Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Boundary stabilization of a reaction-diffusion system weakly coupled at the boundary.

Mohamed Ghattassi * **and Taous Meriem Laleg** *

* *Computer Electrical and Mathematical Science and Engineering (CEMSE) Division, King Abdullah university of science and technology, KSA (e-mail:* mohamed.ghattassi@kaust.edu.sa taousmeriem.laleg@kaust.edu.sa*).*

Abstract: This study analyses the boundary stabilization of a system of two parabolic linear PDEs weakly coupled at the boundary. This model is motivated by heat transfer in a membrane distillation based desalination modeled by a two-dimensional advection diffusion equations coupled at the boundary. Based on some physical assumptions, the 2D model can be formulated as a 1D reaction-diffusion system. Two cases were studied: full and under actuated scenarios. In the full actuated case, a backstepping approach is used to map the plant to an exponentially stable target system. The well-posedness of the kernel equations is proved. Moreover, the actuation of only one of the parabolic equations has been considered. The standard backstepping transformations is again used to transform the initial plant to a desired target system where Lyapunov analysis is adequately used. Finally, a numerical example showing the performance of the proposed control design is presented.

Keywords: Backstepping, reaction-diffusion system, boundary Stabilization, Membrane Distillation, full/under actuated.

1. INTRODUCTION

Water supply is one of the biggest challenges facing the world. Indeed, due to the increasing potable water demand and the limited natural ressources, many countries rely on alternative solutions for their water supply such as water desalination. However, conventional desalination technologies are usually expensive and energy inefficient. More sustainable and efficient water desalination solutions are being investigated. Among the emerging technologies is Membrane Distillation (MD). It is a thermally driven technology, which uses a hydrophobic membrane to filter the water. This technology has been widely studied during the recent years from different perspectives such as mathematical modeling, design and optimization. However very few work has been done in term of control design, where the control problem would be to control the water production to a desired level (or maximum level) while reducing the energy consumption. Some control solutions have been proposed for a Direct Contact Membrane Distillation (DCMD), one the simplest configurations of the MD process, based on a two dimensional coupled advection diffusion model coupled at the boundary in Eleiwi and Laleg-Kirati (2016, 2018), and based on a reduced order approximation given by an algebraic differential equation in Karam et al. (2017); Karam and Laleg-Kirati (2019).

In this paper, we investigate the control problem using a stabilization of a one dimensional approximation of the two dimensional model. This approximation is given by a reaction-diffusion system weakly coupled at the boundary. While the problem of control of PDEs has been well covered, the control of coupled PDE systems has been less investigated. Recently, boundary stabilization for general linear unstable parabolic systems with constant-coefficient and spatially-varying reaction was analyzed Baccoli et al. (2014, 2015); Vazquez and Krstic (2017) . Among the proposed control strategies is the backstepping based control Smyshlyaev and Krstic (2010). For instance, in Vazquez and Krstic (2017), the authors show that the kernels of the reaction-diffusion-convection system with spatially varying coefficients are equivalent to those satisfied by the control kernels for hyperbolic systems. Their proof is based on the results presented within Hu et al. (2016, 2018), showing that the kernel equations of the quasilinear hyperbolic system are well-posed. The authors applied this result in the case of parabolic systems with spatially varying coefficients, and they proved H^1 exponential stability of the closed-loop system. However, in Tsubakino et al. (2013) the stabilization of two coupled reaction-diffusion equations is studied, where the problem is solved using a single control input acting only at the boundary. The authors used a nonconventional backstepping approach using a discontinuous kernel function. More recently, a comparison between the unilateral and bilateral control laws for a class of reaction-diffusion system with an interface relation has been presented in Vazquez and Krstic (2016). The authors proved that the bilateral control law is more accurate for large coefficients. The interface relation presented in this publication does not cover the class of

⋆ This work has been supported by the King Abdullah University of Science and Technology (KAUST) Base Research Fund (BAS/1/1627-01-01) to Taous Meriem Laleg.

systems investigated in our paper. Some specific results about high-dimensional systems of coupled PDEs have been recently considered using the backstepping boundary stabilization techniques, Meurer (2012); Meurer and Kugi (2009); Jadachowski et al. (2015). Moreover, in Vazquez and Krstic (2015) the problem of stabilization of reaction-diffusion equations in arbitrary dimension was studied. However, it is worth noting that several recent publications have studied the controller design for PDE-ODE cascade systems, for example Di Meglio et al. (2018); Deutscher et al. (2019).

The objective of this work is to stabilize two unstable reaction-diffusion equations coupled at the boundary using full and under actuated control law. A particular structure of the kernel exploring the boundary coupling property for the under actuated boundary control of the full system is investigated. This choice reduces the computational complexity of solving the full-kernel system. Under some conditions on the kernel, we prove the L^2 exponential stability of the closed-loop system.

The remainder of the paper is organized as follows. In section 2, the problem of full actuated boundary control of the unstable reaction-diffusion system coupled at the boundary is stated, and the Backstepping transformation is introduced. The under actuated scenarios where the Lyapunov stability analysis is showed in Section 3. Section 4 presents a numerical example illustrating the theoretical result. Finally, section 5 summarizes the paper and gives some future perspectives of this work.

Notations : Throughout this paper, standard notations are used. $L^2(0,1)$ for the Hilbert space of square integrable scalar functions.

$$\|z(.)\|_{2,2}^2 = \left(\|z_1(.)\|_2^2 + \|z_2(.)\|_2^2\right),$$

is adopted for the corresponding norm in $[L^2(0,1)]^2$of a generic vector $z(t,x) = [z_1(t,x),\ z_2(t,x)]^T$.

For a symmetric and positive definite matrix A, $\sigma_m(A)$ and $\sigma_M(A)$ denote the smallest and largest eigenvalues of A, respectively.

2. FULL ACTUATED BOUNDARY STABILIZATION OF PARABOLIC SYSTEM COUPLED AT THE BOUNDARY

Consider the linear parabolic system

$$\begin{cases} q_t(t,x) = \alpha q_{xx}(t,x) + \lambda q(t,x) & t>0 \quad x\in(0,1), \\ q(t,1) = U(t) & t>0, \\ q_x(t,0) = \mathcal{M}q(t,0) & t>0, \\ q(0,x) = q_0(x) & (0,1), \end{cases} \tag{1}$$

with, $\lambda = \begin{bmatrix}\lambda_f & 0\\ 0 & \lambda_p\end{bmatrix}$, $\alpha = \begin{bmatrix}\alpha_f & 0\\ 0 & \alpha_p\end{bmatrix}$, $\mathcal{M} = \begin{bmatrix}\gamma_f & -\gamma_f\\ -\gamma_p & \gamma_p\end{bmatrix}$, $U(t)\in\mathbb{R}^2$ is control law acting on the boundary, $q(t,x) = [q_1(t,x), q_2(t,x)]^T$ is the state of the reaction-diffusion system, and γ_f and γ_p are physical parameters of the device. System (1) can be considered as a simplification to 1D of the two-dimensional DCMD parabolic system; For a cross-section, y-direction is neglected, as shown in the (Ghattassi et al., 2019, figure 1). Moreover, some local reactions in the feed and permeate side can generate an instability of the system. However, the reaction coefficients λ_f, λ_p are assumed sufficiently large such that (1) is open-loop unstable system. To solve the state feedback problem we consider the controller

$$U(t) = \int_0^1 \kappa(x)q(t,x)dx. \tag{2}$$

In order to calculate the state feedback gain $\kappa(x)$ the backstepping approach is used. Thereby, controller coordinates $z(t,x) = [z_1(t,x),\ z_2(t,x)]^T$ are introduced by the backstepping transformation

$$z(t,x) = q(t,x) - \int_0^x K(x,\xi)q(t,\xi)d\xi = \mathcal{T}_b[q(t)](x), \tag{3}$$

where the kernel has the following structure

$$K(x,\xi) = \begin{bmatrix}K^{11}(x,\xi) & K^{12}(x,\xi)\\ K^{21}(x,\xi) & K^{22}(x,\xi)\end{bmatrix}. \tag{4}$$

Let us introduce the target system

$$\begin{cases} z_t(t,x) = \alpha z_{xx}(t,x) - Cz(t,x) & t>0 \quad x\in(0,1), \\ z(t,1) = 0 & t>0, \\ z_x(t,0) = \mathcal{M}z(t,0) & t>0, \\ z(0,x) = z_0(x) & (0,1), \end{cases} \tag{5}$$

where $C = diag(c_{11}, c_{22})$.

Theorem 1. Consider the system (1) with initial condition $q_0 \in [L^2(\Omega)]^2$. The backstteping mapping (3) transfers system (1) into a target system (5), where the kernel matrix $K(x,\xi)$ is a solution from the following hyperbolic system of PDEs:

$$\begin{cases} \alpha K_{xx}(x,\xi) - K_{\xi\xi}(x,\xi)\alpha - CK(x,\xi) - K(x,\xi)\lambda = 0, \\ \alpha K_x(x,x) + \alpha\dfrac{d}{dx}K(x,x) + C + \lambda + K_\xi(x,x)\alpha = 0, \\ \alpha K(x,x) - K(x,x)\alpha = 0, \\ K_\xi(x,0)\alpha = -K(x,0)\alpha\mathcal{M}, \end{cases} \tag{6}$$

where $(x,\xi)\in\mathcal{T} = \{x\in(0,1) \quad 0\leqslant x\leqslant 1, \quad 0\leqslant\xi\leqslant x\}$, and $\frac{d}{dx}K(x,x) = K_x(x,x) + K_\xi(x,x)$.

Proof. From (3) it follows

$$z_t(t,x) = q_t(t,x) - \int_0^x K(x,\xi)q_t(t,\xi)d\xi, \tag{7}$$

$$z_x(t,x) = q_x(t,x) - \int_0^x K_x(x,\xi)q(t,\xi)d\xi - K(x,x)q(t,x),$$

and

$$z_{xx}(t,x) = q_{xx}(t,x) - \int_0^x K_{xx}(x,\xi)q(t,\xi)d\xi$$
$$- K_x(x,x)q(t,x) - K(x,x)q_x(t,x) - \frac{d}{dx}K(x,x)q(t,x),$$

where $\frac{d}{dx}K(x,x) = K_x(x,x) + K_\xi(x,x)$. Thereby, substituting (1) and (5) inside (7), we obtain

$$\alpha z_{xx}(t,x) - Cz(t,x) = \alpha q_{xx}(t,x) + \lambda q(t,x) - \int_0^x K(x,\xi)\left(\alpha q_{\xi\xi}(t,\xi) + \lambda q(t,\xi)\right)d\xi,$$

then,

$$\begin{aligned}
&\alpha\Big(- \int_0^x K_{xx}(x,\xi)q(t,\xi)d\xi - K_x(x,x)q(t,x) \\
&- K(x,x)q_x(t,x) - \frac{d}{dx}K(x,x)q(t,x)\Big) \\
&- Cq(t,x) + C\int_0^x K(x,\xi)q(t,\xi)d\xi \\
&= \lambda q(t,x) - \int_0^x K(x,\xi)\left(\alpha q_{\xi\xi}(t,\xi) + \lambda q(t,\xi)\right)d\xi,
\end{aligned} \tag{8}$$

However, we have

$$\begin{aligned}
&\int_0^x K(x,\xi)\alpha q_{\xi\xi}(t,\xi)d\xi = K(x,x)\alpha q_x(t,x) \\
&\quad - K(x,0)\alpha q_x(t,0) - K_\xi(x,0)\alpha q(t,0) \\
&\quad + \int_0^x K_{\xi\xi}(x,\xi)\alpha q(t,\xi)d\xi + K_\xi(x,x)\alpha q(t,x).
\end{aligned} \tag{9}$$

Consequently, we have

$$\begin{aligned}
&\alpha\Big(- \int_0^x K_{xx}(x,\xi)q(t,\xi)d\xi - K_x(x,x)q(t,x) \\
&- K(x,x)q_x(t,x) - \frac{d}{dx}K(x,x)q(t,x)\Big) \\
&- Cq(t,x) + C\int_0^x K(x,\xi)q(t,\xi)d\xi = \lambda q(t,x) \\
&- K(x,x)\alpha q_x(t,x) + K(x,0)\alpha q_x(t,0) \\
&- \int_0^x K_{\xi\xi}(x,\xi)\alpha q(t,\xi)d\xi + K_\xi(x,x)\alpha q(t,x) \\
&+ K_\xi(x,0)\alpha q(t,0) - \int_0^x K(x,\xi)\lambda q(t,\xi)d\xi.
\end{aligned} \tag{10}$$

Then

$$\begin{aligned}
&\int_0^x \Big(\alpha K_{xx}(x,\xi) - K_{\xi\xi}(x,\xi)\alpha + CK(x,\xi) - K(x,\xi)\lambda\Big) q(t,\xi)d\xi \\
&+ \Big(\alpha K_x(x,x) + \alpha\frac{d}{dx}K(x,x) + C + \lambda + K_\xi(x,x)\alpha\Big) q(t,x) \\
&(\alpha K(x,x) - K(x,x)\alpha)\, q_x(t,x) \\
&+ K(x,0)\alpha q_x(t,0) + K_\xi(x,0)\alpha q(t,0) = 0
\end{aligned} \tag{11}$$

it follows that

$$\alpha K_{xx}(x,\xi) - K_{\xi\xi}(x,\xi)\alpha - CK(x,\xi) - K(x,\xi)\lambda = 0$$

We also need

$$\alpha K_x(x,x) + \alpha\frac{d}{dx}K(x,x) + C + \lambda + K_\xi(x,x)\alpha = 0,$$

and

$$\alpha K(x,x) - K(x,x)\alpha = 0.$$

However, we have

$$\begin{aligned}
&+K_\xi(x,0)\alpha q(t,0) + K(x,0)\alpha q_x(t,0) \\
&\quad = \Big(K_\xi(x,0)\alpha + K(x,0)\alpha\mathcal{M}\Big) q(t,0),
\end{aligned} \tag{12}$$

it leads

$$K_\xi(x,0)\alpha + K(x,0)\alpha\mathcal{M} = 0.$$

Finally, the kernel satisfies (6). By considering the previously relations, it follows that the backstteping mapping (3) transfers system (1) into a target system (5). This finish the proof.

Now, we give the proof of well-posedness of the kernel (6). The demonstration is inspired by the work of Vazquez and Krstic Vazquez and Krstic (2017).

Proposition 1. The kernel system (6) has a solution in the domain $\mathcal{T}$. The transformation (3) is invertible.

Proof. We define

$$S(x,\xi) = K_\xi(x,\xi)\sqrt{\alpha} - K(x,0)\bar{\mathcal{M}}\sqrt{\alpha} - \sqrt{\alpha}K_x(x,\xi). \tag{13}$$

where $\bar{\mathcal{M}}\alpha = \alpha\mathcal{M}$.

Our original system is equivalent to

$$K_\xi(x,\xi)\sqrt{\alpha} + K(x,0)\bar{\mathcal{M}}\sqrt{\alpha} - \sqrt{\alpha}K_x(x,\xi) = S(x,\xi) \tag{14}$$

$$\begin{aligned}
\sqrt{\alpha}S_x(x,\xi) + S_\xi(x,\xi)\sqrt{\alpha} &= -K\lambda - CK \\
&- \sqrt{\alpha}\frac{dK(x,0)}{dx}\bar{\mathcal{M}}\sqrt{\alpha}
\end{aligned} \tag{15}$$

To determine the boundary conditions of K, and S, we will analyze separate the kernel S and K. From the boundary conditions

$$\alpha K(x,x) - K(x,x)\alpha,$$

is automatically satisfied for $i = j$, $i, j \in \{1, 2\}$. Otherwise we need

$$K^{ij}(x,x) = 0, \quad \forall i \neq j.$$

Now, for the boundary condition of S, from (13), we have

$$\begin{aligned}
S(x,\xi)\sqrt{\alpha} &= K_\xi(x,\xi)\alpha + K(x,0)\bar{\mathcal{M}}\alpha - \sqrt{\alpha}K_x(x,\xi)\sqrt{\alpha} \\
&= K_\xi(x,\xi)\alpha + K(x,0)\alpha\mathcal{M} - \sqrt{\alpha}K_x(x,\xi)\sqrt{\alpha}.
\end{aligned} \tag{16}$$

Therefore, we obtain

$$S(x,0)\sqrt{\alpha} = -\sqrt{\alpha}K_x(x,0)\sqrt{\alpha}.$$

Finally, this result shows an equivalence between the kernel equations for this case and the kernel equation for general hyperbolic system, see Hu et al. (2018, 2016). Then, we can deduce the well-posedness criteria of our system. This finishes the proof.

3. UNDERACTUATED BOUNDARY STABILIZATION OF PARABOLIC SYSTEM COUPLED AT THE BOUNDARY

We consider now the unilateral boundary control,where the boundary control input $U(t)$ is given by

$$U = \begin{bmatrix} U_1 \\ 0 \end{bmatrix}. \tag{17}$$

Let us introduce the new variable $z(t,x) = [z_1(t,x),\ z_2(t,x)]^T$, solution of the following target system

$$\begin{cases}
z_t - \alpha z_{xx} = -C(\gamma^*)z + W_1 z(t,0) & t > 0 \quad x \in (0,1), \\
z(t,1) = 0 & t > 0, \\
z_x(t,0) = 0 & t > 0, \\
z(0,x) = z_0(x) & (0,1),
\end{cases} \tag{18}$$

with W_1 is defined further in the paper and $C(\gamma^*)$ is a square matrix depending now on $\gamma^* > 0$, design parameter, is chosen to guarantee the exponential stabilization of the target $z-$system (5).

We assume that the kernel has the following structure

$$K(x,y) = \begin{bmatrix} k(x,y) & 0 \\ 0 & 0 \end{bmatrix}, \tag{19}$$

it leads,

$$z_1(t,x) = q_1(t,x) - \int_0^x k(x,\xi)q_1(t,\xi)d\xi, \tag{20}$$

$$z_2(t,x) = q_2(t,x), \tag{21}$$

where the kernel $k(x,y)$ is solution of

$$\begin{cases} \alpha_f k_{xx}(x,\xi) - \alpha_f k_{\xi\xi}(x,\xi) - \gamma^* k(x,\xi) = 0, & (x,\xi) \in \mathcal{T}, \\ k(x,x) = -\frac{\gamma^*}{2}x, \quad x \neq 0 \\ k(0,0) = -\gamma_f, \\ k_\xi(x,0) - \gamma_f k(x,0) = 0. \end{cases} \tag{22}$$

The explicit form of the kernel is given in Smyshlyaev and Krstic (2004). Therefore, the kernel $k(x,y)$ is given by

$$k(x,y) = \frac{\gamma_f \gamma^*}{\sqrt{\gamma^* + \gamma_f^2}} \int_0^{x-y} e^{-\gamma_f \tau/2} I_0\left(\sqrt{\gamma^*(x+y)(x-y-\tau)}\right) sinh\left(\frac{\sqrt{\gamma^*+\gamma_f^2}}{2}\tau\right) d\tau - \gamma^* x \frac{I_1\left(\sqrt{\gamma^*(x^2-y^2)}\right)}{\sqrt{\gamma^*(x^2-y^2)}}, \tag{23}$$

for $(x,y) \neq (0,0)$, with I_i is a modified Bessel function of order i, for more details see Smyshlyaev and Krstic (2010).

Remark 1. The backstepping transformation (3) is invertible and the inverse is defined by Smyshlyaev and Krstic (2010)

$$q(t,x) = z(t,x) - \int_0^x L(x,\xi) z(t,\xi) d\xi, \tag{24}$$

with the following structure of kernel

$$L(x,y) = \begin{bmatrix} l(x,y) & 0 \\ 0 & 0 \end{bmatrix},$$

such that the kernel PDE of the inverse transformation is given by

$$\begin{cases} \alpha_f l_{xx}(x,\xi) - \alpha_f l_{\xi\xi}(x,\xi) = -\gamma^* l(x,\xi), & (x,\xi) \in \mathcal{T}, \\ l(x,x) = -\frac{\gamma^*}{2}x, \\ l(0,0) = -\gamma_f, \\ l_\xi(x,0) - \gamma_f l(x,0) = 0, \end{cases} \tag{25}$$

where the explicit solution is drawn in Smyshlyaev and Krstic (2004).

Based on the proposed structure of kernel (19), we explore the boundary coupling property of our system to propose the following structure of the matrix $C = C(\gamma^*)$ that depends on the parameter γ^*, where $c_{11} = \alpha_f \gamma^* - \lambda_f$ and $c_{22} = -\lambda_p$. The parameter $\gamma^* > 0$ satisfies the following condition

$$\frac{1}{2\sigma_m(C(\gamma^*))} \alpha_f^2 \gamma_f^2 \gamma^* e^{2\gamma^*} \frac{4}{\pi^2} \leq \sigma_m(\alpha). \tag{26}$$

However, for a very small value of γ^*, the condition (26) holds. This value will be further chosen to guarantee the exponential stability of the target system.

Theorem 2. The Backstteping transformation (3), (20),(21) such that the kernel $k(x,\xi)$ solution of (22), transfers initial system into the target system dynamics (5), where W_1 is given by

$$W_1 = \begin{bmatrix} 0 & -\alpha_f \gamma_f k(x,0) \\ 0 & 0 \end{bmatrix}. \tag{27}$$

Proof. From (3) it follows

$$z_t(t,x) = q_t(t,x) - \int_0^x K(x,\xi) q_t(t,\xi) d\xi, \tag{28}$$

$$z_x(t,x) = q_x(t,x) - \int_0^x K_x(x,\xi) q(t,\xi) d\xi - K(x,x) q(t,x),$$

and

$$z_{xx}(t,x) = q_{xx}(t,x) - \int_0^x K_{xx}(x,\xi) q(t,\xi) d\xi - K_x(x,x) q(t,x) - K(x,x) q_x(t,x) - \frac{d}{dx} K(x,x) q(t,x),$$

where

$$\frac{d}{dx} K(x,x) = K_x(x,x) + K_\xi(x,x).$$

Thereby, substituting (1) and (5) inside (7), we obtain

$$\begin{aligned} \alpha z_{xx}(t,x) - C(\gamma^*) z(t,x) + W_1(t,x) z(t,0) &= \alpha q_{xx}(t,x) + \lambda q(t,x) \\ &- \int_0^x K(x,\xi)\left(\alpha q_{\xi\xi}(t,\xi) + \lambda q(t,\xi)\right) d\xi, \end{aligned}$$

then, from (8), (9), and using the structure of the kernel (19), it follows that

$$\alpha K_{xx}(x,\xi) - K_{\xi\xi}(x,\xi)\alpha - C(\gamma^*) K(x,\xi) - K(x,\xi)\lambda = \begin{bmatrix} \alpha_f k_{xx}(x,\xi) - \alpha_f k_{\xi\xi}(x,\xi) - c_1 k(x,\xi) - \lambda_f k(x,\xi) & 0 \\ 0 & 0 \end{bmatrix},$$

then into (22) we get

$$\alpha K_{xx}(x,\xi) - K_{\xi\xi}(x,\xi)\alpha - CK(x,\xi) - K(x,\xi)\lambda = 0.$$

Therefore it leads

$$\frac{d}{dx} k(x,x) = -\frac{c_1 + \lambda_f}{2\alpha_f} = -\frac{\gamma^*}{2},$$

and

$$c_2 + \lambda_p = 0.$$

Then, it easy to satisfy

$$\alpha K_x(x,x) + \alpha \frac{d}{dx} K(x,x) + C + \lambda + K_\xi(x,x)\alpha = 0.$$

However, for the boundary conditions it follows

$$K_\xi(x,0)\alpha q(t,0) + K(x,0)\alpha q_\xi(t,0) = \begin{bmatrix} \alpha_f\left(k_\xi(x,0) - \gamma_f k(x,0)\right) f(t,0) - \alpha_f \gamma_f k(x,0) p(t,0) \\ 0 \end{bmatrix}. \tag{29}$$

From the backstteping transformation, we have

$$p(t,0) = z_2(t,0),$$

from the invertible backstteping transformation, we deduce

$$q_1(t,x) = z_1(t,x) - \int_0^x l(x,\xi) z_1(t,\xi) d\xi, \tag{30}$$

then

$$q_1(t,0) = z_1(t,0), \tag{31}$$

Thus

$$\begin{bmatrix} q_1(t,0) \\ q_2(t,0) \end{bmatrix} = \begin{bmatrix} z_1(t,0) \\ z_2(t,0) \end{bmatrix}.$$

By considering the previously relations, it follows that the backstteping mapping (3) transfers system (18),(17) into a target system (5).

Based on some hypothesis we now state the stability result of the target system in the space $[L^2(0,1)]^2$. The next theorem specifies the main stability result of this paper.

Theorem 3. Under hypothesis (26), the boundary control input

$$U_1(t) = \int_0^1 k(1,\xi) q_1(t,\xi) d\xi, \tag{32}$$

exponentially stabilizes the system (1) in $[L^2(0,1)]^2$.

Proof. The candidate Lyapunov function is given by

$$V(t) = \frac{1}{2}\|z(t,.)\|_{2,2}^2.$$

The derivative along the solutions of the system (5) takes the form

$$\dot{V}(t) = \int_0^1 z^T(t,x)\left(\alpha z_{xx} - C(\gamma^*)z + W_1(t,x)z(t,0)\right)dx,$$

the two first terms on the right hand side of (3) can be estimated as follows

$$\int_0^1 z^T(t,x)\alpha z_{xx}(t,x)dx \leq -\sigma_m(\alpha)\|z_x(t,.)\|_{2,2}^2,$$

$$-\int_0^1 z^T(t,x)C(\gamma^*)z(t,x)dx \leq -\sigma_m\left(C(\gamma^*)\right)\|z(t,.)\|_{2,2}^2.$$

From (Smyshlyaev and Krstic, 2004, Theorem 2), we have

$$|k(x,\xi)| \leqslant \gamma^* e^{2\gamma^* x} \quad \forall x \in (0,1). \tag{33}$$

From Young's inequality, for all $\epsilon > 0$

$$\begin{aligned}
&\int_0^1 z^T(t,x)W_1(t,x)z(t,0)dx \\
&\qquad = -\alpha_f\gamma_f\int_0^1 z_1(t,x)k(x,0)z_2(t,0)dx \\
&\qquad \leq \frac{1}{2\epsilon}\int_0^1 z_1^2(t,x)dx \\
&\qquad + \frac{\epsilon}{2}\alpha_f^2\gamma_f^2\int_0^1 k(x,0)dx z_2^2(t,0),
\end{aligned}$$

for $\epsilon = \frac{1}{\sigma_m(C(\gamma^*))}$,

$$\int_0^1 z^T(t,x)W_1(t,x)z(t,0)dx \leq \frac{\sigma_m\left(C(\gamma^*)\right)}{2}\int_0^1 z_1^2(t,x)dx$$
$$+ \frac{1}{2\sigma_m\left(C(\gamma^*)\right)}\alpha_f^2\gamma_f^2\int_0^1 k(x,0)dx z_2^2(t,0),$$

Now, using the Poincaré inequality and (33) it leads

$$\begin{aligned}
&\frac{1}{2\sigma_m\left(C(\gamma^*)\right)}\alpha_f^2\gamma_f^2\int_0^1 k(x,0)dx z_2^2(t,0) \\
&\leq \frac{1}{2\sigma_m\left(C(\gamma^*)\right)}\alpha_f^2\gamma_f^2\gamma^* e^{2\gamma^*}\frac{4}{\pi^2}\|z_x(t,.)\|_{2,2}^2.
\end{aligned}$$

Under condition (26), it follows that

$$\dot{V}(t) \leq -\frac{\sigma_m\left(C(\gamma^*)\right)}{2}V(t).$$

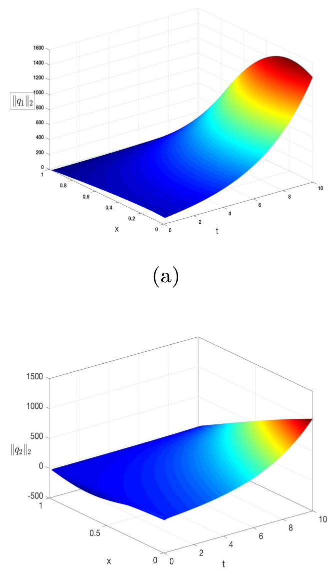
Finally, applying Gronwall's inequality the exponential stability of the target dynamics is proved.

4. NUMERICAL ILLUSTRATIONS

We now describe a numerical example to illustrate the effectiveness of the proposed method for the stabilization of unstable parabolic system coupled at the boundary. This numerical test is carried out in MATLAB. A standard finite-difference approximation method is used in all simulations by discretizing the spatial domain $x \in [0,1]$. Denoting N the number of the spatial nodes, $x_i = ih$, $h = 1/(N+1)$, $i = 1,2,...,N$. The employed value in the numerical example is equal to $N = 40$. The implicit stable scheme for the temporal discretization with time step $\delta t = 10^{-4}$ is used.

The physical parameters of system are as follows: $\alpha_f = 2$ $\alpha_p = 3$ $\lambda_f = 4$, $\lambda_p = 1$, $\gamma_f = 0.3$ and $\gamma_p = 0.5$. The initial conditions are set as $q_{10}(x) = 100cos(\frac{\pi}{2}x)$, $q_{02}(x) = 100cos(\frac{3\pi}{2}x)$. Figures 1-(a), 1-(b) and 3-(a) show the blow-up of the temporal evolutions of the L^2 norms of $\|q_1(.,t)\|_2$ and $\|q_2(.,t)\|_2$ for the open loop case.

The boundary control law U_1 (32), is implemented for the value $\gamma^* = 0.2$ satisfying condition (26) . The convergence of the solution to zeros are proved in figure 2 presenting the exponential decay of $(\|q_1(.,t)\|_2, \|q_2(.,t)\|_2)$ to zero. Figures 2-(a) and 2-(b) show the time evolution of solution (q_1, q_2) for the closed loop system. The time evolution of control is depicted in figure 4. These results demonstrate the applicability of the proposed boundary stabilization method.



(a)

(b)

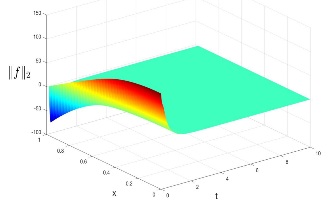
Fig. 1. Spatiotemporal evolution of the state $(\|q_1(.,t)\|_2, \|q_2(.,t)\|_2)$, in the open-loop test.

5. CONCLUSION

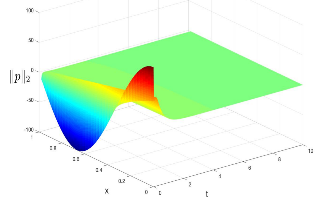
A backstteping approach has been proposed for the boundary stabilization for an unstable two parabolic linear PDEs weakly coupled at the boundary. We propose qui-diffusivity and underactuated boundary control law for the stabilization of the reaction-diffusion system based on the particular form of the kernel matrix which allows to control both equations by only transformation of variable. The boundary coupling property of the system is explored to prove the exponential decay of the closed-loop system in $[L^2(0,1)]^2$. These results open the door to the output feedback design of parabolic system coupled at the boundary including the convection term and with spatially-dependent parameters.

REFERENCES

Baccoli, A., Orlov, Y., and Pisano, A. (2014). On the boundary control of coupled reaction-diffusion equations having the same diffusivity parameters. In *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, 5222–5228. IEEE.

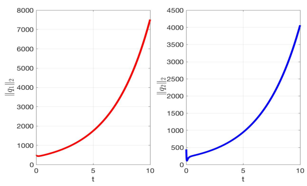


(a)

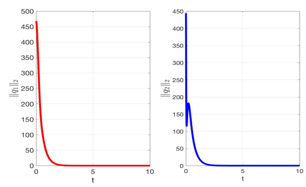


(b)

Fig. 2. Spatiotemporal evolution of the state $(\|q_1(.,t)\|_2, \|q_2(.,t)\|_2)$, in the closed loop test.



(a) Open loop



(b) Closed loop

Fig. 3. Temporal evolution of the norms $(\|q_1(.,t)\|_2, \|q_2(.,t)\|_2)$.

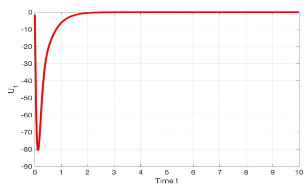


Fig. 4. Time evolution of the control U_1.

Baccoli, A., Pisano, A., and Orlov, Y. (2015). Boundary control of coupled reaction-diffusion processes with constant parameters. *Automatica*, 54, 80–90.

Deutscher, J., Gehring, N., and Kern, R. (2019). Output feedback control of general linear heterodirectional hyperbolic pde-ode systems with spatially-varying coefficients. *International Journal of Control*, 92(10), 2274–2290.

Di Meglio, F., Argomedo, F.B., Hu, L., and Krstic, M. (2018). Stabilization of coupled linear heterodirectional hyperbolic pde–ode systems. *Automatica*, 87, 281–289.

Eleiwi, F. and Laleg-Kirati, T.M. (2016). Nonlinear observer-based lyapunov boundary control of distributed heat transfer mechanisms for membrane distillation plant. *Journal of Process Control*, 47, 78–86.

Eleiwi, F. and Laleg-Kirati, T.M. (2018). Observer-based perturbation extremum seeking control with input constraints for direct-contact membrane distillation process. *International Journal of Control*, 91(6), 1363–1375.

Ghattassi, M., Laleg, T.M., and Vivalda, J.C. (2019). Analysis and output tracking design for the direct contact membrane distillation system. *Preprint,https://arxiv.org/abs/1902.00964.*

Hu, L., Di Meglio, F., Vazquez, R., and Krstic, M. (2016). Control of homodirectional and general heterodirectional linear coupled hyperbolic pdes. *IEEE Transactions on Automatic Control*, 61(11), 3301–3314.

Hu, L., Vazquez, R., Di Meglio, F., and Krstic, M. (2018). Boundary exponential stabilization of 1-d inhomogeneous quasilinear hyperbolic systems. *SIAM (SICON).*

Jadachowski, L., Meurer, T., and Kugi, A. (2015). Backstepping observers for linear pdes on higher-dimensional spatial domains. *Automatica*, 51, 85–97.

Karam, A.M., Alsaadi, A.S., Ghaffour, N., and Laleg-Kirati, T.M. (2017). Analysis of direct contact membrane distillation based on a lumped-parameter dynamic predictive model. *Desalination*, 402, 50–61.

Karam, A.M. and Laleg-Kirati, T.M. (2019). Membrane fouling modeling and detection in direct contact membrane distillation. *Journal of Process Control*, 81, 190–196.

Meurer, T. (2012). *Control of Higher-Dimensional PDEs: Flatness and Backstepping Designs*. Springer Science & Business Media.

Meurer, T. and Kugi, A. (2009). Tracking control for boundary controlled parabolic pdes with varying parameters: Combining backstepping and differential flatness. *Automatica*, 45(5), 1182–1194.

Smyshlyaev, A. and Krstic, M. (2004). Closed-form boundary state feedbacks for a class of 1-d partial integro-differential equations. *IEEE Transactions on Automatic Control*, 49(12), 2185–2202.

Smyshlyaev, A. and Krstic, M. (2010). *Adaptive control of parabolic PDEs*. Princeton University Press.

Tsubakino, D., Krstic, M., and Yamashita, Y. (2013). Boundary control of a cascade of two parabolic pdes with different diffusion coefficients. In *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*, 3720–3725. IEEE.

Vazquez, R. and Krstic, M. (2015). Boundary control of reaction-diffusion pdes on balls in spaces of arbitrary dimensions. *arXiv preprint arXiv:1511.06641.*

Vazquez, R. and Krstic, M. (2016). Bilateral boundary control of one-dimensional first-and second-order pdes using infinite-dimensional backstepping. In *Decision and Control (CDC), 2016 IEEE 55th Conference on*, 537–542. IEEE.

Vazquez, R. and Krstic, M. (2017). Boundary control of coupled reaction-advection-diffusion systems with spatially-varying coefficients. *IEEE Transactions on Automatic Control*, 62(4), 2026–2033.