the steady state is designed as a user-specified point. Hence, the convergence to the compromised point is not preferred, and it is desired that the specified steady state can be assigned as the minimizing point of the state cost function. Moreover, considering the energy efficiency, the feedback gain should be changed between a case where a larger input energy is required and a case where less energy is required, in the transient response. Therefore, in the linear systems,

[1] Artstein and Leizarowitz (1985) obtained the solution for time-variant x_0, but for simplicity we only show the case of constant x_0.

an asymmetric gain is expected to be energy efficient, but the overtaking control law has a constant gain.

This study aims to develop a method that modifies the ill-posed optimal control problem to be well-posed, which satisfies the following properties:

(a) asymptotical stability of the closed-loop system,
(b) easy assignment of the stationary state to a user-specified point $\tilde{x}_0$ satisfying (2), and
(c) realization of energy-efficient asymmetric gain.

To realize the above item **(c)**, the modification of the input-cost term should be as little as possible. In our method, the modified cost term depends on the value of the state vector. When the modification is unnecessary, we adopt the original energy-consumption function minus an offset as the input cost.

3. CONSTRUCTION OF ERROR SYSTEM AND REMOVAL OF COST OFFSET

As well as many control studies, we regard the desired stationary state as the origin by using a coordinate translation. We define the state and input errors from $(\tilde{x}_0, \tilde{u}_0)$ as

$$x = \tilde{x} - \tilde{x}_0, \quad u = \tilde{u} - \tilde{u}_0.$$

The controlled object (1) and the cost functional (3) are rewritten as

$$\dot{x} = f(x) + g(x)u, \tag{4}$$

$$J_{\text{orig}} = \int_0^\infty L_0(x) + r(u + \tilde{u}_0)^2 \, dt, \tag{5}$$

where $f(x) = \tilde{f}(x+\tilde{x}_0)+\tilde{g}(x+\tilde{x}_0)\tilde{u}_0$ and $g(x) = \tilde{g}(x+\tilde{x}_0)$. From the condition (2), we can show that $f(0) = 0$. We call (4) and (5) the error system and the original cost functional, respectively.

By expanding the input cost of the performance criterion (5), we obtain

$$r(u + \tilde{u}_0)^2 = ru^2 + 2ru\tilde{u}_0 + r\tilde{u}_0^2. \tag{6}$$

In many control systems, only the first term of (6) is adopted, but in this way the actual energy consumption is not reflected by the new cost function. The third term $r\tilde{u}_0^2$ of (6) is a constant, which is the main cause of the divergence of the integral in (5). The third term is independent from the input and state variables, and removing this term does not affect the optimal solution. Therefore, we consider a new performance criterion

$$J_{\text{shift}} = \int_0^\infty L_0(x) + ru(u + 2\tilde{u}_0) \, dt, \tag{7}$$

where the third term of (6) is removed.

Unfortunately, the optimal control problems with the new cost functionals (7) remain ill-posed. The existence of an input value that makes the input-cost term $ru(u + 2\tilde{u}_0)$ negative induces the ill-posedness.

4. NECESSITY OF POSITIVE-DEFINITENESS

This section reviews why the state and input cost functions are assumed to be positive-definite in the optimal control theory.

We consider a general performance criterion

$$J_{\text{gen}} = \int_0^\infty L_0(x) + N(u, x) \, dt \tag{8}$$

for the nonlinear system (4). The positive definiteness of the state cost function $L_0(x)$ is assumed. If a smooth value function

$$V(x_0) = \inf_{u(\cdot)} J_{\text{gen}}|_{x(0)=x_0}$$

exists, it satsfies the Hamilton–Jacobi–Bellman (HJB) partial differention equation

$$L_0(x) + \frac{\partial V}{\partial x} f(x) + \inf_{u(\cdot)} \left[N(u, x) + \frac{\partial V}{\partial x} g(x)u \right] = 0. \tag{9}$$

We assume that a minimal value in the calculation of the infimum value in (9) exists. The optimal control law can be derived from the value function $V(x)$ as

$$u = u^*(x) = \operatorname*{argmin}_{u(\cdot)} \left[N(u, x) + \frac{\partial V}{\partial x} g(x)u \right]. \tag{10}$$

We assume that the input cost term $N(u, x)$ is a continuous function and strictly convex with respect to u for any fixed x, and that $N(u, x) \to \pm\infty$ as $u \to \pm\infty$. Then, the optimal feedback law $u^*(x)$ becomes a continuous function with respect to x. The time derivative of the value function under the optimal control law can be calculated as

$$\dot{V} = -L_0(x) - N(u^*(x), x),$$

using the HJB equation (9). If $N(u, x)$ is positive definite with respect to u for any x, the value function $V(x)$ becomes a positive definite function, and the time-derivative of $V(x)$ is negative definite. Hence, under the assumption of the positive-definiteness of $N(u, x)$, the value function $V(x)$ can be regarded as a Lyapunov function, and the asymptotic stability of the closed-loop system can be proven.

To ensure the asymptotic stability, the positive definiteness of the input cost is assumed in most cases. However, strictly speaking, it is not required that $N(u, x)$ is positive definite with respect to u. If $L(x) + N(u^*(x), x)$ is positive definite with respect to x, i.e., if

$$L(x) + N(u^*(x), x) \begin{cases} > 0 & (x \neq 0) \\ = 0 & (x = 0), \end{cases} \tag{11}$$

the asymptotic stability can be guaranteed. The condition (11) is weaker than the positive definiteness of $N(u, x)$ and $L_0(x)$ but cannot be verified without solving the HJB equation (9). Therefore, the condition (11) cannot be ensured a priori. In this study, we adopt a sufficient condition of (11). Namely, we assume that $L(u, x) = L_0(x)+N(u, x)$ is positive definite with respect to both x and u. Under this assumption, $V(x)$ and $\dot{V}(x)$ are positive definite and negative definite, respectively. This assumption indicates that $\boldsymbol{\min_u N(u, x)}$ **may be negative** when $L_0(x)$ has a sufficiently large value. Meanwhile, $N(u, 0)$, which is the input-cost function at $x = 0$, should be positive definite with respect to u.

In the next section, we construct a new input-cost term $N(u, x)$ so that $L(u, x) = L_0(x) + N(u, x)$ is positive definite with respect to both x and u. Notice that the dependency of the input-cost term on the state makes it possible to relax the positive-definiteness condition.

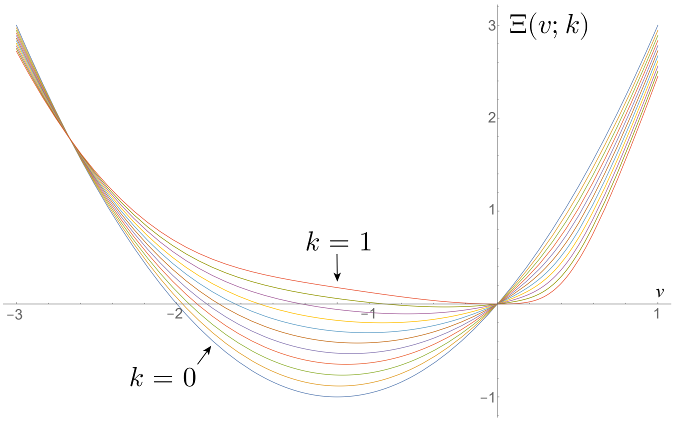

Fig. 1. Shapes of $\Xi(v;k)$.

5. DESIGN OF MODIFIED INPUT-COST FUNCTION

In this section, we will design a new input-cost function. The modified function should approximate the original cost function and satisfy the positive definiteness of $L(u,x)$.

First, we express the original function as

$$ru(u+2\tilde{u}_0) = r\tilde{u}_0^2 \Xi_0(u/\tilde{u}_0),$$
$$\Xi_0(v) = v(v+2).$$

Notice that $\Xi_0(\cdot)$ includes neither r nor $\tilde{u}_0$. Hence, $\Xi_0(v) = v(v+2)$ is a normalized expression of the original cost function. In the following, we modify $\Xi_0(v)$ to a new function, which depends on x and v.

When $x=0$, the new function should be positive definite with respect to v. This case is under the strictest condition, because no negative value is allowed. Let $\Xi_1(v)$ denote the modified function at $x=0$. The function $\Xi_1(v)$ should be close to $\Xi_0(v)$, positive definite, and strictly convex. Moreover, to apply the stable-manifold method for solving HJB equation, $\Xi_1(v)$ is preferred to be differentiable twice. In this paper, we propose to choose the function as

$$\Xi_1(v) = \begin{cases} v\left(v+2-\dfrac{3v+8}{16v^3+4}\right) & (v \geq 0) \\ v\left(v+2+\dfrac{4(3v+8)}{v^3/4-4}\right) & (v<0). \end{cases}$$

The proposed $\Xi_1(v)$ is a twice-differentiable, strictly-convex, and positive-definite function. $\Xi_1(v)$ approximates $\Xi_0(v)$ when $|v|$ is sufficiently large, because

$$\lim_{v\to+\infty} \Xi_0(v) - \Xi_1(v) = 0, \quad \lim_{v\to-\infty} \Xi_0(v) - \Xi_1(v) = 0.$$

The curve in Fig. 1 for $k=1$ indicates the shape of $\Xi_1(v)$, while the curve for $k=0$ is for $\Xi_0(v)$. Of course, other $\Xi_1(v)$ choices are possible, but the proposed $\Xi_1(v)$ satisfies the requirements well.

We linearly interpolate the two functions $\Xi_0(v)$ and $\Xi_1(v)$ by a parameter k $(0 \leq k \leq 1)$ as

$$\Xi(v;k) = (1-k)\Xi_0(v) + k\Xi_1(v)$$

(See Fig. 1). Since any linear combination of two strictly convex functions is always strictly convex, the synthesized function $\Xi(v;k)$ is also strictly convex for any k $(0 \leq k \leq 1)$. From Fig. 2, we can confirm that

$$\min_v \Xi(v;k) \geq k-1 \qquad (0 \leq k \leq 1)$$

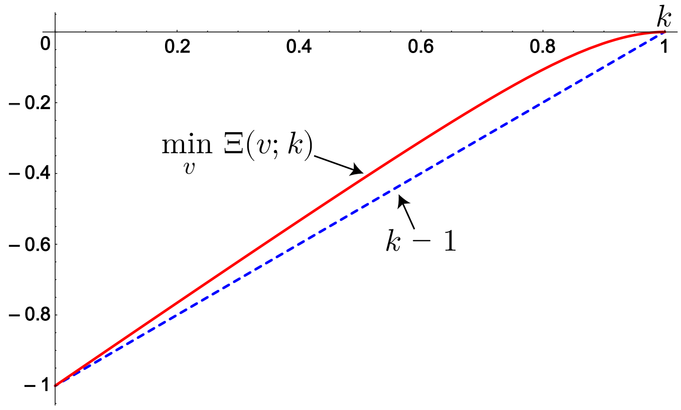

Fig. 2. Minimal value of $\Xi(v;k)$ vs. k.

holds. Hence, to fullfil $L(u,x) = L_0(x) + N(u,x) > 0$ for $(x,u) \neq 0$, we should choose k as a function of x satisfying

$$k(x) - 1 > -\frac{L_0(x)}{r\tilde{u}_0^2}, \quad x \neq 0 \tag{12}$$

and set the input cost to

$$N(u,x) = r\tilde{u}_0^2\, \Xi\left(\frac{u}{\tilde{u}_0}; k(x)\right). \tag{13}$$

In this paper, we propose a method for choosing $k(x)$ as

$$k(x) = \max\left(0, 1 - \frac{L_0(x)}{\eta r\tilde{u}_0^2}\right), \tag{14}$$

which satistfies (12) and $0 \leq k(x) \leq 1$, where η is a positive constant greater than 1. The input cost is determined by (13), and the modified cost functional is given by (8), (13), and (14). The modified function $\Xi(v, k(x))$ is not an even function with respect to v, which makes the feedback gain asymmetric. Hence, the inputs are selected by (10) taking into account actual energy consumption so that energy-efficient control can be achieved during transient responses.

6. EXAMPLE

In this section, we apply the proposed method to an example.

In order to show the effect of this method, we consider a linear system

$$\dot{x} = Ax + bu = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

as a controlled object, and it will be clarified that the generated input sequence behaves like a nonlinear system and increases energy efficiency. An ill-posed performance criterion

$$J_{\text{shift}} = \int_0^\infty x^\top Qx + u(u+2)\, dt,$$
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{15}$$

which should be minimized, is given. We apply the proposed method to this problem and construct an optimal regulator for the modified cost functions. The parameter is chosen as $\eta = 3$. Since the new input cost is not quadratic, we must solve an HJB equation generally, and the derived control law becomes nonlinear. The HJB equation (9) and

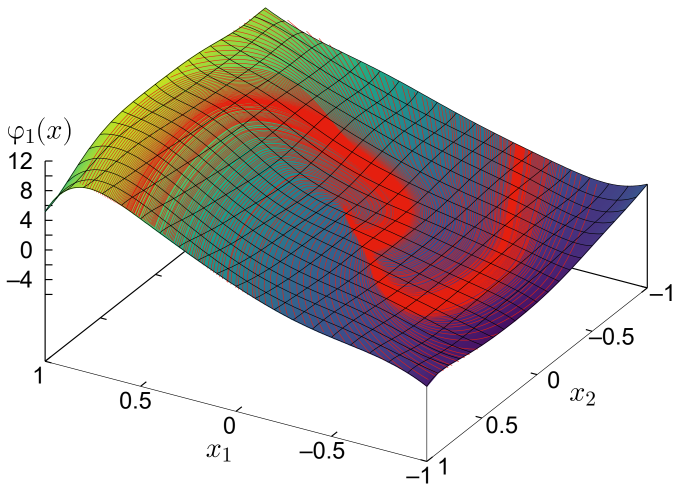

Fig. 3. The 1st costate of the stable manifold of Hamiltonian system.

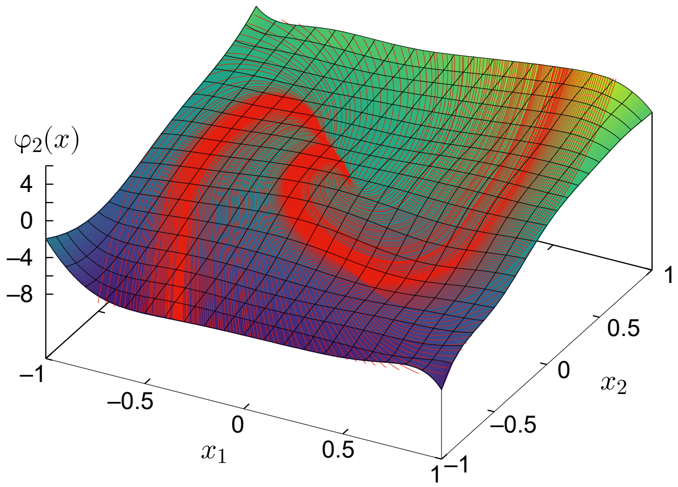

Fig. 4. The 2nd costate of the stable manifold of Hamiltonian system.

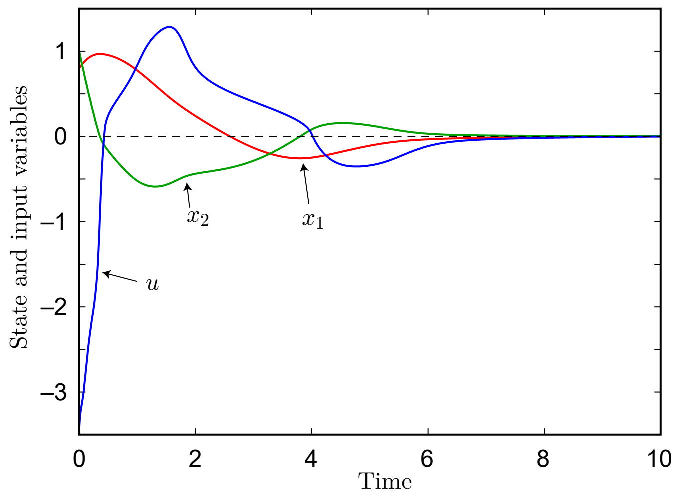

Fig. 5. Time responses of state and input variables of simulation result for the proposed control law.

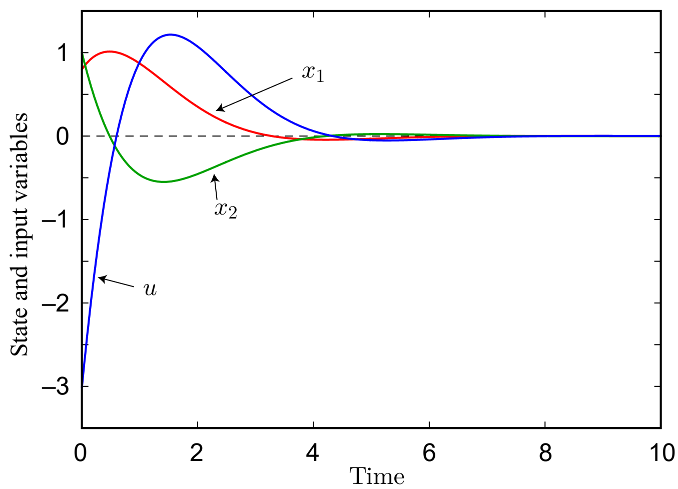

Fig. 6. Time responses of state and input variables of simulation result for LQ regulator.

the control law (10) includes a static input-minimizing sub-problem. However, the sub-problem has a convex optimized function on the one-dimensional space, and therefore can be solved efficiently and quickly online. We can also use a table-lookup method for the online usage of the solution of the sub-problem, because the minimizing point only depends on the values of $L_gV(x)$ $(=(\partial V(x)/\partial x)g(x))$ and $k(x)$.

In this section the HJB equation (9) is solved by a stable manifold method (Yamashita and Shima (1998)). Since $Q > 0$ and the system is controllable, the corresponding Hamiltonian system

$$\begin{aligned}\dot{x} &= f(x) + g(x)\bar{u}(x,p)\\ \dot{p} &= -\left(\frac{\partial L_0(x)}{\partial x}\right)^\top\\ &\quad - \left(\frac{\partial N(u,x) + p^\top(f(x)+g(x)u)}{\partial x}\right)^\top\Bigg|_{u=\bar{u}(x,p)}\\ \bar{u}(x,p) &= \operatorname*{argmin}_u\{N(x,u) + p^\top g(x)u\}\end{aligned}$$

has two-dimensional stable manifold. The stable manifold can be expressed as $p = \varphi(x)$ assuming that it is projective along the direction of x. It is well known that $\varphi(x)^\top$ is integrable, and

$$\varphi(x) = \left(\frac{\partial V(x)}{\partial x}\right)^\top.$$

holds. Therefore, by calculating many trajectories on the stable manifold numerically offline and interpolating them, we can obtain the value of $\partial V(x)/\partial x$. The obtained stable manifold for this example is illustrated by Figs. 3 and 4. Fig. 3 shows the interpolated $\varphi_1(x)$ and sample trajectories, while Fig. 4 is for $\varphi_2(x)$. In these figures, red curves indicate trajectories on the stable manifold of the Hamiltonian system. The surfaces in Figs. 3 and 4 are the results of a Bézier interpolation of these red curves, where the 17th-order (per one variable) Bernstein basis polynomials are used.

We make a simulation for the constructed control law, where the value of $\partial V/\partial x$ is obtained by the interpolated surfaces, and the static minimizing problem in (10) is solved online. As a comparison, a simulation using linear-quadratic regulator (LQR), where the cost function $u(u+$

Table 2. Comparison of the control performance and the consumed energy between the proposed method and LQ regulator.

	Proposed Method	LQR
$\int x^\top Qx\,dt$	1.857	1.913
$\int u(u+2)\,dt$	5.600	6.509
J_{shift}	7.457	8.422

2) in (15) is replaced by a quadratic cost u^2, is also performed.

Fig. 5 shows a simulation result for the proposed method with an initial condition $x(0) = (0.8, 1)^\top$. Similarly, Fig. 6 shows a simulation result for the LQR, with the same initial state as Fig. 5. Figs. 5 and 6 indicate the time-responses of the state variables and the inputs. The state for the proposed controller converges to the origin and the asymptotical stability can be shown. The initial input response of Fig. 5 has a negative large value, which is a natural energy-saving behavior. Notice that the input cost $u(u + 2)$ has the same value for both $u = 1$ and $u = -3$. During the situation when a negative input is required, a somewhat large feedback gain is not energy-wasteful, and the larger feedback gain can speed up the state convergence. Moreover, the input near $t = 5$ has larger negative values than exponential behavior in Fig. 6, which is also the intended behavior.

We compare the state and input costs of the simulation for the proposed method with those for LQR. Table 2 summarizes the costs for both the methods. The input cost for the proposed method is smaller than that for LQR, which shows the energy efficiency of the proposed method. Moreover, we can see that the state cost for the proposed controller is less than for LQR. This result is due to the aggressive use of relatively large negative inputs so that the proposed controller improves convergence speed.

In conclusion, the proposed method can achieve an energy-efficient behavior for this example via the asymmetric gain.

7. EXTENSION TO MULTI-INPUT SYSTEMS

We can extend our method to nonlinear systems with multiple inputs.

Consider a system

$$\dot{x} = f(x) + g(x)u = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m, \quad (16)$$

where $u = (u_1, \ldots, u_m)$ $(\in \mathbb{R}^m)$ is an input vector. The objective functional

$$J_{\text{shift}} = \int_0^\infty L_0(x) + u^\top R(u + 2\tilde{u}_0)\,dt \quad (17)$$

is considered here, where R is a positive-definite matrix and $\tilde{u}_0$ $(\in \mathbb{R}^m)$ is an offset. The matrix R can be decomposed as

$$R = S^\top S, \quad S = \Gamma R^{1/2},$$

where Γ is an orthogonal matrix, i.e., $\Gamma^\top\Gamma = I$, which will be determined later. Let α denote $R^{1/2}\tilde{u}_0$, and let $\bar{\alpha}$ be the normalized α, namely $\bar{\alpha} = (1/|\alpha|)\alpha$. We choose a matrix Γ the first row of which coincides with $\bar{\alpha}^\top$. Since Γ is an orthogonal matrix, the other rows annihilate $\bar{\alpha}$, i.e.,

$$S\tilde{u}_0 = |\alpha|\Gamma\bar{\alpha} = |\alpha|e_1,$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^m.$$

Consider an input-coordinate transformation

$$\zeta = \frac{1}{|\alpha|}Su.$$

The input-cost function can be expressed by ζ as

$$u^\top R(u + \tilde{u}_0) = u^\top S^\top S(u + 2\tilde{u}_0) = |\alpha|^2\zeta^\top(\zeta + 2e_1).$$

Hence, the input cost includes a linear term of ζ_1, where $\zeta = (\zeta_1, \ldots, \zeta_m)^\top$, and the rest becomes a quadratic form with respect to ζ. Namely,

$$\begin{aligned} u^\top R(u + \tilde{u}_0) &= |\alpha|^2\{\zeta_1(\zeta_1 + 2) + \zeta_2^2 + \cdots + \zeta_m^2\} \\ &= |\alpha|^2\{\Xi_0(\zeta_1) + \zeta_r^\top\zeta_r\} \end{aligned}$$

holds, where $\zeta_r = (\zeta_2, \ldots, \zeta_m)^\top$.

Modification of the input-cost function is applied to only the term of ζ_1. As for the single-input cases, $\Xi_0(\zeta_1)$ is replaced by $\Xi(\zeta_1; k(x))$, where $k(x)$ is defined by (14). The modified performance criterion becomes

$$J_{\text{mod}} = \int_0^\infty L_0(x) + |\alpha|^2 \left.\{\Xi(\zeta_1; k(x)) + \zeta_r^\top\zeta_r\}\right|_{\zeta = \frac{1}{|\alpha|}Su} dt.$$

It is obvious from the discussion in the previous sections that the integrand of the new cost functional is positive definite with respect to (x, u).

8. CONCLUSION

When the stationary input corresponding to a prespecified stationary state does not minimize the energy consumption, the proposed method converts the inappropriate optimal-control problem to a well-posed one via a minimal modification of the input cost function. The modified input cost depends on the state also and may have a negative value when the state cost is sufficiently large. Via simulations, we have confirmed that the proposed method contributes energy savings.

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