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Suboptimal Distributed LQR Design for Physically Coupled Systems ⋆

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Abstract: In this paper, we propose a suboptimal distributed LQR control method, applicable to systems coupled through both physical interconnections and the quadratic cost to be minimized. Thanks to a novel suboptimal but distributed cost-to-go matrix update that enforces block-diagonality, the suboptimal LQR gain matrix is structured, making the overall control scheme distributed. Moreover, the proposed control design algorithm is scalable. Theoretical properties of the method, including the stability of the closed-loop system, are investigated. A case study is shown to illustrate the features of the approach.

Keywords: linear systems, distributed control, linear quadratic regulators, block-diagonal upper bound approximation, asymptotic stability.

1. INTRODUCTION

In the last decades, the complexity of engineering systems and the connectivity between plants have continuously increased. Some notable examples of new generation complex systems are power networks (Resende and Peças Lopes, 2011), chemical plants (Farina et al., 2016) and fleets of multi-agent autonomous vehicles (Cortés et al., 2004).

The size of these systems has posed significant challenges in the development of dedicated control systems (Šiljak, 1991; Lunze, 1992) and has motivated several research efforts devoted to the design and implementation of monitoring, fault detection, state estimation, and control algorithms. Approaches based on centralized architectures rely upon the assumption that controllers and estimators are integrated in a monolithic computing unit and a reliable communication network allowing for fast and synchronous data exchange between all the actuators and sensors is available. This paradigm has significant shortcomings. Indeed, as the system scale increases, the control algorithm may become prohibitively complex and demanding in terms of computation and communication requirements.

To overcome these problems, distributed methods have been developed over the years, both for unconstrained (Šiljak, 1991; Lunze, 1992) and constrained (Maestre and Negenborn, 2014) systems. Distributed algorithms assume that the system under control can be regarded as a set of interacting subsystems, and that a local control unit – having both computational and communication capabilities – can be integrated in each subsystem. Among the main challenges of distributed approaches, the following ones have received particular attention: (i) local control algorithms should be integrated in the local computing units and should provide stability properties, robustly for any possible system configuration; (ii) thanks to communication and data exchange between local control units, cooperation and system-wide optimality should be sought for; (iii) design algorithms should be distributed in order to meet the scalability requirements of complex plants as well as to easily accommodate for the addition and removal of subsystems.

In this paper we focus our attention on linear quadratic regulators (LQR), which have been thoroughly studied in the centralized framework, see e.g., Bertsekas (2017). Only few notable works have addressed the design of distributed LQRs, proposing suboptimal solutions, e.g., (Borrelli and Keviczky, 2008; Jiao et al., 2019; Vlahakis and Halikias, 2018; Wang et al., 2016; Zhang et al., 2015; Deshpande et al., 2012). However, the existing works have been developed under the assumption that the subsystems under control are physically decoupled, in the sense that their dynamics are independent with each other. In addition, some methods assume that the quadratic cost function has a special structure. For example, Borrelli and Keviczky (2008) show that if the diagonal submatrices of the state penalty matrix are identical and the off-diagonal submatrices of the state penalty matrix are also identical, the optimal control gain would have all the same sub-

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matrices on the diagonal. Furthermore, the off-diagonal submatrices are equal to each other. The authors then propose a suboptimal controller with a similar structure and prove the stability of the closed-loop system. Jiao et al. (2019) show that for dynamically decoupled systems and a state penalty matrix with the same sparsity structure as the communication topology, the distributed LQR design problem is a simultaneous LQR design problem, requiring the existence of a common control gain to optimize multiple subsystems' performance. They then propose a controller design to guarantee stability and performance for the collective system.

In this paper we propose a distributed LQR control method, applicable to *physically coupled* systems, with *general* quadratic cost functions. Our method relies on a suboptimal but distributed cost-to-go matrix update that enforces block-diagonality. Thank to this, the time-varying LQR gain matrix has a zero pattern that makes the overall control scheme distributed. More precisely, neighbor-to-neighbor bidirectional communication is required for cost-to-go matrix update and for computing the control action for each subsystem. Another feature of the proposed scheme is that it automatically provides a separable Lyapunov function for the closed-loop system. Auxiliary controllers associated with separable Lyapunov functions are explicitly required by the majority of available distributed model predictive control algorithms (Maestre and Negenborn, 2014). The theoretical properties of the proposed method, including convergence and boundedness of the associated Riccati-like equations and stability of the closed-loop system are formally studied. Finally, the performance of the distributed LQR is demonstrated through simulations.

The paper is organized as follows. In Section 2, we give the problem formulation and in Section 3, we study stability and performance of the proposed distributed controller. Simulation examples are given in Section 4, followed by some concluding remarks (Section 5).

Notation: Throughout the paper, all the matrices and vectors are assumed to have compatible dimensions. $\|\cdot\|$ denotes the vector 2-norm or the matrix induced 2-norm. A' denotes the transpose of A. A^{-1} denotes the inverse of A. $[A_{ij}]$ denotes a matrix whose ij-th (block) element is A_{ij}. $\text{diag}\{A_i\}$ denotes a diagonal (block diagonal) matrix whose i-th (block) diagonal element is A_i. $[A]_{ij}$ denotes the ij-th element of A. I denotes the identity matrix. 0 denotes the matrix with all elements to be zero. We say that a matrix X fulfills a structural constraint if certain blocks of X are required to be zero, i.e., $X_{ij} = 0$ for some i, j. Moreover, we say that two matrices X, Y have the same sparsity structure, if $X_{ij} = 0$ implies $Y_{ij} = 0$ and vice versa.

2. PROBLEM FORMULATION

Consider N physically coupled systems, each described by

$$x_i(t+1) = A_{ii}x_i(t) + \sum_{j \neq i} A_{ij}x_j(t) + B_i u_i(t), \qquad (1)$$

for $i = 1, \ldots, N$, where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t) \in \mathbb{R}^{m_i}$ are the state and control input for the i-th system, and $\sum_{j \neq i} A_{ij}x_j(t)$ denotes the physical coupling. The interconnection among systems can be described by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ is the node set and the $\mathcal{E}$ is the edge set. Each node $i \in \mathcal{V}$ represents a system, and the edge $(i,j) \in \mathcal{E}$ exists if and only if $A_{ij} \neq 0$. The in-neighbors' set $\mathcal{N}_i^{\text{in}}$ of agent i is defined as $\{j : (i,j) \in \mathcal{E}\}$. The out-neighbors' set $\mathcal{N}_i^{\text{out}}$ of agent i is defined as $\{j : (j,i) \in \mathcal{E}\}$. The sets $\mathcal{N}_i^{\text{in}}$ and $\mathcal{N}_i^{\text{out}}$ may contain agent i itself if $A_{ii} \neq 0$. The adjacency matrix $\mathcal{A} \in \mathbb{R}^{N \times N}$ for the graph $\mathcal{G}$ is defined by the elements: $[\mathcal{A}]_{ij} = 1$ if $A_{ij} \neq 0$ and $[\mathcal{A}]_{ij} = 0$ if $A_{ij} = 0$.

We can write the collective system in a compact form as

$$x(t+1) = Ax(t) + Bu(t),$$

where $x = [x_1', \ldots, x_N']'$, $A = [A_{ij}]$, $B = \text{diag}\{B_i\}$, $u = [u_1', \ldots, u_N']'$. In this paper, we are interested in the infinite-horizon LQR control problem, i.e., to design a control law $u(t)$ such that the following LQR performance is minimized

$$J = \sum_{t=0}^{\infty} x(t)'Qx(t) + u(t)'Ru(t),$$

where $Q = [Q_{ij}] \geq 0$, $R = \text{diag}\{R_i\} \geq 0$. It is known that the optimal centralized control law is given by $u(t) = K^* x(t)$ with

$$K^* = -(B'S^*B + R)^{-1}B'S^*A, \qquad (2)$$

where S^* is the solution to the following Riccati equation

$$S^* = A'S^*A + Q - A'S^*B(B'S^*B + R)^{-1}B'S^*A.$$

Moreover, the optimal cost is given by $J^* = x(0)'S^*x(0)$.

However, computing the control law u_i from (2) could require the knowledge of x_j for all $j = 1, \ldots, N$. When the number N of subsystems is high, such a centralized controller becomes prohibitive, both in terms of computation and communication requirements.

In this paper, we will study distributed solutions where the computation of u_i requires only system i's neighbors' states. In other words, we would like to design a controller gain K with the same sparsity structure as $\mathcal{A}$. As shown in (Rotkowitz and Lall, 2006), solving optimization problems with controller sparsity structure constraints can be extremely difficult. To circumvent this issue, we will develop a suboptimal approach.

It is clear from (2) that, if S^* was block diagonal, K^* would have the same sparsity structure as $\mathcal{A}$, making also the control law distributed. Therefore, in this paper, we propose a suboptimal controller design, where S^* is approximated by a block diagonal matrix $P(t)$ to generate the controller gain $K(t)$. Moreover, we require that $P(t)$ is an upper bound to the cost-to-go matrix, with which we can guarantee the stability of closed-loop system by studying the boundedness of $P(t)$.

3. MAIN RESULTS

In this section, we first describe the controller design and then analyze the asymptotic stability and performance of the closed-loop system.

3.1 Controller Design

Collectively, the proposed control law is $u(t) = K(t)x(t)$ with

$$K(t) = -(B'P(t)B + R)^{-1}B'P(t)A, \tag{3}$$

where $P(t) = \mathrm{diag}\{P_i(t)\}$ is calculated from the following iteration for $t = 0, 1, 2, \dots$ and $i = 1, \dots, N$

$$P_i(t+1) = \sum_{j\in\mathcal{N}_i^{\text{out}}} A'_{ji}P_j^F(t)A_{ji} + Q_i + \sum_{j\in\mathcal{N}_i^{\text{out}}, j\neq i} \|Q_{ij}\|I + \sum_{k\in\mathcal{N}_i^{\text{out}}} \|A'_{ki}\|\|P_k^F(t)\|(\sum_{j\in\mathcal{N}_k^{\text{in}}, j\neq i} \|A_{kj}\|)I, \tag{4}$$

with

$$P_j^F(t) = P_j(t) - P_j(t)B_j(B'_jP_j(t)B_j + R_j)^{-1}B'_jP_j(t). \tag{5}$$

From the collective control law (3), the control law for each agent i is given by

$$u_i(t) = \sum_{j\in\mathcal{N}_i^{\text{in}}} K_{ij}(t)x_j(t), \tag{6}$$

where

$$K_{ij}(t) = -(B'_iP_i(t)B_i + R_i)^{-1}B'_iP_i(t)A_{ij}. \tag{7}$$

Some remarks are now in order. First notice that, since $P(t)$ is block-diagonal, as previously described, the control law u_i depends only on the states $x_j, j \in \mathcal{N}_i^{\text{in}}$, which must be measured and made available to the controller of system i through a communication network. Second, for computing $P_i(t)$, we require information from the out-neighbors of system i. Indeed, (i) the term $\sum_{j\in\mathcal{N}_i^{\text{out}}} A'_{ji}P_j^F(t)A_{ji}$ in (4) can be updated using the matrices $P_j(t)$ and A_{ji} from all $j \in \mathcal{N}_i^{\text{out}}$; (ii) the term

$$\sum_{k\in\mathcal{N}_i^{\text{out}}} \|A'_{ki}\|\|P_k^F(t)\|(\sum_{j\in\mathcal{N}_k^{\text{in}}, j\neq i} \|A_{kj}\|)I$$

in (4) depends on matrices $P_k(t), A_{ki}, A_{kj}$. $P_k(t)$ and A_{ki} can be obtained from the out-neighbor k of agent i. Moreover, A_{kj} can also be obtained from the out-neighbor k of agent i since agent k has access to the coupling matrix A_{kj} for $j \in \mathcal{N}_k^{\text{in}}$.

The distributed LQR algorithm we propose in this paper is formally described in Algorithm 1.

Remark 1. The proposed control law is a variant of the value iteration in approximate dynamic programming, see (Bertsekas, 2017). In value iteration, the following Centralized Riccati Iteration (CRI) is used to approximate the optimal cost S^*

$$S(t+1) = A'S(t)A + Q - A'S(t)B(B'S(t)B + R)^{-1}B'S(t)A. \tag{8}$$

In contrast, we propose to use a block diagonal matrix $P(t)$ generated from (4) and (5) to approximate S^*. Moreover, similar to the value iteration, the control law $K(t)$ is obtained from the Bellman equation assuming the optimal cost-to-go matrix is approximated by $P(t)$, i.e.,

$$K(t) = \arg\min_K x(t)'Qx(t) + x(t)'KRKx(t) + (Ax(t) + BKx(t))'P(t)(Ax(t) + BKx(t)).$$

Remark 2. If a new system $N+1$ is added at time $\bar{k}$, the new controller can be produced by setting $P_{N+1}(\bar{k}) = 0$ and receiving pieces of information from neighboring systems only. Moreover, the addition of system $N+1$ impacts only the update equations (4) and (5) associated to neighboring systems. Therefore, controllers of systems $j \in \mathcal{V} \setminus \mathcal{N}_{N+1}^{\text{out}}$ are not modified. This guarantees the scalability of the proposed design method.

Algorithm 1 Distributed LQR algorithm

Memory requirements
For $i \in \mathcal{V}$, system i stores in memory the matrices B_i, R_i, A_{ii}, $\{Q_{ij}; j \in \mathcal{N}_i^{\text{out}}\}$, $\{A_{jk}; j \in \mathcal{N}_i^{\text{out}}, k \in \mathcal{N}_j^{\text{in}}, k \neq j\}$, $P_i(t)$, $\{P_j^F(t); j \in \mathcal{N}_i^{\text{out}}\}$, $\{K_{ij}(t); j \in \mathcal{N}_i^{\text{in}}\}$ and the vectors $u_i(t)$, $\{x_j(t); j \in \mathcal{N}_i^{\text{in}}\}$.

Initialization
For $i \in \mathcal{V}$, system i initialize $P_i(0) = 0$.

On-line implementation
At each iteration $t \geq 0$ system i:
1) Measures $x_i(t)$ and broadcasts it to its out-neighbors;
2) Gathers from its in-neighbors the information $\{x_j(t); j \in \mathcal{N}_i^{\text{in}}, j \neq i\}$;
3) Computes $\{K_{ij}(t); j \in \mathcal{N}_i^{\text{in}}\}$ as in (7), and, accordingly, $u_i(t)$ as in (6);
4) Applies $u_i(t)$ as in (1);
4) Computes $P_i^F(t)$ as in (5) and broadcasts it to its in-neighbors;
5) Gathers from its out-neighbors the information $\{P_j^F(t); j \in \mathcal{N}_i^{\text{out}}, j \neq i\}$;
6) Computes $P_i(t+1)$ as in (4).

In the following, we will show that if certain conditions are satisfied, $K(t)$ converges to a stabilizing gain.

3.2 Asymptotic Stability of the Closed-loop System

In this section, we state the main result of the paper, i.e., the conditions guaranteeing the stability of the closed-loop system. The following assumption is needed.

Assumption 3. A_{ii} is invertible for all $i \in \mathcal{V}$.

Under Assumption 3, we can show that if we initialize $P_i(0)$ appropriately and suitable conditions on the coupling between systems are satisfied, the matrices $P(t)$ and $K(t)$ converge to constant values $\bar{P}$ and $\bar{K}$. Moreover, $\bar{K}$ is stabilizing. The result is stated as follows.

Theorem 4. Initialize $P_i(0) = 0$ for all $i \in \mathcal{V}$. Let $F_i = A_{ii} + B_iK_i$ with $K_i \in \mathbb{R}^{m_i\times n_i}$, $F = \mathrm{diag}\{\|F_i\|^2\}$ and define the matrix Γ as

$$[\Gamma]_{ij} = \begin{cases} 1 + \sum_{k\neq i} \|A_{ii}^{-1}\|^2\|A_{ii}\|\|A_{ik}\| & j = i, \\ \|A_{jj}^{-1}A_{ji}\|^2 + \sum_{k\neq i} \|A_{jj}^{-1}\|^2\|A_{ji}\|\|A_{jk}\| & j \neq i. \end{cases}$$

If there exist K_i for $i = 1, \dots, N$ such that

$$\rho(F\Gamma) < 1, \tag{9}$$

then

$$\lim_{t\to-\infty} P(t) = \bar{P}, \quad \lim_{t\to-\infty} K(t) = \bar{K}$$

and $\bar{K}$ is such that $A + B\bar{K}$ is Schur stable.

The proof of Theorem 4 is provided in the Appendix. Verifying the stability conditions in Theorem 4 requires to build the product of the $N \times N$ matrices F and Γ. On the one hand, this computation is centralized. On the other hand, we highlight that the size of matrices F and Γ scales with the number of systems only. The matrix Γ captures the magnitude of coupling between systems. For decoupled systems, one has $\Gamma = I$ and (9) can be always fulfilled if all local systems are stabilizable. The condition (9) also calls for the solution of the nonlinear optimization problem $\min_{K_i} \rho(F\Gamma)$. An alternative is to manually select K_i and then check whether the condition $\rho(F\Gamma) < 1$ is satisfied. A first heuristic method of selecting K_i is to let K_i solve the optimization problem $\min_{K_i} \|A_{ii} + B_i K_i\|$, which can be cast into the following LMI problem

$$\min_{K_i,\gamma} \gamma$$
$$\text{s.t.} \begin{bmatrix} \gamma I & (A_{ii}+B_iK_i)' \\ (A_{ii}+B_iK_i) & I \end{bmatrix} \geq 0.$$

The motivation is that if $\|A_{ii} + B_i K_i\| = 0$ for all i, $\rho(F\Gamma) = 0 < 1$. Therefore, by selecting K_i to make $\|A_{ii} + B_k K_i\|$ as small as possible, we expect that $\rho(F\Gamma) < 1$. Another heuristic method is to run the iteration (4) for a sufficiently long time, use the final $P_i(t)$ to construct $K_i = K_{ii}(t)$ with $K_{ii}(t) = -(B_i' P_i(t) B_i + R_i)^{-1} B_i' P_i(t) A_{ii}$ and use such K_i for verifying $\rho(F\Gamma) < 1$. This procedure is justified by the fact that if $P_i(t)$ converges and $\bar{K}$ is stabilizing, we can expect that the finite horizon approximation $K_{ii}(t)$ is likely to fulfill the stability condition $\rho(F\Gamma) < 1$.

4. SIMULATIONS

We consider a composition of $N = 6$ systems. For $i \in \{1, 2, ..., 6\}$, we set $A_{ii} = \left[\begin{smallmatrix} 0.9 & 0.1 \\ 0.1 & -0.9 \end{smallmatrix}\right]$, $B_i = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and for $i, j \in \{1, \ldots, 6\}$, let $\alpha > 0$ and

$$A_{ij} = \begin{cases} \text{diag}(\alpha, -\alpha) & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For the LQR control problem, we set the symmetric matrices $Q_{ii} = R_i = I$ for $i = 1, \ldots, 6$ and $Q_{ij} = 0$ for $i \neq j$. For each system, we compare its behaviour when using (i) K^* i.e., the static gain from the infinite horizon centralized LQR controller (2), (ii) $K(t)$ as proposed in our paper, (iii) $K_d(t)$ obtained by the dualisation of the Partition-Based Distributed Kalman Filter from (Farina and Carli, 2018), which is given by $K_d(t) = [K_{d,ij}(t)]$ with

$$K_{d,ij}(t) = -(B_i' P_i(t) B_i + R_i)^{-1} B_i P_i(t) A_{ij}, \quad (10)$$
$$P_i(t+1) = \sum_{j \in \mathcal{N}_i^{\text{out}}} \tilde{A}_{ji}' P_j^F(t) \tilde{A}_{ji} + Q_i, \quad (11)$$

where $P_j^F(t)$ is given as in (5); $\tilde{A}_{ji} = \sqrt{|\mathcal{N}_j^{\text{in}}|} A_{ji}$ and $|\mathcal{N}_j^{\text{in}}|$ is the cardinality of $\mathcal{N}_j^{\text{in}}$.

In Figure 1, we show the response for the first system when $\alpha = 0.4$, $T = 100$ and initial conditions of each system are $x = [100, -50]'$. We plot the system response of the first system using K^*, $K(t)$ and $K_d(t)$, where it can be seen that the temporal response converges to $[0, 0]'$ when using K^* and $K(t)$ but diverges when using $K_d(t)$. Besides, the response with K^* and $K(t)$ are close to each other, which demonstrates the effectiveness of the proposed controller. Responses of other systems are not shown since they have a similar evolution.

Figure 2 shows the spectral radius of $A + BK$ with respect to α when K is equal to K^*, $K(T)$ and $K_d(T)$, after $T = 100$ iterations of (4) and (11), respectively. In the considered situations, we can verify that the gains $K(t)$ and $K_d(t)$ converge to constant values after $T = 100$ iterations. From Figure 2, it is clear that in spite of the coupling between systems, the collective system remains stable when using our proposed controller $K(t)$. In contrast, if the controller $K_d(t)$ is used, the coupling α should be sufficiently small (in this case, $\alpha < 0.36$) in order to preserve stability of the closed-loop system.

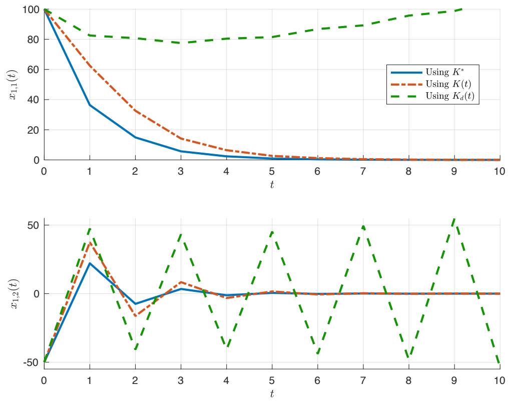

Fig. 1. Temporal response of the first system state variables $x_{1,1}(t)$ and $x_{1,2}(t)$ under different controllers.

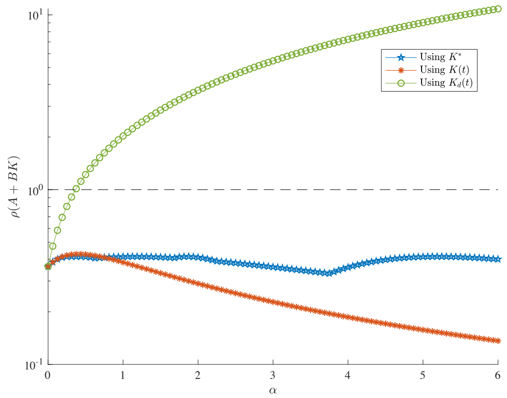

Fig. 2. Values of $\rho(A + BK)$ as a function of α under different controllers.

We verify Theorem 4, by corroborating that if $\rho(F\Gamma) < 1$ then the closed-loop system is stable. In order to obtain $F\Gamma$, we calculate Γ from Theorem 4, and $F = \text{diag}\{\|A_{ii} + B_i K_i\|^2\}$ using $K_i = K_{ii}(T)$ obtained after $T = 100$ iterations of (4). Figure 3 presents $\rho(F\Gamma)$ for $\alpha \in [0, 6]$. As $\rho(F\Gamma) < 1$, from Theorem 4, we can conclude that the asymptotic control gain $\bar{K}$ is stabilizing, which is also reflected in Figure 2.

Finally, we evaluate the finite-horizon cost

$$\sum_{t=0}^{T} x'(t) Q x(t) + u'(t) R u(t)$$

when using the three different control laws for $\alpha = \{0.1, 0.2, 0.3\}$. Table 1 summarises the obtained results, for the finite-horizon LQR with $T = 100$. As it is expected, the performance of our controller $K(t)$ is suboptimal but improves over $K_d(t)$.

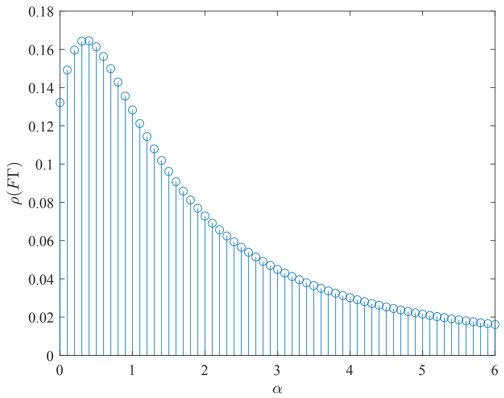

Fig. 3. Values of $\rho(F\Gamma)$ as a function of α.

Table 1. Cost-to-go of the T-step finite-horizon LQR performance, for different coupling values and different controllers

	K^*	$K(t)$	$K_d(t)$
$\alpha = 0.1$	5.47×10^4	7.27×10^4	7.54×10^4
$\alpha = 0.2$	7.76×10^4	11.3×10^4	14.8×10^4
$\alpha = 0.3$	10.6×10^4	17.0×10^4	45.2×10^4

5. CONCLUSIONS

This paper studied the distributed LQR control design for physically coupled systems. Different from other contributions available in the literature, we do not assume any structure for the state penalty matrix in the LQR performance index. We propose a suboptimal distributed control law. Moreover, we study the asymptotic performance of the closed-loop system and show that under mild conditions, asymptotic stability can be guaranteed. Further research will consider the development of distributed output feedback controllers with stability guarantees by combining the partitioned Kalman filter in (Farina and Carli, 2018) with the distributed LQR scheme proposed in this paper.

APPENDIX: PROOF OF THEOREM 4

To prove Theorem 4, we first need to state and prove some preliminary results. The following lemma is needed during the proof and is stated first, which is the Gersgorin theorem for block matrices.

Lemma 5. (Theorem 6.3 of (Varga, 2010)). Consider $A = [A_{ij}]$. Let $\sigma(\cdot)$ denotes the spectrum of a matrix and

$$G_i = \sigma(A_{ii}) \cup \{\lambda \notin \sigma(A_{ii}) : \frac{1}{\|(\lambda I - A_{ii})^{-1}\|} \leq \sum_{j\neq i} \|A_{ij}\|\},$$

then $\sigma(A) \in \cup_i G_i$.

Lemma 6. In the iteration (4), one has

$$P(t+1) \geq A' P^F(t) A + Q,$$

where $P^F(t) = \mathrm{diag}\{P_i^F(t)\}$.

Proof. Since $[A'P^F(t)A]_{ij} = \sum_k A'_{ki} P_k^F(t) A_{kj}$, we have that

$$[P(t+1) - AP^F(t)A' - Q]_{ij}$$
$$= \begin{cases} \sum_{l\neq i}\sum_k \|A'_{ki}\|\|P_k^F(t)\|\|A_{kl}\| I + \sum_{l\neq i}\|Q_{il}\| I & j = i \\ -\sum_k A'_{ki} P_k^F(t) A_{kj} - Q_{ij} & j \neq i \end{cases}$$

In view of the Lemma 5, the eigenvalues of $P(t+1) - A'P^F(t)A - Q$ are in the region

$$\cup_i \left\{ \lambda : \left| \lambda - \sum_{j\neq i} \left(\sum_k \|A'_{ki}\|\|P_k^F(t)\|\|A_{kj}\| + \|Q_{ij}\| \right) \right| \right.$$
$$\left. \leq \sum_{j\neq i} \left\| -\sum_k A'_{ki} P_k^F(t) A_{kj} - Q_{ij} \right\| \right\},$$

which is included in the right half complex plain. Since $P(t+1) - A'P^F(t)A - Q$ is symmetric, we know $P(t+1) - A'P^F(t)A - Q \geq 0$, which concludes the proof. ■

The monotonicity property of the $P(t)$ iteration is stated in the following lemma.

Lemma 7. Let $P_i^A(t) \in \mathbb{R}^{n_i \times n_i}$ and $P_i^B(t) \in \mathbb{R}^{n_i \times n_i}$ be two positive semidefinite matrices. Let $P_i^A(t+1)$ and $P_i^B(t+1)$ be the matrices produced by (4) and (5) when selecting $P_i(t) = P_i^A(t)$ and $P_i(t) = P_i^B(t)$, respectively. Suppose $P_i^A(t) > P_i^B(t)$ for all i. Then $P_i^A(t+1) > P_i^B(t+1)$ for all i.

Proof. From the definition of $P_i(t+1)$ in (4), we only need to prove that $A'_{ji} P_j^F(t) A_{ji}$ and $\|A'_{ki}\|\|P_k^F(t)\|\|A_{kj}\|$ are monotonic with respect to $P_i(t)$. The monotonicity of $A'_{ji} P_j^F(t) A_{ji}$ with respect to $P_i(t)$ follows from Lemma 1.c in (Sinopoli et al., 2004). Therefore, we only need to prove the monotonicity of $\|A'_{ki}\|\|P_k^F(t)\|\|A_{kj}\|$ with respect to $P_i(t)$. Let $P_k^{AF}(t)$ denote the matrix $P_k^F(t)$ when $P_k(t) = P_k^A(t)$. Assume $P_k^{BF}(t)$ is defined similarly. Assume $P_k^A > P_k^B$, since $P_k^F = (P_k^{-1} + B_k R_k^{-1} B'_k)^{-1}$, we have $P_k^{AF} > P_k^{BF}$. Therefore,

$$\|A'_{ki}\|\|P_k^{AF}(t)\|\|A_{kj}\| > \|A'_{ki}\|\|P_k^{BF}(t)\|\|A_{kj}\|,$$

which means $\|A'_{ki}\|\|P_k^F(t)\|\|A_{kj}\|$ is monotonic with respect to $P_i(t)$. The proof is completed. ■

Next, we will show the boundedness of the $P(t)$ iteration. The result is stated in the following lemma.

Lemma 8. If (9) holds, the sequence of matrices $P(t)$ generated from (4) is bounded for all t.

Proof. From the definition of $P_i(t+1)$, we have that

$$P_i(t+1) = P_i^L(t+1) + \Delta_i(t+1) + S_i(t+1) + \tilde{Q}_i,$$

where

$$\Delta_i(t+1) = \sum_{j\neq i} A'_{ji} P_j^F(t) A_{ji},$$

$$S_i(t+1) = \sum_{j\neq i}\sum_k \|A'_{ki}\|\|P_k^F(t)\|\|A_{kj}\| I,$$

$$P_i^L(t+1) = A'_{ii} P_i^F(t) A_{ii}, \quad \tilde{Q}_i = Q_i + \sum_{j\neq i} \|Q_{ij}\| I.$$

Since A_{ii} is invertible, we have that

$$P_i^F(t) = (A'_{ii})^{-1} P_i^L(t+1) A_{ii}^{-1}.$$

Therefore, we obtain

$$\Delta_i(t+1) = \sum_{j\neq i} A'_{ji}(A'_{jj})^{-1}P_j^L(t+1)A_{jj}^{-1}A_{ji},$$
$$S_i(t+1) = \sum_{j\neq i}\sum_{k} \|A'_{ki}\|\|(A'_{kk})^{-1}P_k^L(t+1)A_{kk}^{-1}\|\|A_{kj}\|I,$$

which further implies

$$\begin{aligned}
&P_i^L(t+1) = A'_{ii}P_i^F(t)A_{ii}\\
&= (A_{ii}+B_iK_{ii}(t))'P_i(t)(A_{ii}+B_iK_{ii}(t)) + K_{ii}(t)'R_iK_{ii}(t)\\
&\overset{(a)}{\le} (A_{ii}+B_iK_i)'P_i(t)(A_{ii}+B_iK_i) + K_i'R_iK_i\\
&= F_i'P_i(t)F_i + K_i'R_iK_i\\
&= F_i'(P_i^L(t)+\Delta_i(t)+S_i(t))F_i + K_i'R_iK_i + F_i'\tilde{Q}_iF_i\\
&= F_i'(P_i^L(t)+\Delta_i(t)+S_i(t))F_i + \delta_i, \qquad (12)
\end{aligned}$$

where (a) follows from the fact that $K_{ii}(t)$ minimizes $(A_{ii}+B_iK_i)'P_i(t)(A_{ii}+B_iK_i)+K_i'R_iK_i$ for any K_i and $\delta_i = K_i'R_iK_i + F_i'\tilde{Q}_iF_i$. Since

$$\begin{aligned}
\|\Delta_i(t)\| &= \|\sum_{j\neq i} A'_{ji}(A'_{jj})^{-1}P_j^L(t)A_{jj}^{-1}A_{ji}\|\\
&\le \sum_{j\neq i}\|A_{jj}^{-1}A_{ji}\|^2\|P_j^L(t)\|
\end{aligned}$$

and

$$\begin{aligned}
\|S_i(t)\| &\le \sum_{j\neq i}\sum_{k}\|A_{kk}^{-1}\|^2\|A_{ki}\|\|A_{kj}\|\|P_k^L(t)\|\\
&= \sum_{j}\sum_{k\neq i}\|A_{jj}^{-1}\|^2\|A_{ji}\|\|A_{jk}\|\|P_j^L(t)\|,
\end{aligned}$$

where the last equation is obtained via the swap of index j, k, from (12), we have

$$\begin{aligned}
\|P_i^L(t+1)\| &\le \|F_i\|^2\Bigg(\|P_i^L(t)\| + \sum_{j\neq i}\|A_{jj}^{-1}A_{ji}\|^2\|P_j^L(t)\|\\
&\quad + \sum_{j}\sum_{k\neq i}\|A_{jj}^{-1}\|^2\|A_{ji}\|\|A_{jk}\|\|P_j^L(t)\|\Bigg) + \|\delta_i\|\\
&= \|F_i\|^2(\Gamma_{ii}\|P_i^L(t)\| + \sum_{j\neq i}\Gamma_{ij}\|P_j^L(t)\|) + \|\delta_i\|.
\end{aligned}$$

Therefore, we have that

$$\|P^L(t+1)\| \le F\Gamma\|P^L(t)\| + \|\delta\|,$$

where with a slight abuse of notion $\|P^L\| = [\|P_1^L\|,\ldots,\|P_N^L\|]'$ and $\|\delta\| = [\|\delta_1\|,\ldots,\|\delta_N\|]'$. Therefore, if $\rho(F\Gamma) < 1$, we have that $P_i^L(t+1)$ is bounded, which further implies the boundedness of $P_i(t+1)$. The proof is completed. ■

Now we are in the position to prove Theorem 4.

Proof. Since $P_i(1) = \tilde{Q}_i > P_i(0) = 0$, in view of Lemma 7, we can show that $P_i(t+1) > P_i(t)$ for all t by induction. Therefore $P(t)$ is monotonically increasing with respect to time t. Moreover, since $\rho(F\Gamma) < 1$, $P(t)$ is bounded from Lemma 8. Therefore $P(t)$ and $K(t)$ converge to some constant value $\bar{P}, \bar{K}$ as $t \to \infty$. Besides, in view of Lemma 6, $\bar{P}, \bar{K}$ should satisfy that

$$\bar{P} > (A+B\bar{K})'\bar{P}(A+B\bar{K}).$$

Therefore, $A + B\bar{K}$ is Schur stable. ■

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