Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

On the Stability of Kalman Filter with Random Coefficients ⋆

Die Gan * **ZhiXin Liu** *

* *Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, P. R. China. (E-mails: gandie@amss.ac.cn, lzx@amss.ac.cn).*

Abstract: Inspired by the random packet dropout problem widely existing in the networked control systems, we investigate the stability of Kalman filter with random coefficients. We present an excitation condition about the regression vectors to establish the L_p-stability and L_p-exponential stability of random Riccati equation. Furthermore, we prove the stability of the error equations of Kalman filter under the excitation condition and some conditions on the system matrix and noises, without relying on any stationarity or independence assumptions about the regressors.

Keywords: Kalman filter, L_p-stable, L_p-exponential stable, random Riccati equation, excitation condition.

1. INTRODUCTION

Kalman filter (R. E. Kalman (1960); T. Kailath, A. H. Sayed, B. Hassibi (2000)) is widely used to estimate the states and the parameters in many practical engineering systems (cf., L. Shi, M. Epstein, A. Tiwari, R.M. Murray (2005); Q. S. He, C. Wei, Y. Z. Xu (2017)), such as guidance and navigation of vehicles, signal processing. The investigation of the stability and optimality of Kalman filter and its variants attract much attention of researchers for several decades. With the development of communication and computation, Kalman filter has also great importance in networked control systems, and correspondingly the theoretical study of Kalman filter brings more and more challenge to us.

For the deterministic linear time-invariant systems, a classical result is that the iterative Riccati equation converges to a steady state under some detectable and stabilizable assumptions. More results about deterministic situation can be found in the paper by J. Deyst, C. Price (1968). However, for the networked control systems, data transmission over networks may be randomly dropped or delayed due to the malicious network attacks or the limited computation ability at the sensors, which leads to random evolution of regression vectors. The theoretical results for the deterministic systems can not be used to deal with such a case since the deterministic hypotheses are unsuitable for a general stochastic model as pointed out by Guo (1990). In order to provide theoretical illustration for the feasibility of Kalman filter in a random framework, a large number of literature has been devoted to the study of the properties of Kalman filter, for example, the boundedness of the estimation error covariances for system with independent and identically distributed (i.i.d.) packet dropouts (cf., B. Sinopoli, et al (2004); A. S. Leong, D. E. Quevedo, D. Dolz, & S. Dey (2019)), and Markovian packet dropouts (cf., K. Y. You, M. Y. Fu, L. H. Xie (2011); H. Lin, J. Lam, Z. D. Wang, & H. Lam (2019); M. Y. Huang, S. Dey (2007); L. Xie, L. H. Xie (2019)).

To the best of our knowledge, very few results are obtained concerning with a general stochastic process(cf., P. Bougerol (1993); V. Solo (1996)). But these papers require that the regression vector and system matrix satisfy the stationarity assumptions. For the case where the system matrix is taken as the identity matrix, Guo (1994) establish a general excitation condition to guarantee the stability of Kalman filter without relying on the independence and stationarity assumptions of random regressors. The concept of stochastic observability was proposed by Wang and Guo (1999) for the general system matrix, under which the stability of Kalman filter with random coefficients are established. However, the stochastic observability condition proposed by Wang and Guo (1999) is not valid, even for the i.i.d. packet losses.

In this paper, we focus on the case where the system matrix $\boldsymbol{A}$ is deterministic and time-invariant, and the regression vector $\boldsymbol{\varphi}_k$ is a random vector. It is clear that the framework studied in the current paper can include the system with random packet losses. We extend the excitation condition proposed by Guo (1994) for the standard linear regression system to the model with the system matrix $\boldsymbol{A}$, under which the L_p-stability and L_p-exponential stability of random Riccati equation can be established. Furthermore, the stability of the error equations of Kalman filter can also be obtained under the excitation condition and some conditions on the system matrix and the noises. We remark that the excitation condition in this paer is weaker than the so-called stochastic observability condition proposed by Wang and Guo (1999).

⋆ This work was supported by the National Key R&D Program of China under Grant 2018YFA0703800 and the National Natural Science Foundation of China under grants 11688101, 61621003 and 61873167.

The remainder of this paper is organized as follows. We first introduce some notations and preliminaries on Kalman filter in Section 2. The main results including the conditions for stability of Kalman filter are presented in Section 3, followed by some conclusions in Section 4.

2. PROBLEM FORMULATION

2.1 Some Preliminaries

In this paper, we use $\boldsymbol{X} \in \mathbb{R}^{m\times n}$ to denote an $m \times n$-dimensional matrix. For a matrix $\boldsymbol{X}$, $\|\boldsymbol{X}\|$ denotes the Euclidean norm, i.e., $\|\boldsymbol{X}\| = (\lambda_{max}(\boldsymbol{X}\boldsymbol{X}^T))^{\frac{1}{2}}$, where the notation T denotes the transpose operator and $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of the matrix. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ be two symmetric matrices, then $\boldsymbol{A} \geq \boldsymbol{B}$ means $\boldsymbol{A}$-$\boldsymbol{B}$ is a positive semidefinite matrix. In order to proceed our discussions, we need to introduce some definitions (see, Guo (1994)).

Definition 1. A random matrix sequence $\{\boldsymbol{A}_k, k \geq 0\}$ defined on the basic probability space $(\Omega, \mathscr{F}, P)$ is called L_p-stable ($p > 0$) if $\sup_{k\geq 0} E\|\boldsymbol{A}_k\|^p < \infty$. We define $\|\boldsymbol{A}_k\|_{L_p} \triangleq (E\|\boldsymbol{A}_k\|^p)^{\frac{1}{p}}$ as the L_p-norm of the random matrix $\boldsymbol{A}_k$.

Definition 2. A sequence of $n \times n$ random matrices $\boldsymbol{A} = \{\boldsymbol{A}_k, k \geq 0\}$ is called L_p-exponentially stable ($p \geq 0$) with parameter $\lambda \in [0, 1)$, if it belongs to the following set

$$S_p(\lambda) = \Big\{\boldsymbol{A} : \Big\| \prod_{i=j+1}^{k} \boldsymbol{A}_i \Big\|_{L_p} \leq M\lambda^{k-j}, \forall k \geq j, \quad \forall j \geq 0, for\ some\ M > 0\Big\}.$$

For convenience of discussions, we introduce the subclass of $S_1(\lambda)$ for a scalar sequence $a = \{a_k, k \geq 0\}$:

$$S^0(\lambda) = \Big\{a : a_k \in [0, 1), E\prod_{i=j+1}^{k} a_i \leq M\lambda^{k-j}, \forall k \geq j, \quad \forall j \geq 0, for\ some\ M > 0\Big\}.$$

Remark 1. It is clear that if there exist a constant $a_0 \in [0, 1)$ such that $a_k \leq a_0$, then $\{a_k\} \in S^0(a_0)$. More properties about the set $S^0(\lambda)$ are introduced by L. Guo (1993).

Definition 3. Let $\{\boldsymbol{A}_k\}$ be a matrix sequence and $\{b_k\}$ be a positive scalar sequence. Then by $\boldsymbol{A}_k = O(b_k)$ we mean that there exists a constant $M > 0$ such that

$$\|\boldsymbol{A}_k\| \leq M b_k \quad \forall k \geq 0.$$

2.2 Kalman Filter

In this paper, we consider the following discrete-time linear dynamical system:

$$\begin{cases} \boldsymbol{\theta}_{k+1} = \boldsymbol{A}\boldsymbol{\theta}_k + \boldsymbol{w}_{k+1} \\ y_k = \boldsymbol{\varphi}_k^T\boldsymbol{\theta}_k + v_k \end{cases}, \tag{1}$$

where $\boldsymbol{\theta}_k \in \mathbb{R}^n (k \geq 0)$ is the state to be estimated and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is a deterministic matrix, y_k is a scalar observation and $\boldsymbol{\varphi}_k$ is an n-dimensional random regression vector, $\{\boldsymbol{w}_k, v_k, k \geq 0\}$ is an independent noise process and satisfies:

$$E(\boldsymbol{w}_k\boldsymbol{w}_k^T) = \boldsymbol{Q} > 0,\ E(v_k^2) = R > 0,\ E(v_k\boldsymbol{w}_k^T) = 0.$$

The initial condition $\boldsymbol{\theta}_0$ is a random vector with mean $\hat{\boldsymbol{\theta}}_0$ and covariance matrix $\boldsymbol{P}_0' \geq 0$, and is independent of the sequence $\{\boldsymbol{w}_k, v_k, k \geq 0\}$.

First, let us define

$$\begin{aligned} \hat{\boldsymbol{\theta}}_k' &\triangleq E(\boldsymbol{\theta}_k|\mathscr{F}_{k-1}),\ \hat{\boldsymbol{\theta}}_k \triangleq E(\boldsymbol{\theta}_k|\mathscr{F}_k), \\ \boldsymbol{P}_k' &\triangleq E((\hat{\boldsymbol{\theta}}_k' - \boldsymbol{\theta}_k)(\hat{\boldsymbol{\theta}}_k' - \boldsymbol{\theta}_k)^T|\mathscr{F}_{k-1}), \\ \boldsymbol{P}_k &\triangleq E((\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k)(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k)^T|\mathscr{F}_k)), \end{aligned}$$

where $\mathscr{F}_k \triangleq \sigma\{y_i, \boldsymbol{\varphi}_i, i \leq k\}$.

Then the Kalman filter algorithm of system (1) can be stated as follows (cf., G. Welch, G. Bishop (2000)).

State prediction process:

$$\hat{\boldsymbol{\theta}}_{k+1}' = \boldsymbol{A}\hat{\boldsymbol{\theta}}_k, \tag{2}$$

$$\boldsymbol{P}_{k+1}' = \boldsymbol{A}\boldsymbol{P}_k\boldsymbol{A}^T + \boldsymbol{Q}, \tag{3}$$

Measurement update process:

$$\boldsymbol{P}_k^{-1} = \boldsymbol{P}_k'^{-1} + \boldsymbol{\varphi}_k R^{-1}\boldsymbol{\varphi}_k^T, \tag{4}$$

$$\hat{\boldsymbol{\theta}}_k = \hat{\boldsymbol{\theta}}_k' + \boldsymbol{L}_k(\boldsymbol{y}_k - \boldsymbol{\varphi}_k^T\hat{\boldsymbol{\theta}}_k'), \tag{5}$$

$$\boldsymbol{L}_k = \boldsymbol{P}_k'\boldsymbol{\varphi}_k(\boldsymbol{\varphi}_k^T\boldsymbol{P}_k'\boldsymbol{\varphi}_k + R)^{-1}.$$

Remark 2. From the above Kalman filter algorithm, we obtain the following random Riccati equation,

$$\boldsymbol{P}_{k+1}' = \boldsymbol{A}(\boldsymbol{I} - \boldsymbol{L}_k\boldsymbol{\varphi}_k^T)\boldsymbol{P}_k'(\boldsymbol{I} - \boldsymbol{L}_k\boldsymbol{\varphi}_k^T)^T\boldsymbol{A}^T + \boldsymbol{Q}_k \tag{6}$$

with $\boldsymbol{Q}_k = \boldsymbol{A}\boldsymbol{L}_k R \boldsymbol{L}_k^T\boldsymbol{A}^T + \boldsymbol{Q}$.

What we concern in this paper is the L_p stability of the random Riccati equation. Our purpose is to establish the conditions on the regressors $\{\boldsymbol{\varphi}_k\}$ and the system matrix $\boldsymbol{A}$ to guarantee the boundedness of the random process $\{\boldsymbol{P}_k'\}$.

3. STABILITY OF KALMAN FILTER

3.1 L_p-Stability of Random Riccati Equation

In this section, the L_p-stability of random Riccati equation is analyzed. To this end, we first introduce the following preliminary assumptions.

Assumption 1. The system matrix $\boldsymbol{A}$ is invertible.

Assumption 2. (Excitation Condition) There exists an positive integer h such that $\{1 - \lambda_k\} \in S^0(\lambda)$ for some $\lambda \in (0, 1)$, where λ_k is defined by

$$\lambda_k \triangleq \lambda_{\min}\left\{E\left[\frac{1}{1+h}\boldsymbol{G}((k+1)h, kh)|\mathscr{F}_{kh-1}\right]\right\},$$

with

$$\boldsymbol{G}((k+1)h, kh) \triangleq \sum_{i=kh}^{(k+1)h-1} \frac{(\boldsymbol{A}^{i-kh})^T\boldsymbol{\varphi}_k\boldsymbol{\varphi}_k^T\boldsymbol{A}^{i-kh}}{1 + \|\boldsymbol{\varphi}_k^T\boldsymbol{A}^{i-kh}\|^2}.$$

Remark 3. It can be proved that the following two typical cases satisfy Assumption 2,

i) The regression vector $\{\boldsymbol{\varphi}_k, k \geq 0\}$ is deterministic time-invariant, and $(\boldsymbol{A}, \boldsymbol{\varphi}_k)$ satisfy the observable condition.

ii) The regression vector $\{\boldsymbol{\varphi}_k, k \geq 0\}$ is generated by an i.i.d. packet dropout with some observable conditions.

Assumption 3. The norm of the system matrix $\boldsymbol{A}$ satisfies

$$2^{\frac{p-1}{ph}}\|\boldsymbol{A}\|^2 \cdot \lambda^{\frac{1}{8nph^2(1+R)}} < 1 \quad for\ some\ p \geq 1,$$

where n is the dimension of $\boldsymbol{\theta}_k$, λ and h are defined in Assumption 2.

Remark 4. Wang and Guo (1999) proposed the following random observability condition, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$P\left\{\lambda_{min}\left[\sum_{k+1}^{k+h_0}(\boldsymbol{A}^{i-k})^T\boldsymbol{\varphi}_k\boldsymbol{\varphi}_k^T\boldsymbol{A}^{i-k}\right] < \delta \middle| \mathscr{F}_{k-1}\right\} < \varepsilon, \quad (7)$$

where h_0 is an integer. We can prove that under the condition $\sup_k E\|\boldsymbol{\varphi}_k|\mathscr{F}_{k-1}\|^4 < \infty$, the above random observability condition implies Assumption 2, which means that Assumption 2 is in some sense weaker than (7).

To proceed our analysis, we first introduce two lemmas.

Lemma 1. (Guo (1994)) Let $\{1-\beta_k\} \in S^0(\lambda)$, and $\beta_k \leq \beta < 1$, where β is a positive constant. Then for any $\varepsilon \in (0,1)$, $\{1-\varepsilon\beta_k\} \in S^0(\lambda^{(1-\beta)\varepsilon})$.

Lemma 2. (Guo (1994)) Let $\{z_m, \mathscr{F}_m\}$ be an adapted process with $z_m \geq 1$, and

$$z_{m+1} \leq \alpha_{m+1}z_m + \eta_{m+1}, \quad m \geq 0, \quad Ez_0^2 < \infty,$$

where $\{\alpha_m, \mathscr{F}_m\}$ and $\{\eta_m, \mathscr{F}_m\}$ are two adapted nonnegative process with properties:

$$\alpha_m \geq \varepsilon_0 > 0, \quad \forall m,$$
$$E[\eta_{m+1}^2|\mathscr{F}_m] \leq N < \infty, \quad \forall m,$$
$$\left\|\prod_{m=j}^{k}E[\alpha_{m+1}^4]|\mathscr{F}_m\right\| \leq M\eta^{k-j+1}, \quad \forall k \geq j, \quad \forall j,$$

where ε_0, M, N and $\eta \in (0,1)$ are constants. Then we have

$$(i) \quad \left\|\prod_{m=j}^{k}\alpha_m\right\|_{L_2} \leq M^{\frac{1}{4}}\eta^{\frac{1}{4}(k-j+1)}, \quad \forall k \geq j, \quad \forall j;$$
$$(ii) \quad \sup_m E\|z_m\| < \infty;$$
$$(iii) \quad \left\{1 - \frac{1}{z_k}\right\} \in S^0(\lambda)\ for\ some\ \lambda \in (0,1).$$

Remark 5. In Lemma 2, for the conclusions of (i) and (ii), the condition $z_m \geq 1$ is not critical because we can use a new process $\{z_m + 1\}$ to substitute $\{z_m\}$.

Theorem 1. Let $\{\boldsymbol{P}_k'\}$ be generated by (6), then under Assumption 1, we have

$$T_{m+1} \leq (1 - a_{m+1})\|\boldsymbol{A}\|^{2h}T_m + b_{\boldsymbol{A}}. \quad (8)$$

where

$$T_{m+1} \triangleq \sum_{k=mh}^{(m+1)h-1} tr(\boldsymbol{A}^{mh-(k+1)}\boldsymbol{P}_{k+1}'(\boldsymbol{A}^{mh-(k+1)})^T),$$
$$T_0 = \ 0,$$
$$a_{m+1} \triangleq \frac{tr[(\boldsymbol{S}_{mh,h})^2\boldsymbol{G}((m+1)h, mh)]}{h(1+R)tr(\boldsymbol{S}_{mh,h})[1+\lambda_{\max}(\boldsymbol{S}_{mh,h})]},$$
$$\boldsymbol{S}_{mh,h} \triangleq \boldsymbol{P}_{mh}' + \sum_{j=1}^{h}\boldsymbol{A}^{-j}\boldsymbol{Q}(\boldsymbol{A}^{-j})^T,$$
$$b_{\boldsymbol{A}} \triangleq \ htr[\sum_{j=0}^{h}\boldsymbol{A}^j\boldsymbol{Q}(\boldsymbol{A}^j)^T] + 2h\sum_{j=1}^{h}tr[\boldsymbol{A}^{-j}\boldsymbol{Q}(\boldsymbol{A}^{-j})^T].$$

Proof. Note that by (4), we get $\boldsymbol{P}_k \leq \boldsymbol{P}_k'$. Combining this with (3), we have for any $k \in [mh, (m+1)h-1]$,

$$\begin{aligned}\boldsymbol{P}_k' &\leq \boldsymbol{A}\boldsymbol{P}_{k-1}'\boldsymbol{A}^T + \boldsymbol{Q}\\ &\leq \boldsymbol{A}(\boldsymbol{A}\boldsymbol{P}_{k-2}'\boldsymbol{A}^T + \boldsymbol{Q})\boldsymbol{A}^T + \boldsymbol{Q}\\ &= \boldsymbol{A}^2\boldsymbol{P}_{k-2}'(\boldsymbol{A}^2)^T + \boldsymbol{A}\boldsymbol{Q}\boldsymbol{A}^T + \boldsymbol{Q}\\ &\leq \cdots\\ &\leq \boldsymbol{A}^{k-mh}\boldsymbol{P}_{mh}'(\boldsymbol{A}^{k-mh})^T\\ &\quad + \sum_{j=1}^{k-mh}\boldsymbol{A}^{k-mh-j}\boldsymbol{Q}(\boldsymbol{A}^{k-mh-j})^T \qquad (9)\\ &\leq \boldsymbol{A}^{k-mh}\boldsymbol{P}_{mh}'(\boldsymbol{A}^{k-mh})^T\\ &\quad + \sum_{j=1}^{h}\boldsymbol{A}^{k-mh-j}\boldsymbol{Q}(\boldsymbol{A}^{k-mh-j})^T. \qquad (10)\end{aligned}$$

For convenience, we denote

$$\boldsymbol{S}_{k,mh}^1 = \boldsymbol{A}^{k-mh}\boldsymbol{P}_{mh}'(\boldsymbol{A}^{k-mh})^T,$$
$$\boldsymbol{S}_{k,mh}^2 = \sum_{j=1}^{h}\boldsymbol{A}^{k-mh-j}\boldsymbol{Q}(\boldsymbol{A}^{k-mh-j})^T.$$

Then by (10), we have

$$\boldsymbol{P}_k' \leq \boldsymbol{S}_{k,mh}^1 + \boldsymbol{S}_{k,mh}^2.$$

Hence by the matrix inverse formula, we have

$$\begin{aligned}\boldsymbol{P}_{k+1}' &= \boldsymbol{A}(\boldsymbol{P}_k'^{-1} + \boldsymbol{\varphi}_kR^{-1}\boldsymbol{\varphi}_k^T)^{-1}\boldsymbol{A}^T + \boldsymbol{Q}\\ &\leq \boldsymbol{A}[(\boldsymbol{S}_{k,mh}^1 + \boldsymbol{S}_{k,mh}^2)^{-1} + \boldsymbol{\varphi}_kR^{-1}\boldsymbol{\varphi}_k^T]^{-1}\boldsymbol{A}^T + \boldsymbol{Q}\\ &= \boldsymbol{S}_{k+1,mh}^1 + \boldsymbol{S}_{k+1,mh}^2 + \boldsymbol{Q} -\\ &\quad \frac{\boldsymbol{A}(\boldsymbol{S}_{k,mh}^1 + \boldsymbol{S}_{k,mh}^2)\boldsymbol{\varphi}_k\boldsymbol{\varphi}_k^T(\boldsymbol{S}_{k,mh}^1 + \boldsymbol{S}_{k,mh}^2)\boldsymbol{A}^T}{R + \boldsymbol{\varphi}_k^T(\boldsymbol{S}_{k,mh}^1 + \boldsymbol{S}_{k,mh}^2)\boldsymbol{\varphi}_k}.\end{aligned} \quad (11)$$

By the definition of $\boldsymbol{S}_{mh,h}$, it is easy to see that

$$\boldsymbol{S}_{mh,h} = \boldsymbol{A}^{mh-k}(\boldsymbol{S}_{k,mh}^1 + \boldsymbol{S}_{k,mh}^2)(\boldsymbol{A}^{mh-k})^T.$$

for any $k \in [mh, (m+1)h-1]$ holds. Moreover, by (11), we have for any $k \in [mh, (m+1)h-1]$

$$
\begin{aligned}
&\boldsymbol{A}^{mh-(k+1)}\boldsymbol{P}'_{k+1}(\boldsymbol{A}^{mh-(k+1)})^T\\
\leq& \boldsymbol{S}_{mh,h}+\boldsymbol{A}^{mh-(k+1)}\boldsymbol{Q}(\boldsymbol{A}^{mh-(k+1)})^T\\
&-\frac{\boldsymbol{S}_{mh,h}(\boldsymbol{A}^{k-mh})^T\boldsymbol{\varphi}_k\boldsymbol{\varphi}_k^T\boldsymbol{A}^{k-mh}\boldsymbol{S}_{mh,h}}{R+\boldsymbol{\varphi}_k^T\boldsymbol{A}^{k-mh}\boldsymbol{S}_{mh,h}(\boldsymbol{A}^{k-mh})^T\boldsymbol{\varphi}_k}\\
\leq& \boldsymbol{S}_{mh,h}+\sum_{j=1}^{h}\boldsymbol{A}^{-j}\boldsymbol{Q}(\boldsymbol{A}^{-j})^T\\
&-\frac{\boldsymbol{S}_{mh,h}\frac{(\boldsymbol{A}^{k-mh})^T\boldsymbol{\varphi}_k\boldsymbol{\varphi}_k^T\boldsymbol{A}^{k-mh}}{1+\|\boldsymbol{\varphi}_k^T\boldsymbol{A}^{k-mh}\|^2}\boldsymbol{S}_{mh,h}}{(1+R)\cdot\lambda_{\max}(\boldsymbol{S}_{mh,h})}\\
\leq& \boldsymbol{S}_{mh,h}+\sum_{j=1}^{h}\boldsymbol{A}^{-j}\boldsymbol{Q}(\boldsymbol{A}^{-j})^T\\
&-\frac{\boldsymbol{S}_{mh,h}\frac{(\boldsymbol{A}^{k-mh})^T\boldsymbol{\varphi}_k\boldsymbol{\varphi}_k^T\boldsymbol{A}^{k-mh}}{1+\|\boldsymbol{\varphi}_k^T\boldsymbol{A}^{k-mh}\|^2}\boldsymbol{S}_{mh,h}}{(1+R)\cdot(1+\lambda_{\max}(\boldsymbol{S}_{mh,h}))}\\
&\cdot\frac{htr(\boldsymbol{P}'_{mh})}{htr((\boldsymbol{S}_{mh,h}))} \qquad (12)
\end{aligned}
$$

Summing both sides of (12), by the definition of T_{m+1}, $\boldsymbol{S}_{mh,h}$, and a_{m+1}, we obtain

$$
\begin{aligned}
T_{m+1}\leq& htr\boldsymbol{P}'_{mh}-a_{m+1}htr\boldsymbol{P}'_{mh}\\
&+2h\sum_{j=1}^{h}tr[\boldsymbol{A}^{-j}\boldsymbol{Q}(\boldsymbol{A}^{-j})^T], \qquad (13)
\end{aligned}
$$

Again by (3) and (4), for any $i\in[(m-1)h,mh-1]$, we have

$$
\begin{aligned}
\boldsymbol{P}'_{mh}\leq& \boldsymbol{A}^{mh-(i+1)}\boldsymbol{P}'_{i+1}(\boldsymbol{A}^{mh-(i+1)})^T\\
&+\sum_{j=1}^{mh-(i+1)}\boldsymbol{A}^{mh-(i+1)-j}\boldsymbol{Q}(\boldsymbol{A}^{mh-(i+1)-j})^T.
\end{aligned}
$$

Then

$$
\begin{aligned}
htr\boldsymbol{P}'_{mh}=&\sum_{i=(m-1)h}^{mh-1}tr\boldsymbol{P}'_{mh}\\
\leq& tr\left(\boldsymbol{A}^h\sum_{i=(m-1)h}^{mh-1}\boldsymbol{S}^1_{(m-1)h,i+1}(\boldsymbol{A}^h)^T\right)\\
&+tr\left(\sum_{i=(m-1)h}^{mh-1}\sum_{j=1}^{mh-(i+1)}\boldsymbol{A}^{mh-(i+1)-j}\right.\\
&\left.\cdot\boldsymbol{Q}(\boldsymbol{A}^{mh-(i+1)-j})^T\right)\\
\leq& \|\boldsymbol{A}\|^{2h}T_m+htr(\sum_{j=0}^{h}\boldsymbol{A}^j\boldsymbol{Q}(\boldsymbol{A}^j)^T).
\end{aligned}
$$

Substituting this into (13), we obtain the results of the theorem. ∎

Theorem 2. Under Assumptions 1-3, the random Riccati equation defined in (6) is L_p-stable, i.e., $\sup_{k\geq 0}E\|\boldsymbol{P}'_k\|^p<\infty$.

Proof. If $\|\boldsymbol{A}\|<1$, then by (3) and (4), it is easy to see that

$$
\begin{aligned}
\|\boldsymbol{P}'_k\|\leq& \|\boldsymbol{A}\boldsymbol{P}'_{k-1}\boldsymbol{A}^T\|+\|\boldsymbol{Q}\|\\
\leq& \|\boldsymbol{A}\|^2\|\boldsymbol{P}'_{k-1}\|+\|\boldsymbol{Q}\|\\
\leq& \|\boldsymbol{A}\|^{2k}\|\boldsymbol{P}'_0\|+\sum_{j=0}^{k-1}\|\boldsymbol{A}\|^{2j}\|\boldsymbol{Q}\|\\
\leq& \|\boldsymbol{P}'_0\|+\frac{\|\boldsymbol{Q}\|}{1-\|\boldsymbol{A}\|^2}<\infty.
\end{aligned}
$$

If $\|\boldsymbol{A}\|\geq 1$, denote

$$c_{m+1}=2^{p-1}(1-a_{m+1})\|\boldsymbol{A}\|^{2ph}[I(tr\boldsymbol{P}'_{mh}\geq 1)].$$

By Theorem 1, C_r-inequality, (13), we have

$$
\begin{aligned}
T^p_{m+1}=& T^p_{m+1}[I(tr\boldsymbol{P}'_{mh}\geq 1)]+T^p_{m+1}[I(tr\boldsymbol{P}'_{mh}<1)]\\
\leq& \Big[(1-a_{m+1})\|\boldsymbol{A}\|^{2h}T_m+b_{\boldsymbol{A}}\Big]^p[I(tr\boldsymbol{P}'_{mh}\geq 1)]\\
&+[(1-a_{m+1})htr\boldsymbol{P}'_{mh}+b_{\boldsymbol{A}}]^p[I(tr\boldsymbol{P}'_{mh}<1)]\\
\leq& \Big[2^{p-1}(1-a_{m+1})\|\boldsymbol{A}\|^{2ph}T^p_m+2^{p-1}b^p_{\boldsymbol{A}}\Big]\\
&\cdot[I(tr\boldsymbol{P}'_{mh}\geq 1)]+[2^{p-1}h^p+2^{p-1}b^p_{\boldsymbol{A}}]\\
\leq& c_{m+1}T^p_m+(2b_{\boldsymbol{A}})^p+2^{p-1}h^p. \qquad (14)
\end{aligned}
$$

We denote $\mathscr{G}_m=\mathscr{F}_{mh-1}$, then $\boldsymbol{P}'_{mh}\in\mathscr{G}_m$. By the definition of a_{m+1} and the fact $tr(\boldsymbol{S}_{mh,h})^2\geq n^{-1}(tr\boldsymbol{S}_{mh,h})^2$ and $Etr\boldsymbol{B}=trE\boldsymbol{B}$ for any matrix $\boldsymbol{B}$, then we have

$$
\begin{aligned}
&E(a_{m+1}|\mathscr{G}_m)\\
=&\frac{tr\Big[(\boldsymbol{S}_{mh,h})^2E[\boldsymbol{G}((m+1)h,mh)]\Big|\mathscr{G}_m\Big]}{h(1+R)\cdot\Big[1+\lambda_{\max}(\boldsymbol{S}_{mh,h})\Big]tr(\boldsymbol{S}_{mh,h})}\\
\geq&\frac{(1+h)\lambda_m tr(\boldsymbol{S}_{mh,h})^2}{h(1+R)\cdot\Big[1+\lambda_{\max}(\boldsymbol{S}_{mh,h})\Big]tr(\boldsymbol{S}_{mh,h})}\\
\geq&\frac{(1+h)\lambda_m\cdot tr(\boldsymbol{S}_{mh,h})}{nh(1+R)\cdot\Big[1+\lambda_{\max}(\boldsymbol{S}_{mh,h})\Big]}\\
\geq&\frac{(1+h)\lambda_m}{2nh(1+R)}\qquad on\ \ tr(\boldsymbol{P}'_{mh}\geq 1).
\end{aligned}
$$

Hence by the definition of c_{m+1}, we have

$$
\begin{aligned}
E(c^4_{m+1}|\mathscr{G}_m)\leq& E\Big(16^{p-1}(1-a_{m+1})\|\boldsymbol{A}\|^{8ph}\\
&\cdot[I(tr\boldsymbol{P}'_{mh}\geq 1)]\Big|\mathscr{G}_m\Big)\\
\leq& 16^{p-1}\|\boldsymbol{A}\|^{8ph}\left(1-\frac{(1+h)\lambda_m}{2nh(1+R)}\right)\\
&\cdot[I(tr\boldsymbol{P}'_{mh}\geq 1)]. \qquad (15)
\end{aligned}
$$

Denote

$$
d_{m+1}=\begin{cases}c_{m+1}, & tr\boldsymbol{P}'_{mh}\geq 1;\\ 2^{p-1}\|\boldsymbol{A}\|^{2ph}\left(1-\dfrac{(1+h)\lambda_m}{2nh(1+R)}\right), & otherwise.\end{cases}
$$

By (14), we have

$$T^p_{m+1}\leq d_{m+1}T^p_m+(2b_{\boldsymbol{A}})^p+2^{p-1}h^p.$$

Hence by (15) and the definition of d_{m+1}, we have

$$\Big\| \prod_{m=j}^{k} E(d_{m+1}^4|\mathscr{G}_m)\Big\|_{L_1}$$
$$\leq \left\| \prod_{m=j}^{k} \left[16^{p-1}\|\boldsymbol{A}\|^{8ph}\left(1-\frac{(1+h)\lambda_m}{2nh(1+R)}\right)\right]\right\|_{L_1}. \quad (16)$$

By Assumption 2, we have $\{1-\lambda_m\} \in S^0(\lambda)$ for some $\lambda \in (0,1)$. Applying Lemma 1 and the fact $\lambda_m \leq \frac{h}{1+h}$, we obtain $\left\{1-\frac{(1+h)\lambda_m}{2nh(1+R)}\right\} \in S^0(\lambda^{\frac{1}{2nh(1+R)}})$. Hence by (16) and Assumption 3, we see that there exists a constant M such that

$$\Big\| \prod_{m=j}^{k} E(d_{m+1}^4|\mathscr{G}_m)\Big\|_{L_1} \leq M{\lambda_1}^{k-j+1}, \quad (17)$$

where $\lambda_1 \triangleq 16^{p-1}\|\boldsymbol{A}\|^{8ph}\lambda^{\frac{1}{2nh(1+R)}} \in (0,1)$. By Lemma 2 (ii), we have $\sup_m ET_m^p < \infty$.

Now, we prove the stability of random Ricatti equation. By the definition of T_m, we have

$$\begin{aligned}
&ET_{m+1}^p \\
&= \sum_{k=mh}^{(m+1)h-1} E\Big[tr(\boldsymbol{A}^{mh-(k+1)}\boldsymbol{P}'_{k+1}(\boldsymbol{A}^{mh-(k+1)})^T)\Big]^p \\
&\geq \sum_{k=mh}^{(m+1)h-1} E\left[\frac{\|\boldsymbol{P}'_{k+1}\|^p}{\lambda_{\max}^p\Big(\boldsymbol{A}^{(k+1)-mh}(\boldsymbol{A}^{(k+1)-mh})^T\Big)}\right] \\
&= \sum_{k=mh}^{(m+1)h-1} E\left[\frac{\|\boldsymbol{P}'_{k+1}\|^p}{\|\boldsymbol{A}^{(k+1)-mh}\|^{2p}}\right] \\
&\geq \sum_{k=mh}^{(m+1)h-1} E\left[\frac{\|\boldsymbol{P}'_{k+1}\|^p}{\|\boldsymbol{A}\|^{2p[(k+1)-mh]}}\right] \\
&\geq \frac{1}{\|\boldsymbol{A}\|^{2ph}} \sum_{k=mh}^{(m+1)h-1} E\|\boldsymbol{P}'_{k+1}\|^p.
\end{aligned}$$

Hence we can obtain that $\sup_k E\|\boldsymbol{P}'_k\|^p < \infty$. This completes the proof. ■

3.2 L_p-Exponential Stability of Random Riccati Equation

In this section, we establish the L_p-exponential stability of random Riccati equation.

Theorem 3. Under Assumptions 1-3, there exits a constant $\lambda_2 \in (0,1)$ such that $\left\{1-\frac{1}{1+\|\boldsymbol{Q}^{-1}\|\|\boldsymbol{P}'_k\|}\right\} \in S^0(\lambda_2)$.

Proof. Denote

$$\begin{aligned}
x_m &= h + M_{\boldsymbol{A}}\|\boldsymbol{Q}^{-1}\|T_m, \\
y_m &= \sum_{k=(m-1)h}^{mh-1}\left[1+\|\boldsymbol{Q}^{-1}\|tr\boldsymbol{P}'_{k+1}\right],
\end{aligned}$$

where $M_{\boldsymbol{A}} \triangleq \max\{1, \|\boldsymbol{A}\|^{2h}\}$ and T_k is defined in Theorem 1. Then by Theorem 1, we have

$$\begin{aligned}
x_{m+1} &= h + M_{\boldsymbol{A}}\|\boldsymbol{Q}^{-1}\|T_{m+1} \\
&\leq h + M_{\boldsymbol{A}}\|\boldsymbol{Q}^{-1}\|\big[(1-a_{m+1})\|\boldsymbol{A}\|^{2h}T_m + b_{\boldsymbol{A}}\big] \\
&\leq (1-a_{m+1})\|\boldsymbol{A}\|^{2h}(h + M_{\boldsymbol{A}}\|\boldsymbol{Q}^{-1}\|T_m) + h \\
&\quad + M_{\boldsymbol{A}}\|\boldsymbol{Q}^{-1}\|b_{\boldsymbol{A}} \\
&= (1-a_{m+1})\|\boldsymbol{A}\|^{2h}x_m + h + M_{\boldsymbol{A}}\|\boldsymbol{Q}^{-1}\|b_{\boldsymbol{A}}.
\end{aligned}$$

Denote

$$f_{m+1} = \begin{cases} (1-a_{m+1})\|\boldsymbol{A}\|^{2h}I(tr\boldsymbol{P}'_{mh}\geq 1), & tr\boldsymbol{P}'_{mh}\geq 1; \\ \|\boldsymbol{A}\|^{2ph}\left(1-\dfrac{(1+h)\lambda_m}{2nh(1+R)}\right), & otherwise. \end{cases}$$

Note that $a_m \in [0, \frac{1}{1+R}]$. Similar to the analysis of (17), there exists a constant M_1 and $\lambda_3 \in (0,1)$ such that

$$\Big\| \prod_{m=j}^{k} E[f_{m+1}^4|\mathscr{G}_m]\Big\|_{L_1} \leq M_1\lambda_3^{k-j+1}.$$

Then by Lemma 2 (iii), we have $\{1-\frac{1}{x_m}\} \in S^0(\gamma)$ for some $\gamma \in (0,1)$. By the definition of T_m,

$$\begin{aligned}
x_m &= \sum_{k=(m-1)h}^{mh-1}\Big[1 + M_{\boldsymbol{A}}\|\boldsymbol{Q}^{-1}\| \\
&\qquad \cdot tr\Big(\boldsymbol{A}^{(m-1)h-(k+1)}\boldsymbol{P}'_{k+1}(\boldsymbol{A}^{(m-1)h-(k+1)})^T\Big)\Big] \\
&\geq \sum_{k=(m-1)h}^{mh-1}\Big[1 + M_{\boldsymbol{A}}\|\boldsymbol{Q}^{-1}\| \\
&\qquad \lambda_{\min}\Big(\boldsymbol{A}^{(m-1)h-(k+1)}(\boldsymbol{A}^{(m-1)h-(k+1)})^T\Big)tr\boldsymbol{P}'_{k+1}\Big] \\
&\geq \sum_{k=(m-1)h}^{mh-1}\left[1 + \frac{M_{\boldsymbol{A}}\|\boldsymbol{Q}^{-1}\|tr\boldsymbol{P}'_{k+1}}{\|\boldsymbol{A}\|^{2[(k+1)-(m-1)h]}}\right] \\
&\geq \sum_{k=(m-1)h}^{mh-1}\left[1+\|\boldsymbol{Q}^{-1}\|tr\boldsymbol{P}'_{k+1}\right] = y_m.
\end{aligned}$$

Hence we have $\{1-\frac{1}{y_m}\} \in S^0(\gamma)$. Using the proof idea of Lemma 5 given by Guo (1990), it is easy to see that $\{1-\frac{1}{1+\|\boldsymbol{Q}^{-1}\|tr\boldsymbol{P}'_k}\} \in S^0(\lambda_2)$ for some $\lambda_2 \in (0,1)$, which completes the proof of this theorem. ■

Theorem 4. Under Assumptions 1-3, the sequence $\{\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{L}_k\boldsymbol{\varphi}_k^T),\ k \geq 0\}$ is L_p-exponentially stable, where p is defined in Assumption 3.

We omit the proof of this theorem due to space limitations.

In the following, we consider the stability of the Kalman filter. We first introduce the conditions about the initial value $\boldsymbol{\theta}_0$ and noises $\boldsymbol{w}_{k+1}$ and v_k.

Assumption 4. The initial value and noises satisfy the following conditions,

$$E\|\boldsymbol{\theta}_0\|^p < \infty, \quad \sup_k E[\|\boldsymbol{w}_{k+1}\|^p + \|v_k\|^{2p}] < \infty.$$

Theorem 5. Under Assumptions 1-4, the estimate error $\widetilde{\boldsymbol{\theta}}'_k \triangleq \boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}'_k$ satisfies $\sup_k \|\widetilde{\boldsymbol{\theta}}'_k\|_{L_{\frac{p}{2}}} < \infty$.

Proof. By (2) and (5), we have

$$\hat{\boldsymbol{\theta}}'_{k+1} = \boldsymbol{A}\hat{\boldsymbol{\theta}}'_k + \boldsymbol{A}\boldsymbol{L}_k(\boldsymbol{y}_k - \boldsymbol{\varphi}_k^T\hat{\boldsymbol{\theta}}'_k).$$

Combining this with (1), we obtain the following error equation

$$\widetilde{\boldsymbol{\theta}}'_{k+1} = \boldsymbol{A}(\boldsymbol{I} - \boldsymbol{L}_k\boldsymbol{\varphi}_k^T)\widetilde{\boldsymbol{\theta}}'_k + \boldsymbol{w}_{k+1} - \boldsymbol{A}\boldsymbol{L}_k v_k. \qquad (18)$$

Denote $\boldsymbol{\xi}_{k+1} = \boldsymbol{w}_{k+1} - \boldsymbol{A}\boldsymbol{L}_k v_k$. Then from (18), we have

$$\begin{aligned}\widetilde{\boldsymbol{\theta}}'_{k+1} = &\prod_{j=0}^{k}(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{L}_j\boldsymbol{\varphi}_j^T))\widetilde{\boldsymbol{\theta}}'_0\\ &+\sum_{j=1}^{k+1}\prod_{i=j}^{k}(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{L}_i\boldsymbol{\varphi}_i^T))\boldsymbol{\xi}_{i+1}.\end{aligned}$$

By Theorem 4 and the Hölder inequality, we obtain

$$\begin{aligned}\|\widetilde{\boldsymbol{\theta}}'_{k+1}\|_{L_{\frac{p}{2}}} \le &\left\|\prod_{j=0}^{k}(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{L}_j\boldsymbol{\varphi}_j^T))\right\|_{L_p}\|\widetilde{\boldsymbol{\theta}}'_0\|_{L_p}\\ &+\sum_{j=1}^{k+1}\left\|\prod_{i=j}^{k}(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{L}_j\boldsymbol{\varphi}_j^T))\right\|_{L_p}\|\boldsymbol{\xi}_{j+1}\|_{L_p}\\ \le &M\lambda_4^{k+1}O(1) + \sum_{j=1}^{k+1}M\lambda_4^{k-j+1}\|\boldsymbol{\xi}_{j+1}\|_{L_p}\\ = &O(1) + M\sum_{j=0}^{k}\lambda_4^j\|\boldsymbol{\xi}_{k-j+2}\|_{L_p}\end{aligned}$$

with $\lambda_4 \in (0,1)$. Then by Assumption 4, we have $\sup_j \|\boldsymbol{\xi}_j\|_{L_p} < \infty$ since $\|\boldsymbol{L}_k\| \le \frac{\|\boldsymbol{P}'_k\|^{\frac{1}{2}}}{(2\sqrt{R})}$ is holds. Therefore,

$$\|\widetilde{\boldsymbol{\theta}}'_{k+1}\|_{L_{\frac{p}{2}}} = O(1) + O(1)\sum_{j=0}^{k}\lambda^j = O(1),$$

which completes the proof. ■

4. CONCLUSION

In this paper, we studied the stability of Kalman filter under a general random framework including the widely investigated packet dropout problems. The theoretical results on the L_p-stability and L_p-exponentially stability of Kalman filter were established under the excited condition we proposed, without requiring on the independence and stationarity assumptions on the regressors. And the boundedness of the state estimation error can also be obtained under the excitation condition. Some interesting problems deserve to be further investigated, e.g., relaxation of the the assumptions on system matrix (Assumptions 1 and 3), performance analysis of the Kalman filter under this general framework, and the extension to the distributed Kalman filter.

ACKNOWLEDGEMENTS

We would like to express our sincere thanks to Prof. Lei Guo for his discussions and valuable suggestions.

REFERENCES

L. Guo. Estimating time-varying parameters by Kalman filter based algorithm: stability and convergence. *IEEE Transactions on Automatic Control*, 35(2):141–147, 1990.

L. Guo. Stability of recrsive stochastic tracking algorithms. *SIAM J. Control and Optimization*, 32(5):1195–1225, 1994.

Y. Wang, L. Guo. On stability of random Riccati equations. *Science in China (Series E)*, 42(2):136–148, 1999.

G. Welch, G. Bishop. An introduction to the Kalman filter. *Dept. of Computer Science, Univ. of North Carolina at Chapel Hill*, technical report, 2000.

J. Deyst, C. Price. Conditions for asymptotic stability of the discrete minimum-variance linear estimator. *IEEE Transactions on Automatic Control*, 13(6):702–705, 1968.

B. Sinopoli, L. Schenato, M. Franceschetti, et al. Kalman filtering with intermittent observation. *IEEE Transactions on Automatic Control*, 49(9):1453–1464, 2004.

K. Y. You, M. Y. Fu, L. H. Xie. Mean square stability for Kalman filtering with Markovian packet losses. *Automatica*, 47:2467–2657, 2011.

A. S. Leong, D. E. Quevedo, D. Dolz, & S. Dey. Transmission scheduling for remote state estimation over packet dropping links in the presence of an eavesdropper. *IEEE Transactions on Automatic Control*, 64(9):3732–3738, 2019.

H. Lin, J. Lam, Z. D. Wang, & H. Lam. State estimation over non-acknowledgment networks with Markovian packet dropouts. *Automatica*, 109:1–10, 2019.

M. Y. Huang, S. Dey. Stability of Kalman filtering with Markovian packet looses. *Automatica*, 43:598–607, 2007.

P. Bougerol Kalman filtering with random cofficients and contractions. *SIAM J. Control and Optimization*, 31(4): 942–959, 1993.

V. Solo. Stability of the Kalman filter with stochastic time-varying parameters. *Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan, December, 1996.

L. Xie, L. H. Xie. Stability of a random Riccati equation with markovian binary switching. *IEEE Transactions on Automatic Control*, 53(7):1759–1764, 2008.

Q. S. He, C. Wei, Y. Z. Xu. An improved adaptive Kalman filtering algorithm for balancing vehicle. *Proceedins of the 2017 Chinese Automation Congress*, Jinan, China, October, 2017.

T. Kailath, A. H. Sayed, B. Hassibi. Linear estimation. Upper Saddle River, NJ: Prentice-Hall, 2000.

R. E. Kalman. A new approach to linear filtering and prediction problems. *Trans. ASME-Journal of Basic Engineering* , 82 (Series D):35–45, 1960

L. Guo. Time-Varying Stochastic systems. *Jilin Scientific and Technological Press*, 1993.

L. Shi, M. Epstein, A. Tiwari, R.M. Murray. Estimation with information loss: asymptotic analysis and error bounds. *Proceedings of the 44th IEEE Conference on Decision and Control and Europen Control Conference*, Seville, Spain, 1215–1221, 2005.