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Distributed min–max MPC for dynamically coupled nonlinear systems: A self-triggered approach

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Abstract: This paper considers the formation stabilization problem of dynamically coupled nonlinear systems with parametric uncertainties and additive disturbances. We develop a distributed min–max model predictive control (MPC) framework, in which each subsystem adopts the local optimal control action by solving the constrained optimization problem. As a main contribution of this paper, a self-triggered strategy is presented within the proposed framework for resource-constrained coupled systems. By implementing the distributed self-triggered scheduler, the communication burden is significantly alleviated while ensuring comparable control performance. In addition, for each subsystem the latest information is transmitted to its neighbors asynchronously at the local triggering time instants. Moreover, the resulting distributed self-triggered min–max MPC framework ensures the constraints satisfaction and the closed-loop stability. Finally, the numerical experiments are performed to verify the theoretical results.

Keywords: Self-triggered, Min–max DMPC, Asynchronous communication, Coupled systems.

1. INTRODUCTION

Recent years have witnessed a growing trend to develop distributed model predictive control (DMPC) for large-scale systems and multi-agent systems (MAS), such as autonomous vehicles, power systems and process systems (Maestre et al. (2014)). The problem of interest in this work is to propose a distributed controller for the large-scale dynamically coupled nonlinear perturbed systems which are driven to stabilization in a cooperative fashion. The DMPC scheme is advocated here, because of computational savings and also its excellent ability to handle various constraints, such as actuator limits and state constraint (Dunbar and Murray (2006)). Some interesting results on DMPC for large-scale nonlinear systems have been developed for decoupled nonlinear systems (Dunbar and Murray (2006); Wang and Ding (2014)) and for dynamically coupled nonlinear systems (Dunbar (2007)). In practice, the systems are inevitably subject to disturbances. Different robust DMPC methods are proposed to accommodate external disturbances, for example, tube-based DMPC (Trodden and Richards (2010)), DMPC with robustness constraint (Li and Shi (2013)). It is worth noting that the above-mentioned robust DMPC methods only consider the additive disturbances instead of the parametric uncertainties. How to guarantee the robustness of coupled systems with parametric uncertainties and external disturbances remains challenging. Motivated by these facts, we aim to propose a min–max DMPC strategy for dynamically coupled nonlinear systems with parametric uncertainties and additive disturbances.

The aforementioned DMPC approaches have in common that the system state measurement, the calculated control inputs and information exchanged among subsystems are operated periodically. However, this may lead to bursts of communication and a nontrivial consumption in term of energy. Moreover, the subsystems may have limited bandwidth communication networks. To address this challenging issue, triggered control is proposed to reduce the communication burden of MASs, see (Heemels et al. (2012)) and the papers listed there. Triggered MPC aims to reduce the communication burden while achieve a desired level of control performance of MASs with control and state constraint. It makes a balance between the control performance and the communication cost. Some interesting results for the single agent have been reported in (Gommans and Heemels (2015); Gommans et al. (2014); Yao et al. (2018)). For MASs, a self-triggered distributed model predictive consensus approach is proposed in (Zhan et al. (2019)), where the control input and the triggering interval are jointly optimized. (Zou et al. (2019)) develops an event-triggered DMPC, where the neighbors information involved event-triggered condition is based on the novel robustness constraint. Motivated by these observations, in order to reduce the communication burden, the self-triggered robust DMPC is proposed for the dynamically coupled systems, where the states are aperiodically sampled and asynchronously broadcast.

In this work, we propose a distributed self-triggered min–max MPC scheme for large-scale dynamically coupled nonlinear systems with parametric uncertainties and disturbances. The main contributions of this work are as follows:

(1) A distributed self-triggered min–max MPC scheme is proposed to achieve comparable control performance as the periodic MPC method and decrease the communication burden. Under this framework, each subsystem asynchronously broadcasts its aperiodically updated state information to its neighbors.
(2) The conventional tube-based MPC method is difficult to deal with nonlinear systems with the parametric uncertainties (Mayne et al. (2011)). The min–max MPC considers the worst case of all possible additive disturbances and parametric uncertainties, ensuring the robustness of coupled systems.
(3) The proposed algorithm is proved to be recursively feasible. The closed-loop stability of the overall system is input-to-state practically stable (ISpS) (Jiang et al. (1994); Jiang and Wang (2001)) at triggering time instants in its region of attraction.

The paper is organized as follows: Section II formulates the control problem. In Section III, the main results are presented including the distributed min-max MPC design, the self-triggered scheduler, the algorithm. Section IV gives the theoretical analysis of the feasibility and the closed-loop stability. Section V presents the simulation results. Finally, the conclusion is given in Section VI.

2. PROBLEM FORMULATION

We make use of the following notation. The set $\mathbb{I}_{\geq 0}$ denotes the nonnegative integers and $\mathbb{I}_{m:n}$ denotes integers in the interval $[m, n]$ satisfying $m \leq n$. For any vector $x \in \mathbb{R}^n$, the Euclidean norm is $\|x\|$. $\|x\|_P^2$ denotes the weighted Euclidean norm $x^{\mathrm{T}} P x$, and P is a positive definite matrix. The superscript 'T' denotes the transposition.

The systems includes a group of M discrete-time perturbed nonlinear interconnected subsystems which are dynamically coupled. Agent $i, i \in \mathcal{M}, \mathcal{M} := \{1, \ldots, M\}$, is characterized by

$$S_i: \ x_i(t_{k+1}) = f_i(x_i(t_k), x_{-i}(t_k), u_i(t_k), d_i(t_k)), \quad (1)$$

where $x_i(t_k) \in \mathbb{X}_i \subseteq \mathbb{R}^n$, $x_{-i}(t_k)$, $u_i(t_k) \in \mathbb{U}_i \subseteq \mathbb{R}^m$ and $d_i(t_k) = \mathrm{col}(w_i(t_k), v_i(t_k)) \in \mathbb{D}_i \subseteq \mathbb{R}^d$ are the local state, neighboring states, control input and time-varying uncertainty,respectively. $w_i(t_k) \in \mathbb{W}_i \subseteq \mathbb{R}^w$ represents the additive disturbance and $v_i(t_k) \in \mathbb{V}_i \subseteq \mathbb{R}^v$ denotes the parametric uncertainty, where $\mathbb{W}_i$ and $\mathbb{V}_i$ are compact sets and contain the origin in their interiors. Each subsystem is assumed to have the same sampling period, i.e., $t_{k+1} = t_k + 1$. The directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is deployed to describe the systems and the information exchange between subsystems, where $\mathcal{V} = \{S_1, \cdots, S_M\}$ is the set of nodes and $\mathcal{E} = \{(S_i, S_j) \subset \mathcal{V} \times \mathcal{V}\}$ is the set of all directed edges that characterize the information from the node i to the node j. Each subsystem i, corresponding to the node i of the $\mathcal{G}$, has local control input u_i and local state x_i. Let $\mathcal{N}_i := \{j | (S_i, S_j) \in \mathcal{E}\}$ denote the set of indices of subsystem i's neighbors. Here the graph $\mathcal{G}$ is assumed to be strongly connected. To distinguish different trajectories, we denote the following trajectories: $x_i(s; t_k), s \in \mathbb{I}_{[0,N]}$ is the predictive state trajectory, $x_{-i}(s; t_k)$ concatenates the state trajectories of $j, j \in \mathcal{N}_i$, i.e., $x_{-i}(s; t_k) = (\ldots, x_j(s; t_k), \ldots)$; $x_i^*(s; t_k)$ is the optimal state trajectory; $x_i^b(s; t_k)$ is the newest broadcasting state of subsystem i. Similarly, $u_i(s; t_k)$ is the predictive control input, $u_i^*(s; t_k)$ is the optimal control input.

Definition 1. A set $\Omega \subseteq \mathbb{R}^n$ is a robust positive invariant (RPI) of system, $\forall x \in \Omega$, $f(x, \mathbb{D}) \subseteq \Omega$.

Lemma 2. (Limón et al. (2006), Theorem 1). For the system $x(t_k+1) = f(x(t_k), d(t_k))$ with an RPI set Ω, $\forall x \in \Omega$, $d \in \mathbb{D}$, a positive definite function $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ satisfy $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + c_1$; $V(f(x, d)) - V(x) \leq -\alpha_3(x) + \gamma(\|w\|) + c_2$, with $c_1, c_2 \geq 0$, $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)$ being $\mathcal{K}_\infty$ function and $\gamma(\cdot)$ being $\mathcal{K}$ function, then the system is ISpS in Ω with respect to w.

Our goal in this paper is to propose a distributed robust self-triggered MPC strategy for coupled systems with parametric uncertainty and external disturbances, such that the subsystems are cooperatively stabilized toward the origin,

$$u_i(t) = h_i(x_i(t_k), \hat{x}_{-i}(t_k), t - t_k), \ t \in [t_k, t_{k+1}), \quad (2)$$

where h_i denotes the control input generated by solving the distributed min–max MPC optimization problem, $\hat{x}_{-i}(t_k)$ represents a collection of the latest state information of subsystem $j, j \in \mathcal{N}_i$, which will be discussed later.

For subsystem $i, i \in \mathcal{M}$, $\{t_{k_i}^i\}$ is an independent sampling/triggering instant sequence that is determined by using its local self-triggered scheduler, i.e.,

$$t_{k_i+1}^i = t_{k_i}^i + H^i(x_i(t_{k_i}^i)), H^i(x_i(t_{k_i}^i)) \in \mathbb{I}_{\geq 1}. \quad (3)$$

For a given fixed prediction horizon $N \in \mathbb{I}_{>1}$, subsystem i broadcasts its newest state information to its neighbors at $t_{k_i}^i, t_{k_i}^i \in \mathbb{I}_{\geq 0}$. $\{t_{k_{-i}}^{-i}\}$ is a collection of the triggering instant sequence of neighbors $t_{k_{ij}}^j, j \in \mathcal{N}_i$. At $t_{k_i}^i, k_i \in \mathbb{I}_{\geq 0}$, its neighboring newest triggering instant $t_{k_{ij}}^j$ is defined as $t_{k_{ij}}^j := \arg\min_{t_{k_j}^j} \{t_{k_i}^i - t_{k_j}^j | \text{ s.t. } t_{k_j}^j \leq t_{k_i}^i\}$, where $k_j \in \mathbb{I}_{\geq 0}$.

For subsystem $i, i \in \mathcal{M}$, the cost function at its triggering instant $t_{k_i}^i$ is defined as

$$\begin{aligned} &J_{i,N}^{H^i}(x_i(t_{k_i}^i), u_i(s; t_{k_i}^i)) \\ &= \sum_{s=0}^{H^i-1} \frac{1}{\hbar_i} L_i(x_i(s; t_{k_i}^i), u_i(s; t_{k_i}^i)) \\ &\quad + \sum_{s=H^i}^{N-1} L_i(x_i(s; t_{k_i}^i), u_i(s; t_{k_i}^i)) + F_i(x_i(N; t_{k_i}^i)) \end{aligned} \quad (4)$$

where $s \in \mathbb{I}_{[0,N)}$, $\hbar_i \geq 1$ is a fixed constant. The local stage cost for the subsystem i is defined as

$$\begin{aligned} &L_i(x_i(s; t_{k_i}^i), u_i(s; t_{k_i}^i)) \\ =&\|x_i(s; t_{k_i}^i)\|_{Q_i}^2 + \|x_i(s; t_{k_i}^i) - x_i^b(s + H^{i*}(t_{k_i-1}^i); t_{k_i-1}^i)\|_{Q_i'}^2 \\ &+ \|u_i(s; t_{k_i}^i)\|_{R_i}^2 \end{aligned}$$

where $Q_i = Q_i^{\mathrm{T}} \succeq 0$, $Q_i' = Q_i'^{\mathrm{T}} \succeq 0$, and $R_i = R_i^{\mathrm{T}} \succ 0$, $x_i^b(s; t_{k_i}^i)$ is the broadcasting state sequence at last triggering time instant and the stage cost is assumed to be continuous with $L_i(0, 0) = 0$. Similarly, the local terminal cost is continuous with $F_i(0) = 0$ and can be defined as

$$F_i(x_i(N; t_{k_i}^i)) = \|x_i(N; t_{k_i}^i)\|_{P_i}^2,$$

where $P_i = P_i^{\mathrm{T}} \succeq 0$ is the weighting matrix.

Remark 3. Different from the consistency constraint in Zhan et al. (2019); Zheng et al. (2016), the local stage cost term $\|x_i(s;t^i_{k_i}) - x^b_i(s+H^{i*}(t^i_{k_i-1});t^i_{k_i-1})\|^2_{Q'_i}$ enforces a degree of consistency between what subsystem i plans to do and what neighbors believe subsystem i will do. In this way, the constraint is transformed into a soft constraint, and the feasible solution is easy to construct.

3. SELF-TRIGGERED MIN–MAX DMPC

In this section, the distributed robust MPC optimization problem for the dynamically coupled systems is defined. Then, we give the self-triggered asynchronous communication mechanism and the distributed self-triggered min–max MPC algorithm. Finally, the recursive feasibility and the closed-loop stability are analyzed.

3.1 Distributed min–max optimization problem

The local min–max optimization problem $\mathcal{P}_i$ for each subsystem $i, i \in \mathcal{M}$ at the triggering instant $t^i_{k_i}$ is defined in the following

$$\min_{u_i(s;t^i_{k_i})} \Big\{ \max_{d_i(s;t^i_{k_i})} \{J^{H^i}_{i,N}(x_i(t^i_{k_i}), u_i(s;t^i_{k_i}))\}, \text{such that}$$
$$x_i(H^i;t^i_{k_i}) \in \mathbb{X}_{i,N-H^i}(\Omega_i), \forall d_i(s;t^i_{k_i}) \in \mathbb{D}_i \Big\}$$
$$\text{s.t. } x_i(s+1;t^i_{k_i}) = f_i(x_i(s;t^i_{k_i}), \hat{x}_{-i}(t^{-i}_{k_{-i}}), u_i(s;t^i_{k_i}), d_i(s;t^i_{k_i})) \tag{5a}$$
$$x_i(0;t^i_{k_i}) = x_i(t^i_{k_i}) \tag{5b}$$
$$u_i(s;t^i_{k_i}) \in \mathbb{U}_i \tag{5c}$$
$$x_i(s;t^i_{k_i}) \in \mathbb{X}_i \tag{5d}$$

where $s \in \mathbb{I}_{[0,H^i)}$, $J^{H^i}_{i,N}(x_i(s;t^i_{k_i}), u_i(s;t^i_{k_i})) = \sum_{s=0}^{H^i-1} \frac{1}{\bar{h}_i} L_i (x_i(s;t^i_{k_i}), u_i(s;t^i_{k_i})) + V_{i,N-H^i}(x_i(H^i;t^i_{k_i}))$, $\mathbb{X}_l(\Omega_i)$ is l-step robust stabilizable set, i.e., any $x \in \mathbb{X}_l$ can be robustly steered into Ω_i within l steps. $V^{H^i}_{i,N}(x^*_i(s;t^i_{k_i}))$ denotes the optimal value of $\mathcal{P}_i$, and

$$V_{i,l}(x_i(l;t^i_{k_i})) = \min_{\mu_i(l;t^i_{k_i})} \Big\{ \max_{d_i(l;t^i_{k_i})} \{L(x_i(l;t^i_{k_i}), \mu_i(l;t^i_{k_i})) + V_{i,l-1}(x_i(l-1;t^i_{k_i}))\}, \text{such that}$$
$$f_i(x_i(l;t^i_{k_i}), \hat{x}_{-i}(l;t^{-i}_{k_{-i}}), \mu_i(l;t^i_{k_i}), \mathbb{D}_i) \subseteq \mathbb{X}_{i,l-1}(\Omega_i) \Big\},$$

where $l = N - s \in \mathbb{I}_{[1,N-H^i)}$ denotes steps to go and the boundary condition is $V_{i,0}(x_i(N;t^i_{k_i})) := F(x_i(N;t^i_{k_i}))$. Besides, the optimal predictive control input is denoted as $\mathbf{u}^*_i(t^i_{k_i}) = (u^*_i(0;t^i_{k_i}), \ldots, u^*_i(H^{i*}-1;t^i_{k_i}), \mu_i(H^{i*};t^i_{k_i}), \ldots, \mu_i(N-1;t^i_{k_i}))$, where $u^*_i(\cdot;t^i_{k_i})$ is the open-loop optimal control action and $\mu_i(\cdot;t^i_{k_i})$ is the feedback control law.

3.2 Distributed self-triggered scheduler

Consider the distributed self-triggered scheduler of subsystem $i, i \in \mathcal{M}$ at the triggering instant $t^i_{k_i}$,

$$t^i_{k_i+1} = t^i_{k_i} + H^{i*}(t^i_{k_i}), \tag{6a}$$
$$H^{i*}(t^i_{k_i}) = \max\{H^i \in \mathbb{I}_{[1,\bar{H}]} | V^{H^i}_{i,N}(x^*_i(s;t^i_{k_i})) \le V^1_{i,N}(x^*_i(s;t^i_{k_i}))\}, \tag{6b}$$

where $H^{i*}(t^i_{k_i})$ is the optimal triggering interval and $V^1_{i,N}(x^*_i(s;t^i_{k_i}))$ denotes the optimal value of $\mathcal{P}_i$ at $t^i_{k_i}+1$. $\bar{H}$ is the maximum triggering interval. The control input is defined as

$$u^{\text{mpc}}_i(t^i_{k_i}) := u^*_i(s;t^i_{k_i}),\ s \in \mathbb{I}_{[0,H^{i*}(t^i_{k_i}))}, \tag{7}$$

which is generated by solving the problem $\mathcal{P}_i$.

As shown in Fig. 1, three subsystems are included in the coupled system. The asynchronous broadcasting communication mechanism is built on the distributed self-triggered scheduler. The sequence of triggering instants of subsystem i is denoted as $t^i_{0_i}, t^i_{1_i}, t^i_{2_i}, \cdots$. For example, the controller of subsystem $i, i \in \mathcal{M}$ calculates the optimal control input and its next triggering instant $t^i_{1_i}$ at instant $t^i_{0_i}$ based on its local state $x_i(t^i_{0_i})$ and its neighboring systems' assumed state $\hat{x}_{-i}(t^{-i}_{0_{-i}})$, and broadcasts the newest updated state $x^b_i(s;t^i_{0_i}), s \in \mathbb{I}_{[0,N+H^i(t^i_{0_i})]}$ to its neighbors, applies the control input $u^{\text{mpc}}_i(t^i_{0_i})$ to the subsystem i and then it re-computes its optimal control input immediately at the next triggering instant $t^i_{1_i}$. The newest broadcasting state sequence of subsystem i at $t^i_{k_i}$ is constructed as

$$x^b_i(s;t^i_{k_i}) = \begin{cases} x^*_i(s;t^i_{k_i}) & s \in \mathbb{I}_{[0,N]}, \\ \tilde{x}_i(s;t^i_{k_i}) & s \in \mathbb{I}_{(N,H^{i*}(t^i_{k_i})+N]}, \end{cases} \tag{8}$$

where $\tilde{x}_i(s+1;t^i_{k_i})$ is the feasible system state under the local controller $\kappa_i(\tilde{x}_i(s;t^i_{k_i}))$.

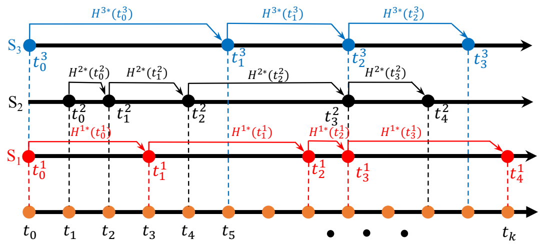

Fig. 1. An example of the asynchronous communication mechanism. The dots denote the triggering time instants. The bottom line is the global time sequence.

With the asynchronous broadcasting communication, the assumed latest neighboring state information received by subsystem i can be constructed as follows, $\hat{x}_{-i}(t^i_{k_i}) = \text{col}(\ldots, \hat{x}_j(t^i_{k_i}), \ldots), j \in \mathcal{N}_i$ and

$$\hat{x}_j(t^i_{k_i}) = [x^b_j(0;t^i_{k_i}), x^b_j(1;t^i_{k_i}), \cdots, x^b_j(N;t^i_{k_i})]. \tag{9}$$

3.3 Distributed self-triggered min–max MPC algorithm

The proposed distributed self-triggered min–max MPC algorithm for subsystem $i, i \in \mathcal{M}$ is specified as follows.

3.4 Theoretical analysis

Before introducing the main results, the following assumption and lemma are given.

Assumption 4. (Robust stability assumption) For subsystem $i, i \in \mathcal{M}$, the local decoupled terminal region $\Omega_i \subseteq \mathbb{X}_i$ is an RPI set for the system in (1) with a local feedback controller $\kappa_i(x_i(t^i_{k_i})) \in \mathbb{U}_i$ at time instant $t^i_{k_i}$. There exist $\mathcal{K}_\infty$ functions $\underline{\alpha}_L$, $\underline{\alpha}_F$ and $\bar{\alpha}_F$ and a non-negative constant ϵ such that: 1). $\underline{\alpha}_L(\|x_i(t^i_{k_i})\|) \le L_i(x_i(t^i_{k_i}), u_i(t^i_{k_i}))$,

Algorithm 1 Distributed self-triggered min–max MPC

1: **Initialization**: For each subsystem $i, i \in \mathcal{M}$, give the initial states $x_i(t^i_{k_i})$, $x_j(t^i_{k_i})$, the initial feasible control $\tilde{u}_i(\cdot, t^i_{k_i})$ and other design parameters. Set $k_i = 0$.
2: If $x_i(t^i_{k_i}) \in \Omega_i$, apply $u_i^{\mathrm{mpc}}(t^i_{k_i}) = \kappa_i(x_i(t^i_{k_i}))$, else go to Step 3:
3: Samples system state $x_i(t^i_{k_i})$ and receives the newest broadcasting information of its neighbors $x^b_j(s; t^j_{k_j}), s \in \mathbb{I}_{[0,N]}, j \in \mathcal{N}_i$;
4: Solves the optimization problem $\mathcal{P}_i$ and (6b), generating $\mathbf{u}^*_i(t^i_{k_i})$ and $H^{i*}(t^i_{k_i})$, and broadcasts the newest states $x^b_i(s; t^i_{k_i}), s \in \mathbb{I}_{[0,N+H^{i*}(t^i_{k_i})]}$ to its neighbors;
5: Applies the control input sequence $u_i^{\mathrm{mpc}}(t^i_{k_i})$;
6: $k_i = k_i + 1$, $t^i_{k_i} = t^i_{k_i} + H^{i*}(t^i_{k_i})$ and go to Step 2; if $t = T_{\mathrm{sim}}$, stop.

$\forall\, x_i(t^i_k) \in \mathbb{X}_i$, $\forall\, u_i(t^i_{k_i}) \in \mathbb{U}_i$; 2). $\underline{\alpha}_F(\|x_i(t^i_{k_i})\|) \le F_i(x_i(t^i_{k_i})) \le \bar{\alpha}_F(\|x_i(t^i_{k_i})\|)$, $\forall\, x_i(t^i_k) \in \Omega_i$; 3). $F_i(x_i(t^i_{k_i} + 1)) - F_i(x_i(t^i_{k_i})) \le -L_i(x_i(t^i_{k_i}), \kappa_i(x_i(t^i_{k_i}))) + \epsilon$, where $\forall\, x_i(t^i_{k_i}) \in \mathbb{X}_i$ and $\epsilon := \max_{d_i}\{L_i(x_i(t^i_{k_i}), \kappa_i(x_i(t^i_{k_i}))) + F_i(x_i(t^i_{k_i} + 1)) - F_i(x_i(t^i_{k_i}))\}$.

Lemma 5. Suppose that *Assumption 5* holds. Then, $V_{i,l+1}(x_i(t^i_{k_i})) - V_{i,l}(x_i(t^i_{k_i})) \le V_{i,l}(x_i(t^i_{k_i} + 1)) - V_{i,l-1}(x_i(t^i_{k_i} + 1))$, $x_i(t^i_{k_i}) \in \mathbb{X}_{i,l}(\Omega_i)$. Also $V_{i,l}(x_i(t^i_{k_i})) - V_{i,l-1}(x_i(t^i_{k_i})) \le \epsilon$ and $V_{i,l}(x_i(t^i_{k_i})) \le V_{i,0}(x_i(t^i_{k_i})) + l\epsilon$, where $x_i(t^i_{k_i}) \in \Omega_i$.

Proof. For $x_i(t^i_{k_i}) \in \Omega_i$, by *Assumption 5*, and hence

$$\begin{aligned} & V_{i,1}(x_i(t^i_{k_i})) - V_{i,0}(x_i(t^i_{k_i})) \\ =& \min_{u_i(t^i_{k_i})} \max_{d_i(t^i_{k_i})} \{-V_{i,0}(x_i(t^i_{k_i})) + L_i(x_i(t^i_{k_i}), u_i(t^i_{k_i})) \\ & + V_{i,0}(x_i(t^i_{k_i} + 1))\} \\ \le& \epsilon. \end{aligned} \tag{10}$$

For $x_i(t^i_{k_i}) \in \mathbb{X}_{i,l}(\Omega_i)$, the control action $u_i(t^i_{k_i}) = h(x(t^i_{k_i}))$ is assumed to be a feasible solution to the optimization problem $\mathcal{P}_i$, hence we have $x_i(t^i_{k_i} + 1) = f(x_i(t^i_{k_i}), \hat{x}_{-i}(t^i_{k_i}), h(x_i(t^i_{k_i})), d_i(t^i_{k_i}))$. Moreover,

$$\begin{aligned} & V_{i,l+1}(x_i(t^i_{k_i})) - V_{i,l}(x_i(t^i_{k_i})) \\ \le& \max_{d_i(t^i_{k_i})} \{L_i(x_i(t^i_{k_i}), h(x_i(t^i_{k_i}))) \\ & + V_{i,l}(f(x_i(t^i_{k_i}), \hat{x}_{-i}(t^i_{k_i}), h(x_i(t^i_{k_i})), d_i(t^i_{k_i})))\} \\ & - \max_{d_i(t^i_{k_i})} \{L_i(x_i(t^i_{k_i}), h(x_i(t^i_{k_i}))) \\ & + V_{i,l-1}(f(x_i(t^i_{k_i}), \hat{x}_{-i}(t^i_{k_i}), h(x_i(t^i_{k_i})), d_i(t^i_{k_i})))\} \\ \le& \max_{d_i(t^i_{k_i})} \{V_{i,l}(x_i(t^i_{k_i} + 1)) - V_{i,l-1}(x_i(t^i_{k_i} + 1))\} \\ \le& \epsilon. \end{aligned} \tag{11}$$

From (10) and (11), it follows that $V_{i,l}(x_i(t^i_{k_i})) \le V_{i,0}(x_i(N; t^i_{k_i})) + l\epsilon$, $\forall x_i(t^i_{k_i}) \in \Omega_i$. This is the monotonicity property of the value function for the constrained distributed min–max MPC optimization problem.

Theorem 6. For the dynamically coupled perturbed nonlinear systems with asynchronous communication graph $\mathcal{G}$, suppose that $x_i(t^i_{k_i}) \in \mathbb{X}_i$ and *Assumption 5* and *Lemma 6* are satisfied. Then, Problem $\mathcal{P}_i$ of **Algorithm 1** is recursively feasible. Furthermore, the closed-loop perturbed system in (1) with the distributed self-triggered min–max MPC strategy is ISpS with respect to the set Ω_i.

Proof. *Part I: Recursive Feasibility:* We first establish the recursive feasibility of **Algorithm 1**. Assume that there exists a feasible solution for problem $\mathcal{P}_i$ at $t^i_{0_i}$ and recursive feasibility for all subsequent triggering instants is proven by induction. The optimal control sequence obtained at $t^i_{k_i}$ is denoted as $\mathbf{u}^*_i(t^i_{k_i}) = (u^*_i(0; t^i_{k_i}), \ldots, u^*_i(H^{i*} - 1; t^i_{k_i}), \mu_i(H^{i*}; t^i_{k_i}), \ldots, \mu_i(N-1; t^i_{k_i}))$. The first H^{i*} open-loop control actions are applied to the system i. At the next triggering time instant $t^i_{k_i+1} = t^i_{k_i} + H^{i*}(t^i_{k_i})$, a candidate control $\tilde{u}_i(s; t^i_{k_i+1})$ can be constructed as

$$\tilde{u}_i(s; t^i_{k_i+1}) = \begin{cases} \mu_i(H^{i*} + s; t^i_{k_i}) & s \in \mathbb{I}_{[0,N-H^{i*})} \\ \kappa_i(\tilde{x}_i(H^{i*} + s; t^i_{k_i})) & s \in \mathbb{I}_{[N-H^{i*},N)}. \end{cases}$$

where $H^{i*}(t^i_{k_i})$ is abbreviated as H^{i*}. By *Assumption 5*, it follows that $\tilde{u}_i(s; t^i_{k_i+1}) \in \mathbb{U}_i, s \in \mathbb{I}_{[H^{i*}(t^i_{k_i}), H^{i*}(t^i_{k_i})+N)}$, the control input constraint (5c) is satisfied. Hence, we only need to prove the state constraint is fulfilled at next triggering instant $t^i_{k_i+1}$. For $s \in \mathbb{I}_{[H^{i*}(t^i_{k_i}),N)}$, we have $\tilde{x}_i(s; t^i_{k_i+1}) = x^*_i(s; t^i_{k_i}) \in \mathbb{X}_i$; then, for $s \in \mathbb{I}_{[N,H^{i*}(t^i_{k_i})+N)}$, under the local controller $\kappa_i(\tilde{x}_i(s; t^i_{k_i+1}))$, the subsystem states always belong to the robust invariant set Ω_i. Hence, the subsystem state constraint (5d) is satisfied.

Part II: Stability: Now the stability of the closed-loop system i is analyzed. Follow the idea in (Liu et al. (2018)), we give the upper bound of the value function. As *Assumption 5* holds, then the candidate ISpS-type Lyapunov function for subsystem $i, i \in \mathcal{M}$, $V^{H^i}_{i,N}(x_i(t^i_{k_i})) \ge L(x_i(t^i_{k_i}), u_i(t^i_{k_i})) \ge \underline{\alpha}_L(\|x_i(t^i_{k_i})\|)$. Define a set $B_{i,r} = \{x_i(t^i_{k_i}) \in \mathbb{R}^n | \|x_i(t^i_{k_i})\| \le r_i\} \subseteq \Omega_i$. Due to the compactness of $\mathbb{X}_i$ and $\mathbb{U}_i$, the optimal value of the min–max MPC cost function is upper bounded, i.e., $V^{H^i}_{i,N}(x_i(t^i_{k_i})) \le \bar{V}_{i,N}$.

If $x_i(t^i_{k_i}) \in \Omega_i$, according to *Assumption 5*, we have

$$F_i(x_i(t^i_{k_i} + 1)) - F_i(x_i(t^i_{k_i})) \le -L_i(x_i(t^i_{k_i}), \kappa_i(x_i(t^i_{k_i}))) + \epsilon. \tag{12}$$

By summing up (12) from $s = 0$ to N, we get

$$\begin{aligned} & F_i(x_i(N; t^i_{k_i})) + \sum_{s=0}^{N-1} L_i(x_i(s; t^i_{k_i}), \kappa_i(x_i(s; t^i_{k_i}))) \\ \le& F_i(x_i(0; t^i_{k_i})) + N\epsilon. \end{aligned} \tag{13}$$

As the self-triggered condition (6b) is satisfied, i.e., $V^{H^i}_{i,N}(x_i(t^i_{k_i})) \le V^1_{i,N}(x_i(t^i_{k_i}))$. From (13) follows that,

$$V^{H^i}_{i,N}(x_i(t^i_{k_i})) \le V_{i,N}(x_i(t^i_{k_i})) \le \bar{\alpha}_F(\|x_i(t^i_{k_i})\|) + N\epsilon.$$

Next, if $x_i(t^i_{k_i}) \in \mathbb{X}_{i,N}(\Omega_i) \backslash \Omega_i$, which implies that $\bar{\alpha}_F(\|x_i(t^i_{k_i})\|) \ge \bar{\alpha}_F(r_i)$. And thus

$$V^{H^i}_{i,N}(x_i(t^i_{k_i})) \le \bar{V}_{i,N} \frac{\bar{\alpha}_F(\|x_i(t^i_{k_i})\|)}{\bar{\alpha}_F(r_i)} \le \theta_i \bar{\alpha}_F(\|x_i(t^i_{k_i})\|) + N\epsilon$$

where $\theta_i = \max\{1, \frac{\bar{V}_{i,N}}{\bar{\alpha}_F(r_i)}\}$.

If the control action generated by **Algorithm 1** $u_i(t^i_{k_i}) = u_i^{\mathrm{mpc}}(t^i_{k_i})$ is applied, then the system evolves to $x_i(t^i_{k_i+1}) =$

$f(x_i(t_k^i), \hat{x}_{-i}(t_{k_i}^i), u_i^{\text{mpc}}(t_{k_i}^i), d_i(t_{k_i}^i))$. Based on the monotonicity of the value function in *Lemma 6*, we obtain

$$\begin{aligned}
&V_{i,N}^{H^{i*}(t_{k_i+1}^i)}(x_i(t_{k_i+1}^i)) - V_{i,N}^{H^{i*}(t_{k_i}^i)}(x_i(t_{k_i}^i)) \\
\leq& V_{i,N}^1(x_i(t_{k_i+1}^i)) - V_{i,N}^{H^{i*}}(x_i(t_{k_i}^i)) \\
\leq& V_{i,N}(x_i(t_{k_i+1}^i)) - \max_{d_i \in \mathbb{D}_i}\{V_{i,N-H^{i*}}(x_i^*(H^{i*}; t_{k_i}^i)) \\
&+ \sum_{s=0}^{H^{i*}-1} \frac{1}{\hbar_i} L_i(x_i^*(s; t_{k_i}^i), u_i^*(s; t_{k_i}^i))\} \\
\leq& V_{i,N}(x_i(t_{k_i+1}^i)) - V_{i,N-H^{i*}}(x_i(t_{k_i+1}^i)) \\
&- \sum_{s=0}^{H^{i*}-1} \frac{1}{\hbar_i} L_i(x_i^*(s; t_{k_i}^i), u_i^*(s; t_{k_i}^i)) \\
\leq& - \sum_{s=0}^{H^{i*}-1} \frac{1}{\hbar_i} L_i(x_i^*(s; t_{k_i}^i), u_i^*(s; t_{k_i}^i)) + H^{i*}(t_{k_i}^i)\epsilon
\end{aligned} \tag{14}$$

Clearly, the function $V_{i,N}^{H_i^i}(x_i(t_{k_i}^i))$ is the ISpS-type Lyapunov function, which means that the closed-loop system i is ISpS with respect to Ω_i. This concludes the proof.

4. NUMERICAL EXAMPLE

Consider a three-agent system that comprises three discrete nonlinear cart-spring-damper system (Liu et al. (2014)). Each subsystem i is characterized by

$$\begin{aligned}
x_{i1}(t_k+1) =& x_{i1}(t_k) + Tx_{i2}(t_k) \\
x_{i2}(t_k+1) =& x_{i2}(t_k) + \frac{T}{m_i}(-k_0 e^{-x_{i1}(t_k)} x_{i1}(t_k) \\
&- h_d x_{i2}(t_k) - k_c(x_{i1}(t_k) - x_{j1}(t_k)) \\
&+ u_i(t_k) + v_i(t_k)x_{i2}(t_k) + w_i(t_k))
\end{aligned} \tag{15}$$

where x_{i1} and x_{i2} are the displacement and the velocity, respectively, the mass $m_i = 1$kg, the local nonlinear spring steady-state stiffness $k_0 = 0.33$N/m, the interconnecting linear spring stiffness $k_c = 0.01$N/m, the local viscous damping $h_d = 0.3$Ns/m, the sampling period is 0.3s. The communication topology of systems adopts the neighbor-to-neighbor method as shown in Fig. 2. The disturbances and parametric uncertainties are bounded by $-0.1 \leq w_i(t_k) \leq 0.1$, $-0.05 \leq v_i(t_k) \leq 0.05$, respectively. The control input requires $-2\text{N} \leq u_i(t_k) \leq 2\text{N}$ and the system state requires $-2\text{m} \leq x_{i1}(t_k) \leq 2\text{m}$. The initial states of three subsystems are $x_1(t_0) = [1.5, 0.5]^{\text{T}}$, $x_2(t_0) = [-1.6, -0.8]^{\text{T}}$, $x_3(t_0) = [-0.65, 1.0]^{\text{T}}$.

The parameters of distributed min–max MPC optimization problem are chosen as: the prediction horizon is selected as $N = 5$ and the maximum triggering interval $\bar{H} = 4$. The weighting matrices are chosen as $Q_i = \text{diag}(0.64, 0.64)$, $P_i = [4.5678, 3.2018; 3.2018, 4.3500]$ and $R_i = 1$. The trade-off parameter $\hbar_i = 1.1$. According to (Liu et al. (2018); Magni et al. (2006)) the terminal conditions for min–max MPC are designed as follows: $\Omega_i = \{x : \|x\|_{P_i} \leq \sqrt{3.8}\}$ and the terminal control law is chosen as $\kappa_i(x_i) = [-0.7797, -1.1029]x_i$. The feedback control policy $\mu_i(x_i) = a\kappa_i(x_i) + b\|x_i\|^2 + c$, with $a, b, c \in \mathbb{R}$. The disturbances of each subsystem are shown in Fig. 5. Two group of numerical experiments are conducted, i.e., the distributed periodic min–max MPC and the distributed self-triggered min–max MPC. The constrained distributed min–max MPC optimization problem is solved by using built-in function `fminimax` in MATLAB.

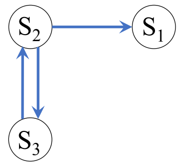

Fig. 2. Interconnection topology of three subsystems.

The displacements, velocities, self-triggered instants and control signals of three subsystems are depicted in Figs. 3-4. Fig. 3 illustrates the displacements, velocities trajectories and triggering time instants of three subsystems. The control input signals of subsystems with periodic sampling and self-triggered sampling are shown in Fig. 4. To further compare the control performance and communication cost, the average sampling time and the average cost are summarized in TABLE. 1. The average cost is defined as $\bar{J} = \frac{J_1+J_2+J_3}{3}$, $J_i = \frac{\sum_{t_k=0}^{T_{\text{sim}}} \|x_i(t_k)\|_{Q_i} + \|u_i\|_{R_i}}{T_{\text{sim}}}$, where T_{sim} is the total simulation time. As can be seen, the communication cost is significantly reduced by using the proposed algorithm while achieving comparable control performance of distributed periodic min–max MPC.

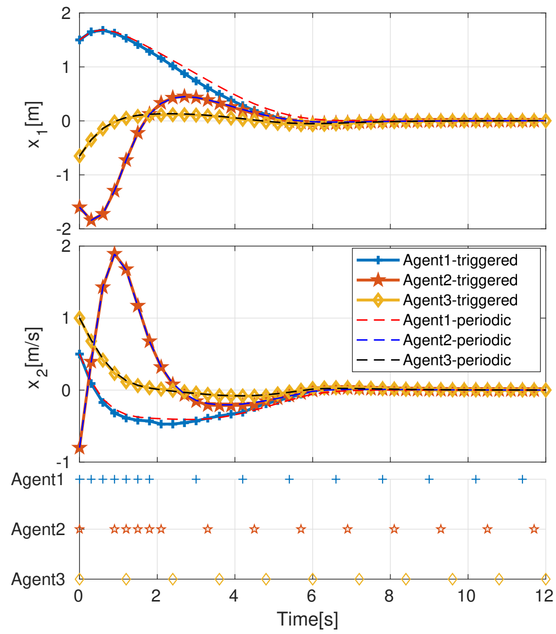

Fig. 3. State trajectories of three subsystems with periodic sampling and self-triggered sampling and the triggering time instants of three subsystems. **Top:** The displacement trajectory of three subsystems. **Middle:** The velocities. **Bottom:** The triggering instants.

Table 1. Performance comparison.

Method	Average sampling time	Average cost
Periodic	0.3000	1.4702
self-triggered	**0.9065**	1.5600

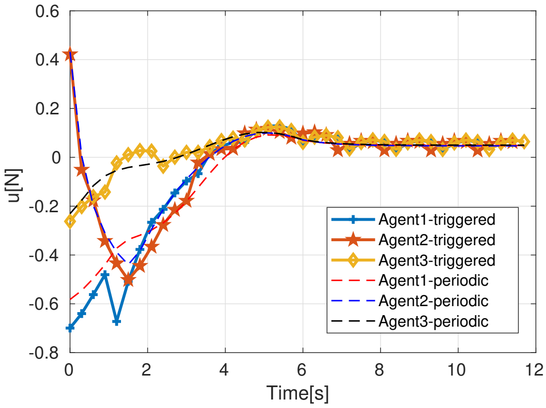

Fig. 4. Trajectories of control u_i of three subsystems with the periodic and self-triggered setting.

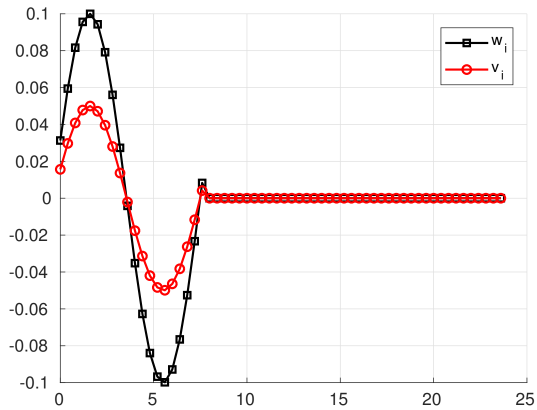

Fig. 5. Disturbances trajectories of three subsystems.

5. CONCLUSION

In this paper, we propose a distributed self-triggered min–max MPC algorithm for the formation stabilization of the nonlinear dynamically coupled systems with parametric uncertainties and external disturbances. Each subsystem aperiodically samples its states and asynchronously broadcasts its newest states to its neighbors based on the designed distributed self-triggered scheduler. The recursive feasibility and closed-loop stability of the proposed method are rigorously analyzed. The simulation results verified the effectiveness of the proposed method.

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