Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Output Feedback Consensus of Multi-agent Systems with Generalized Uniformly Joint-Connected Switching Networks

Youfeng Su *

* *College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350116, P. R. China. (e-mail: yfsu@fzu.edu.cn).*

Ti-Chung Lee **

** *Department of Electrical Engineering, Minghsin University of Science and Technology, Hsinchu 304, Taiwan, R.O.C. (e-mail: tc1120@ms19.hinet.net).*

Abstract: This paper presents an output feedback synthesis for leader-follower consensus of linear multi-agent systems with switching networks. We first establish uniform global exponential stability (UGES) for a class of cascaded linear switched systems by adopting weak zero-state detectibility (WZSD). Then a distributed output feedback controller is proposed based on the certainty equivalent principle, employing the neighborhood output estimation error only. A generalized uniformly joint-connected condition for switching networks is provided to check WZSD without any dwell-time condition.

Keywords: Multi-agent systems; Switched systems; Output feedback; Generalized uniform joint-connectivity

1. INTRODUCTION

Over the last decade, control of multi-agent systems has been extensively studied by the control community, see the survey Olfati-Saber, Fax, and Murray [2007] and books Qu [2009], Ren and Beard [2008]. Reaching consensus is one of important control targets for various multi-agent systems, see Jadbabaie, Lin, and Morse [2003], Olfati-Saber and Murray [2004]. The network should be employed for describing the interconnection of agents, and plays a key role as it induces a cooperative (or distributed) form of the multi-agent control. One of the significant and interesting networks is the so-called switching network, which is able to formulate the communication failures or changes in practice more effectively than the static network. It is known that the cooperative control over a switching network induces a switched closed-loop system. In particular, when the network is not connected at any time but merely satisfies the so-called uniformly jointly-connected (UJC) condition, the closed-loop embodies essentially a switched system with marginally stable (or even unstable) switching modes. Such a situation makes the closed-loop stability analysis nontrivial even in the field of switched systems.

So far, many efforts have been devoted to the consensus studies over UJC switching networks. One attempt is based on the stochastic matrix analysis. Jadbabaie, Lin, and Morse [2003] first applied this idea for studying the single integrator multi-agent systems with undirect networks. Such analysis was later extended to the direct network case in Hu and Hong [2007], Ren and Beard [2005] and linear system with an input matrix of full row rank in Qin, Gao, and Yu [2014]. Notice that this method requires the dwell-time condition of the switching signal to guarantee a well-defined transition matrix, and may be no easy to be extended to other agent models due to the complexity of transition matrix.

The other attempt is based on Lyapunov analysis. Since the network may by disconnected at some instants, it is usually difficult to construct the strict Lyapunov function for the closed-loop switched system. Instead, one may only find the weak Lyapunov function, and hence, additional tools of switched systems have to be explored for reaching the convergence, such as non-smooth analysis (Lin [2005], Yang et.al. [2016]), small-gain theorem (Liu and Jiang [2014]), generalized Barbalat's Lemma (Su and Huang [2012b,d]), LaSalle's invariance principle (Cheng, Wang, and Hu [2008]), Krasovskii-LaSalle theorem (Lee, Xia, Su, and Huang [2018]). However, all of these approaches still rely on the dwell-time condition of switching signals. Notice that a dwell-time condition cannot be associated with the link failures instantaneously, so it is important to consider approaches that avoid dwell-time constraints. For this purpose, Lee, Tan, and Mareels [2017] provided an analytic condition that is without dwell-time condition.

It is worth mentioning that all of the aforementioned references worked only on the full-state feedback control. Regarding the output feedback control, some attempts have been devoted to static networks by means of the famous separation principle for the linear time-invariant

⋆ This work was supported in part by National Natural Science Foundation of China, under Grant No. 61773122 and in part by the MOST, Taiwan, R.O.C., under contract MOST 106-2221-E-159-001-MY2. *(Corresponding author: Ti-Chung Lee)*

system, see You and Xie [2011], Zhang, Lewis, and Das [2011], while, the studies for the GUJC (or UJC) switching networks is quite difficult and has been few touched, see Su and Huang [2012c]. There are two main challenges that has to be overcome. First, there are no results on the separation principle for switched systems with a specified group of switching signals. Second, the closed-loop system can hardly admit a weak Lyapunov function even though some partial subsystems can.

Motivating by this situation, this paper aims to investigate the output feedback synthesis for leader-follower consensus of linear multi-agent systems subject to GUJC switching networks. Our contribution is of the following aspects.

- First, the uniform global exponential stability of a class of cascaded linear switched systems is presented. It is interesting to see that the presented system may hardly admit a (weak/strict) Lyapunov function. It is either not sure on the construction of the converse Lyapunov function with respect to the switched system with a specified group of switching signals. These difficulties are overcome by adopting the so-called weak zero-state detectibility (WZSD).
- Second, the WZSD property for the switched closed-loop system is interpreted by the GUJC condition of switching networks without any dwell-time condition.

2. STABILITY RESULTS ON SWITCHED SYSTEMS

This section reviews some terminologies for switched linear systems, adopting from Lee [2018], Lee, Tan, and Mareels [2017, 2020]. The concept of WZSD and a generalized Krasovskii-LaSalle theorem are recalled. The uniform global exponential stability of a class of cascaded linear switched systems is then established by adopting WZSD.

2.1 Generalized Krasovskii-LaSalle Theorem: Changing Dynamics and Outputs

Consider the switched linear system

$$\begin{aligned} \dot{x} &= \mathbf{A}_{\lambda(t)}x, \\ y &= \mathbf{C}_{\lambda(t)}x \end{aligned} \tag{1}$$

where $x \in \mathbb{X}$ is the state with $\mathbb{X}$ being a nonempty closed subset of $\mathbb{R}^p$, $y \in \mathbb{R}^q$ is the output, $\lambda : \mathbb{R}_{\geq 0} \to \Lambda$ is the Λ-valued switching signal with Λ being a finite index set, $\mathbf{A}_\zeta \in \mathbb{R}^{p\times p}$, $\zeta \in \Lambda$, is the system matrix, and $\mathbf{C}_\zeta \in \mathbb{R}^{p\times q}$ is the output matrix. Here a switching signal is defined as a piecewise constant and right-continuous function with finitely many discontinuous points in any finite time interval. We use Θ to denote a set of possible switching signals.

Let (x, λ) be a forward complete solution pair (see Lee, Tan, and Mareels [2017, 2020]) with $x : \mathbb{R}_{\geq 0} \to \mathbb{X}$ being locally absolutely continuous and $\lambda : \mathbb{R}_{\geq 0} \to \lambda$ being a switching signal, where $\mathbb{X}$ is *scaling-invariant*, i.e., for any $\rho > 0$ and any $x \in \mathbb{X}$, $\rho x \in \mathbb{X}$. Let $\Phi(\Theta)$ be the set of all forward complete solution pairs (x, λ) with $\lambda \in \Theta$. To depict the limiting behavior of the solution of (1), it is necessary to recall the concept of *limiting zeroing-output solution*, see Lee, Tan, and Mareels [2017].

Definition 1. A continuous function $\bar{z} : \mathbb{R} \to \mathbb{X}$ is said to be a limiting zeroing-output solution of (1) w.r.t. $\Phi(\Theta)$ if, there exist sequences $\{(z_n, \lambda_n)\} \subseteq \Phi(\Theta)$ and $\{t_n\} \subseteq \mathbb{R}_{\geq 0}$ with $t_n \geq 2n$, such that the following hold: $\{z_n(\cdot + t_n) : [-n, n] \to \mathbb{X}\}$ converges uniformly to $\bar{z}$ on every compact subset of $\mathbb{R}$; and for almost all $t \in \mathbb{R}$, $\lim_{n\to+\infty} \mathbf{C}_{\lambda_n(t+t_n)}\bar{z}(t) = 0$.

The key tool to describe the limiting zeroing-output solution is the so-called *zeroing pair*, see Lee, Tan, and Mareels [2020]. It is defined as: given two matrices $\mathbf{M} \in \mathbb{R}^{\hat{q}\times p}$ and $\tilde{\mathbf{M}} \in \mathbb{R}^{\tilde{q}\times p}$, $(\mathbf{M}, \tilde{\mathbf{M}})$ is said to be a zeroing pair w.r.t. $\mathbb{X}$ if for any $x \in \mathbb{X}$, $\mathbf{M}x = 0$ implies $\tilde{\mathbf{M}}x = 0$. As shown in [Lee, Tan, and Mareels, 2020, Theorem 2], if there exist switching matrices $\hat{\mathbf{A}}_\zeta \in \mathbb{R}^{p\times p}$ and $\hat{\mathbf{C}}_\zeta \in \mathbb{R}^{\hat{q}\times p}$ such that for any $\zeta \in \Lambda$, $(\mathbf{C}_\zeta, \mathbf{A}_\zeta - \hat{\mathbf{A}}_\zeta)$ and $(\mathbf{C}_\zeta, \hat{\mathbf{C}}_\zeta)$ are both zeroing pairs w.r.t. $\mathbb{X}$, then, every bounded limiting zeroing-output solution $\bar{z} : \mathbb{R} \to \mathbb{X}$ of (1) w.r.t. $\Phi(\Theta)$ satisfies the following conditions where $\{\lambda_n\} \subseteq \Theta$ and $\{t_n\} \subseteq \mathbb{R}_{\geq 0}$ with $t_n \geq 2n$:

$$\bar{z}(t) = \bar{z}(0) + \lim_{n\to+\infty} \int_0^t \hat{\mathbf{A}}_{\lambda_n(\tau+t_n)}\bar{z}(\tau)\mathrm{d}\tau, \text{ for all } t \in \mathbb{R} \tag{2}$$

and

$$\lim_{n\to+\infty} \hat{\mathbf{C}}_{\lambda_n(t+t_n)}\bar{z}(t) = 0, \text{ for almost all } t \in \mathbb{R}. \tag{3}$$

Conversely, it is possible to show that any solution to the limiting equation (2), subject to (3) and lying within $\mathbb{X}$, is also the limiting zeroing-output solution of (1) w.r.t. $\Phi(\Theta)$. This fact is summarized by the following lemma, where its proof is omitted due to the space limit.

Lemma 1. Consider the switched system (1), where $\mathbb{X}$ is scaling-invariant. Assume that Θ satisfies $\lambda(\cdot + s) \in \Theta$ for all $s \geq 0$ and all $\lambda \in \Theta$, and the origin of (1) is uniformly globally stable (UGS) w.r.t. $\Phi(\Theta)$. Suppose that there exists a bounded continuous function $\bar{z} : \mathbb{R} \to \mathbb{X}$ such that (2) holds with $\hat{\mathbf{A}}_\zeta = \mathbf{A}_\zeta$ for all $\zeta \in \Lambda$. Then there exist sequences $\{(z_m, \lambda_m)\} \subseteq \Phi(\Theta)$ and $\{s_m\} \subseteq \mathbb{R}_{\geq 0}$ with $s_m \geq 2m$, such that $\{z_m(\cdot + s_m) : [-m, m] \to \mathbb{X}\}$ converges uniformly to $\bar{z}$ on every compact subset of $\mathbb{R}$. Moreover, if (3) holds with $\hat{\mathbf{C}}_\zeta = \mathbf{C}_\zeta$ for all $\zeta \in \Lambda$, then $\bar{z}$ is a limiting zeroing-output solution of (1) w.r.t. $\Phi(\Theta)$.

Now weak zero-state detectability can be defined, and the generalized Krasovskii-LaSalle theorem can be established accordingly, see in Lee, Tan, and Mareels [2020], where by the scaling-invariant property of $\mathbb{X}$, uniform global exponential stability of the origin is equivalent to uniform global asymptotic stability of the origin, see [Lee and Jiang, 2008, Lemma 1]:

Definition 2. System (1) is said to be *weakly zero-state detectable* (WZSD) w.r.t. Θ if every bounded limiting zeroing-output solution $\bar{z} : \mathbb{R} \to \mathbb{X}$ of (1) w.r.t. $\Phi(\Theta)$ satisfies $\inf_{t\in\mathbb{R}} \|\bar{z}(t)\| = 0$.

Theorem 1. [Generalized Krasovskii-LaSalle Theorem] Consider the switched system (1), where $\lambda \in \Theta$. Suppose that the origin is UGS w.r.t. $\Phi(\Theta)$ and the bounded-output-energy condition

$$\int_s^{+\infty} \|\mathbf{C}_{\lambda(\tau)}x(\tau)\|^2\mathrm{d}\tau \leq \mu(x(s)), \ \forall s \geq 0 \tag{4}$$

holds for any $\lambda \in \Theta$, with some continuous function $\mu : \mathbb{R}^p \to \mathbb{R}_{\geq 0}$. Then the origin is uniformly globally exponential stable (UGES) w.r.t. $\Phi(\Theta)$, i.e., there exist

$a > 0$ and $b > 0$ such that for any $(x, \lambda) \in \Phi(\Theta)$, $\|x(t)\| \leq a e^{-b(t-s)} \|x(s)\|$ for all $t \geq s \geq 0$, provided that (1) is WZSD w.r.t Θ.

In the remaining of this section, the terminologies of UGES, UGS, and limiting zeroing output solution are all with respect to $\Phi(\Theta)$ and the terminology of WZSD is with respect to Θ, where we omit "w.r.t." for simplicity.

2.2 UGES of cascaded linear switched systems

We need the following lemma.

Lemma 2. Consider three functions $\kappa_i : [s, +\infty) \to \mathbb{R}_{\geq 0}$, $i = 1, 2, 3$, for some $s \in \mathbb{R}$. Suppose that κ_1 is locally absolutely continuous, κ_2 is measurable, and κ_3 is Lebesgue integrable. If $\dot{\kappa}_1(t) \leq -\kappa_2(t) + \kappa_3(t)(1 + \kappa_1(t))$ for almost all $t \in [s, +\infty)$, then, for all $t \geq s$, the following inequalities hold:

$$\kappa_1(t) \leq e^{\kappa_4(s)}(1 + \kappa_1(s)), \tag{5}$$

$$\int_s^{+\infty} \kappa_2(\tau) \mathrm{d}\tau \leq \kappa_4(s) e^{\kappa_4(s)} + (1 + \kappa_4(s) e^{\kappa_4(s)}) \kappa_1(s) \tag{6}$$

where $0 \leq \kappa_4(s) \triangleq \int_s^{+\infty} \kappa_3(\tau) \mathrm{d}\tau < +\infty$.

Now let us explore the UGES property of the following general cascaded form

$$\dot{\xi}_1 = \mathcal{M}_{1\lambda(t)} \xi_1 + \mathcal{M}_{3\lambda(t)} \xi_2, \tag{7a}$$

$$\dot{\xi}_2 = \mathcal{M}_{2\lambda(t)} \xi_2 \tag{7b}$$

where, for $i = 1, 2$, $\xi_i \in \mathbb{R}^{p_i}$ is the state, and for $\zeta \in \Lambda$, $\mathcal{M}_{1\zeta} \in \mathbb{R}^{p_1 \times p_1}$, $\mathcal{M}_{2\zeta} \in \mathbb{R}^{p_2 \times p_2}$ and $\mathcal{M}_{3\zeta} \in \mathbb{R}^{p_1 \times p_2}$. Regarding system (7), the following theorem holds.

Theorem 2. Consider the switched linear system (7) and an interesting set Θ of switching signals satisfying that Θ satisfies $\lambda(\cdot + s) \in \Theta$ for all $s \geq 0$ and all $\lambda \in \Theta$. Suppose that there exist proper quadratic positive definite functions $V_i : \mathbb{R}^{p_i} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, such that for all $\xi_i \in \mathbb{R}^{p_i}$ and all $\zeta \in \Lambda$,

$$\frac{\partial V_i(\xi_i)}{\partial \xi_i} \mathcal{M}_{i\zeta} \xi_i \leq -||\mathcal{N}_{i\zeta} \xi_i||^2 \tag{8}$$

and the systems

$$\dot{\xi}_1 = \mathcal{M}_{1\lambda(t)} \xi_1, \ Y_1 = \mathcal{N}_{1\lambda(t)} \xi_1, \tag{9a}$$

$$\dot{\xi}_2 = \mathcal{M}_{2\lambda(t)} \xi_2, \ Y_2 = \mathcal{N}_{2\lambda(t)} \xi_2 \tag{9b}$$

are WZSD. Then the origin of (7) is UGES.

Proof: Since $V_i(\cdot)$ is quadratic and positive definite, there exist $\mu_j > 0$, $j = 1, 2, 3$, such that for $i = 1, 2$, $\mu_1 ||\xi_i||^2 \leq V_i(\xi_i) \leq \mu_2 ||\xi_i||^2$ and $||\partial V_i(\xi_i)/\partial \xi_i|| \leq \mu_3 ||\xi_i||$.

Step-1: to show the origin of subsystem (7b) is UGES. From (8), we know that the origin of subsystem (7b) is UGS. Define a virtual output $Y_2(t)$ as that in (9b). By (8) and applying (6) of Lemma 2 with $\kappa_1(t) = V_2(\xi_2(t))$, $\kappa_2(t) = ||\mathcal{N}_{2\lambda(t)} \xi_2(t)||^2$, and $\kappa_3(t) = 0$, it holds that $\int_s^{+\infty} ||\mathcal{N}_{2\lambda(\tau)} \xi_2(\tau)||^2 \mathrm{d}\tau \leq V_2(\xi_2(s)) \leq \mu_2 ||\xi_2(s)||^2$. Notice that system (9b) is WZSD. Following Theorem 1, the origin of subsystem (7b) is UGES, i.e., there exist $a_2 > 0$ and $b_2 > 0$ such that for all $(\xi_2, \lambda) \in \Phi(\Theta)$, we have

$$\|\xi_2(t)\| \leq a_2 e^{-b_2(t-s)} \|\xi_2(s)\| \tag{10}$$

for all $t \geq s \geq 0$.

Step-2: to show the origin of the whole system (7) is UGES. From (8), it holds that for all $\zeta \in \Lambda$,

$$\dot{V}_1(\xi_1) \leq -||\mathcal{N}_{1\zeta} \xi_1||^2 + \frac{\partial V_1(\xi_1)}{\partial \xi_1} \mathcal{M}_{3\zeta} \xi_2. \tag{11}$$

Notice that

$$\left\| \frac{\partial V_1(\xi_1)}{\partial \xi_1} \right\| \leq \mu_3 \left(1 + ||\xi_1||^2\right) \leq \mu_3 \max\{1, \frac{1}{\mu_1}\}(1 + V_1(\xi_1)).$$

From (10) and (11), it holds that

$$\dot{V}_1(\xi_1) \leq -||\mathcal{N}_{1\zeta} \xi_1||^2 + (1 + V_1(\xi_1)) \gamma(t) \tag{12}$$

where $\gamma(t) \triangleq a_2 \mu_3 \max\{1, 1/\mu_1\} \max_{\zeta \in \Lambda}\{||\mathcal{M}_{3\zeta}||\} \|\xi_2(s)\| e^{-b_2(t-s)}$, which satisfies $\int_s^{+\infty} \gamma(\tau) \mathrm{d}\tau = \hat{\gamma} \|\xi_2(s)\|$ with $\hat{\gamma} \triangleq (a_2/b_2) \mu_3 \max\{1, 1/\mu_1\} \max_{\zeta \in \Lambda}\{||\mathcal{M}_{3\zeta}||\}$. Then applying (5) of Lemma 2 with $\kappa_1(t) = V_1(\xi_1(t))$, $\kappa_1(t) = ||\mathcal{N}_{1\lambda(t)} \xi_1(t)||^2$, and $\kappa_3(t) = \gamma(t)$ gives that $V_1(\xi_1(t)) \leq \hat{\gamma} \|\xi_2(s)\| (1 + V_1(\xi_1(s))) \leq \hat{\gamma} \|\xi_2(s)\| \left(1 + \mu_2 ||\xi_1(s)||^2\right)$ for all $t \geq s \geq 0$. It yields that $\|\xi_1(t)\| \leq \mu_1^{-\frac{1}{2}} (\hat{\gamma} \|\xi_2(s)\| (1 + \mu_2 ||\xi_1(s)||^2))^{\frac{1}{2}}$ for all $t \geq s \geq 0$. By Cauchy inequality, it holds that

$$\|\xi(t)\| \leq \sqrt{2} \left(\|\xi_1(t)\| + \|\xi_2(t)\|\right) \leq \alpha(\|\xi(s)\|)$$

for all $t \geq s \geq 0$, where $\alpha(l) \triangleq \sqrt{2}(\mu_1^{-\frac{1}{2}} (\hat{\gamma} l (1 + \mu_2 l^2))^{\frac{1}{2}} + l)$ is of class $\mathcal{K}_\infty$. Therefore, the origin of (7) is UGS.

Now applying (6) of Lemma 2 with $\kappa_1(t) = V_1(\xi_1(t))$, $\kappa_2(t) = ||\mathcal{N}_{1\lambda(t)} \xi_1(t)||^2$, and $\kappa_3(t) = \gamma(t)$ gives that

$$\int_s^{+\infty} ||\mathcal{N}_{1\lambda(\tau)} \xi_1(\tau)||^2 \mathrm{d}\tau \leq \mu(\xi_1(s), \xi_2(s)) \tag{13}$$

where $\mu(u_1, u_2) = \hat{\gamma} \|u_2\| e^{\hat{\gamma} \|u_2\|} + (1 + \hat{\gamma} \|u_2\| e^{\hat{\gamma} \|u_2\|}) \mu_2 ||u_1||^2$, for all $u_k \in \mathbb{R}^{p_k}$, $k = 1, 2$. Let us define the virtual output for the whole system (7) as

$$Y(t) = \begin{bmatrix} \mathcal{N}_{1\lambda(t)} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} \mathcal{N}_{1\lambda(t)} & 0 \\ 0 & I_{p_2} \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}. \tag{14}$$

By (10) and (13), it holds that $\int_s^{+\infty} \|Y(t)\|^2 \mathrm{d}\tau \leq \frac{a_2^2}{2b_2} \|\xi_2(s)\|^2 + \mu(\xi_1(s), \xi_2(s))$ for all $t \geq s \geq 0$. Therefore, the bounded-output-energy condition for the system (7) holds.

Since for all $\zeta \in \Lambda$

$$\left(\begin{bmatrix} \mathcal{N}_{1\zeta} & 0 \\ 0 & I_{p_2} \end{bmatrix}, \begin{bmatrix} \mathcal{M}_{1\zeta} & \mathcal{M}_{3\zeta} \\ 0 & \mathcal{M}_{2\zeta} \end{bmatrix} - \begin{bmatrix} \mathcal{M}_{1\zeta} & 0 \\ 0 & \mathcal{M}_{2\zeta} \end{bmatrix} \right)$$

are zeroing pairs w.r.t. $\mathbb{R}^{p_1 + p_2}$, there exist $\{\lambda_n\} \subseteq \Theta$ and $\{t_n\} \subseteq \mathbb{R}_{\geq 0}$ with $t_n \geq 2n$ such that every bounded limiting solution of (7) satisfies $\bar{\xi}_2(t) \equiv 0$ and

$$\bar{\xi}_1(t) = \bar{\xi}_1(0) + \lim_{n \to +\infty} \int_0^t \mathcal{M}_{1\lambda_n(\tau + t_n)} \bar{\xi}_1(\tau) \mathrm{d}\tau, \text{ for all } t \in \mathbb{R} \tag{15}$$

and

$$\lim_{n \to +\infty} \mathcal{N}_{1\lambda_n(t + t_n)} \bar{\xi}_1(t) = 0, \text{ for almost all } t \in \mathbb{R}. \tag{16}$$

According to Lemma 1, (15) and (16) imply that $\bar{\xi}_1$ is a limiting zeroing-output solution of system (9a). Since system (9a) is WZSD, $\inf_{t \in \mathbb{R}} ||\bar{\xi}_1(t)|| = 0$. Thus, the system composed of (7) and (14) is WZSD.

Therefore, by Theorem 1, the origin of (7) is UGES. The proof is thus completed. □

Remark 1. Theorem 2 essentially establishes a separation principle for a class of switched linear systems with a cascaded structure by means of UGES property of its "diagonal" switched linear subsystems.

3. AGENT DYNAMICS AND SWITCHING NETWORKS

This section gives first the problem formulation of leader-follower consensus. Then a new connectivity condition, namely, generalized uniform joint connectivity (GUJC) is depicted for switching networks. Such a condition is without the so-called dwell-time condition, and hence is weaker than those conditions in the literatures. Moveover, the WZSD property is related to the GUJC condition for two classes of linear switched systems determined by the switching networks.

3.1 System Model and Problem Formulation

Consider a group of $N+1$ agents consisting of N follower systems with linear dynamics

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i,\ i = 1, \dots, N \end{aligned} \tag{17}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $y_i \in \mathbb{R}^p$ are the state, input, and output of the ith follower, respectively, and the leader system with autonomous linear dynamics

$$\dot{x}_0 = Ax_0,\ y_0 = Cx_0 \tag{18}$$

where $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^p$ are the state and output of the leader, respectively.

All of these $N+1$ agents are interconnected by a switching network that is defined by a switching graph $\bar{\mathcal{G}}_\zeta = (\bar{\mathcal{V}}, \bar{\mathcal{E}}_\zeta)$, where $\bar{\mathcal{V}} = \{0, 1, \dots, N\}$ represents the N agents, $\bar{\mathcal{E}}_\zeta \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$ for all $\zeta \in \Lambda$. Here we assume that all the subgraph $\mathcal{G}_\zeta$ of $\bar{\mathcal{G}}_\zeta$ by removing the node 0 and the corresponding edges is undirected. Λ represents all possible graphs which are not necessary to be connected. For a Λ-valued switching signal $\lambda : \mathbb{R}_{\geq 0} \to \Lambda$ with Λ denoting a set of possible graphs having the node set $\bar{\mathcal{V}}$, the edge $(j, i) \in \bar{\mathcal{E}}_{\lambda(t)}$ if and only if the control of the i-th agent can make use of the information of the j-th agent for feedback at the time instant t. It is assumed for convenience that no edges $(j, 0)$ exist since the leader system needs no input.

The leader-follower consensus is said to be achieved if *we can find a distributed output feedback controller such that for all initial states of systems* (17) *and* (18)*, all solutions of the closed-loop system exist and are bounded for all* $t \in \mathbb{R}_{\geq 0}$ *with* $\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0$.

Here, without loss of generality, the initial times of the closed-loop state and the switching signal are always assumed to be zero. In contrast with the state feedback case studied in the previous references, say Cheng, Wang, and Hu [2008], Jadbabaie, Lin, and Morse [2003], Lin [2005], Ren and Beard [2005], Su and Huang [2012b], Yang et.al. [2016], the distributed observer based output feedback synthesis will be considered here.

3.2 Generalized Uniform Joint Connectivity

This section presents a general condition for the network connectivity. For any $\tau_a > 0$ and any switching signal λ, the τ_a-joint graph over a time interval $[t_1, t_2)$ is defined as

$$\bar{\mathcal{G}}_\lambda^{\tau_a}([t_1, t_2)) = \left(\bar{\mathcal{V}}, \bigcup_{\zeta \in \lambda_{\tau_a}[t_1, t_2)} \bar{\mathcal{E}}(\zeta)\right)$$

where

$$\lambda_{\tau_a}[t_1, t_2) = \left\{\zeta \in \Lambda \,\middle|\, \int_{t_1}^{t_2} \lambda^\zeta(\tau)\mathrm{d}\tau \geq \tau_a\right\}$$

with $\lambda^\zeta(\cdot)$ being the indicator function defined as

$$\lambda^\zeta(\tau) = \begin{cases} 1, \text{ if } \lambda(\tau) = \zeta \\ 0, \text{ if } \lambda(\tau) \neq \zeta \end{cases}$$

for the given switching signal λ. In this paper, the switching graph $\bar{\mathcal{G}}_{\lambda(t)}$ is assumed to satisfy the following generalized uniformly jointly connected (GUJC) condition in the leader-follower sense, where Θ is an interested set of switching signals.

Assumption 1. There exists a constant pair (τ_a, T) with $T \geq \tau_a > 0$ such that for all $\lambda \in \Theta$ and any $t \in \mathbb{R}_{\geq 0}$, the τ_a-joint graph $\bar{\mathcal{G}}_\lambda^{\tau_a}([t, t+T))$ contains a spanning tree with the node 0 as the root.

Notice that the set Θ of switching signals satisfies $\lambda(\cdot + s) \in \Theta$ for all $s \geq 0$ and all $\lambda \in \Theta$ automatically provided that Assumption 1 holds.

Remark 2. It is worth mentioning that Lee, Tan, and Mareels [2017] presents an equivalent analytic condition (see condition (C2) in Lee, Tan, and Mareels [2017]) for evaluating the WZSD of the system (19). Comparing with that analytic condition, the GUJC condition here is easier to be verified.

3.3 WZSD via GUJC

This section interprets the WZSD property of two classes of switched linear systems.

For all $\zeta \in \Lambda$, define the H-matrix $\mathcal{H}_\zeta = [h_{ij}^\zeta] \in \mathbb{R}^{N\times N}$ associated with the graph $\bar{\mathcal{G}}_\zeta$, where $h_{ij}^\zeta = -a_{ij}^\zeta$, and $h_{ii}^\zeta = \sum_{j=0}^N a_{ij}^\zeta$. Here for all $\zeta \in \Lambda$, a_{ij}^ζ satisfy for $i, j = 0, 1, \dots, N$, $a_{ii}^\zeta = 0$ and $a_{ij}^\zeta = a_{ji}^\zeta > 0$ if and only if $(j, i) \in \bar{\mathcal{E}}_\zeta$. Notice that for all $\zeta \in \Lambda$, $\mathcal{H}_\zeta$ is symmetric and positive semi-definite.

For a Λ-valued switching signal $\lambda(t)$ and the corresponding switching matrix $\mathcal{H}_{\lambda(t)}$, consider linear switched autonomous systems

$$\begin{aligned} \dot{\xi} &= ((I_N \otimes A) - \delta\mathcal{H}_{\lambda(t)} \otimes (BB^\mathsf{T}P))\xi, \\ y &= \begin{bmatrix} I_N \otimes (-PA - A^\mathsf{T}P)^{1/2} \\ \sqrt{2\delta}\mathcal{H}_{\lambda(t)}^{1/2} \otimes B^\mathsf{T}P \end{bmatrix} \xi \end{aligned} \tag{19}$$

and

$$\begin{aligned} \dot{\xi} &= (I_N \otimes A - \delta\mathcal{H}_{\lambda(t)} \otimes (P^{-1}C^\mathsf{T}C))\xi, \\ y &= \begin{bmatrix} I_N \otimes (-PA - A^\mathsf{T}P)^{1/2} \\ \sqrt{2\delta}\mathcal{H}_{\lambda(t)}^{1/2} \otimes C \end{bmatrix} \xi \end{aligned} \tag{20}$$

where $\xi \in \mathbb{R}^{nN}$, $\delta > 0$, $A, P \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ satisfying that (A, B, C) is stabilizable and detectable, and P is positive definite with $PA + A^\mathsf{T}P \leq 0$. Then the following lemma holds, where its proof is omitted due to the space limit.

Lemma 3. Under Assumption 1, systems (19) and (20) are both WZSD.

Remark 3. Lemma 3 interprets the WZSD property by means of the GUJC condition. As a consequence, it is possible to use the separation principle providing in Section 2 for designing the distributed observer based output feedback controller.

4. DISTRIBUTED OUTPUT FEEDBACK SYNTHESIS

This section provides a distributed observer based output feedback synthesis for the leader-follower consensus problem. The stability analysis relies on the WZSD from UGJC as discussion in previous section.

4.1 Distributed Observer and Output Feedback Controller

For $i = 1, \ldots, N$, we define the distributed output feedback controller of the following form:

$$u_i = -K \sum_{j=0}^{N} a_{ij}^{\lambda(t)} (\hat{x}_i - \hat{x}_j), \tag{21a}$$

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - L \sum_{j=0}^{N} a_{ij}^{\lambda(t)} [C(\hat{x}_i - \hat{x}_j) - (y_i - y_j)] \tag{21b}$$

where K and L are the gain matrices to be determined. Here like Zhang, Lewis, and Das [2011], it is supposed that the leader state can be measured by those agents that are link to the leader and hence we let $\hat{x}_0 = x_0$ for simplicity.

Remark 4. The observer (21b) is Luenberger type based on the neighborhood output estimation error. It has been employed in Zhang, Lewis, and Das [2011] for solving the leader-follower consensus subject to static networks by means of the famous separation principle for the linear time-invariant system. In contrast, by making use of the separation principle for the switched linear system (see Remark 1), we are able to develop the leader-follower consensus subject to the switching networks satisfying the proposed GUJC condition.

4.2 Stability of the Closed-loop System

To depict the closed-loop system, we define $\tilde{x}_i = x_i - \hat{x}_i$, and $e_i = x_i - x_0,\ i = 1, \ldots, N$. Letting $e = [e_1^\top, \ldots, e_N^\top]^\top$ and $\tilde{x} = [\tilde{x}_1^\top, \ldots, \tilde{x}_N^\top]^\top$ gives that

$$\dot{e} = [I_N \otimes A - \delta\mathcal{H}_{\lambda(t)} \otimes (BB^\top)P]e + \delta\mathcal{H}_{\lambda(t)} \otimes (BBP^\top)\tilde{x}, \tag{22a}$$

$$\dot{\tilde{x}} = [I_N \otimes A - \delta\mathcal{H}_{\lambda(t)} \otimes (P^{-1}C^\top C)]\tilde{x}, \tag{22b}$$

Theorem 3. Suppose that (A, B, C) is stabilizable and detectable and P is positive definite with $PA + A^\top P \leq 0$. Let $K = \delta B^\top P$ and $L = \delta P^{-1}C^\top$ with $\delta > 0$ chosen arbitrarily. Under Assumption 1, the closed-loop system (22) is UGES. Thus, the leader-follower consensus problem is solved.

Proof: It is seen that (22) is of the cascaded form (7) with $\xi_1 = e$, $\xi_2 = \tilde{x}$, and

$$\mathcal{M}_{1\zeta} = I_N \otimes A - \delta\mathcal{H}_\zeta \otimes (BB^\top P),$$
$$\mathcal{M}_{2\zeta} = I_N \otimes A - \delta\mathcal{H}_\zeta \otimes (P^{-1}C^\top C),$$
$$\mathcal{M}_{3\zeta} = \delta\mathcal{H}_\zeta \otimes (BB^\top P)$$

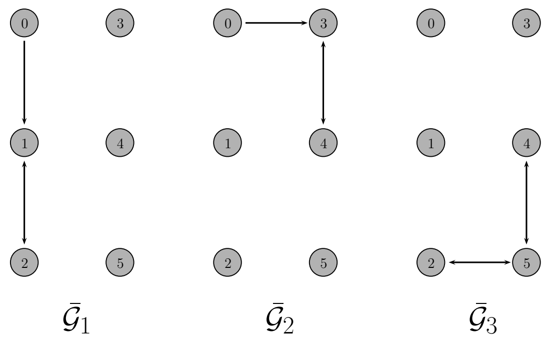

Fig. 1. The graphs $\bar{\mathcal{G}}_i$, $i = 1, 2, 3$.

Consider the proper positive definite functions $V_1(e) = e^\top(I_N \otimes P)e$ and $V_2(\tilde{x}) = \tilde{x}^\top(I_N \otimes P)\tilde{x}$. Then it holds that

$$\frac{\partial V_1(e)}{\partial e}[I_N \otimes A - \delta\mathcal{H}_\zeta \otimes (BB^\top P)]e$$
$$= e^\top[I_N \otimes (PA + A^\top P)]e - 2\delta e^\top[\mathcal{H}_\zeta \otimes (PBB^\top P)]e.$$

and

$$\frac{\partial V_2(\tilde{x})}{\partial \tilde{x}}[I_N \otimes A - \delta\mathcal{H}_\zeta \otimes (P^{-1}C^\top C)]\tilde{x}$$
$$= \tilde{x}^\top[I_N \otimes (PA + A^\top P)]\tilde{x} - 2\delta\tilde{x}^\top[\mathcal{H}_\zeta \otimes (C^\top C)]\tilde{x}.$$

Thus, we can define

$$\mathcal{N}_{1\zeta} = \begin{bmatrix} I_N \otimes (-PA - A^\top P)^{1/2} \\ \sqrt{2\delta}\mathcal{H}_\zeta^{1/2} \otimes (B^\top P) \end{bmatrix},$$
$$\mathcal{N}_{2\zeta} = \begin{bmatrix} I_N \otimes (-PA - A^\top P)^{1/2} \\ \sqrt{2\delta}\mathcal{H}_\zeta^{1/2} \otimes C \end{bmatrix}.$$

Then (8) holds. Moreover, from Lemma 3, both systems of (9) are WZSD. In view of Theorem 2 , it can be concluded that the origin of the system (22) is UGES. This completes the proof. □

4.3 Simulation

An example is provided here to illustrate the aforementioned control design. Consider a multi-agent system with one leader and five followers, where the agent dynamics are of the form (17) and (18) with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = [0\ 1\ -1]$$

and the switching graph $\mathcal{G}(\zeta)$, with $\zeta \in \{1, 2, 3\}$, is shown in Fig. 1. The switching signal $\lambda(t)$ is defined as follows:

$$\lambda(t) = \begin{cases} 1, & \text{if } (k + \frac{l}{k+1})T' \leq t < (k + \frac{l+1/3}{k+1})T', \\ 2, & \text{if } (k + \frac{l+1/3}{k+1})T' \leq t < (k + \frac{l+2/3}{k+1})T' \\ 3, & \text{if } (k + \frac{l+2/3}{k+1})T' \leq t < (k + \frac{l+1}{k+1})T' \end{cases}$$

where $k \in \mathbb{Z}_+$, $l = 0, 1, \ldots, k$, and T' is chosen arbitrarily. It is of interest to see that the proposed switching signal $\lambda(t)$ does not satisfy any dwell-time conditions. However, $\mathcal{G}_{\lambda(t)}$ is GUJC since Assumption 1 holds with $\tau_a = T'/6$ and $T = T'$. Notice that (A, B, C) is stabilizable and detectable, and satisfies $PA + A^\top P \leq 0$ with a positive definite matrix $P = [2, 0.5, 0.5; 0.5, 1, 0; 0.5, 0, 1]$.

Applying Theorem 3, a simulation with $\delta = 3$, $K = [0.5, 0, 1]$, $L = [0, 1, -1]^\top$, $T' = 10$ is reported in Fig.

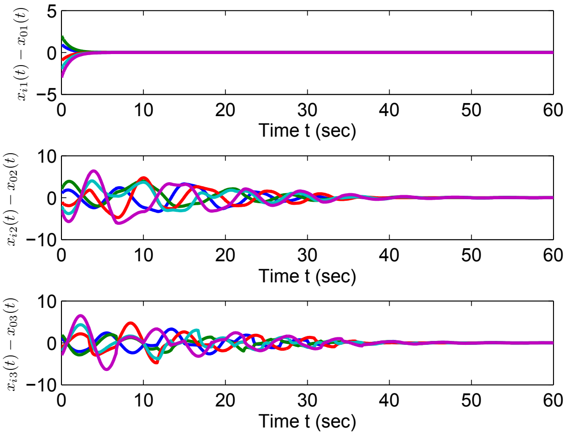

Fig. 2. Time responses of the consensus errors.

2, where the responses of the consensus errors have been shown. It can be seen that satisfactory converging behavior is obtained.

5. CONCLUSION

This paper has established the uniform exponential stability of a class of cascaded linear switched systems. With its aid, a distributed output feedback controller has been developed for solving the leader-follower consensus of linear multi-agent systems with undirect switching networks satisfying a generalized uniformly joint-connected condition without any dwell-time condition. The further work will toward the study on direct switching networks.

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