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A Characterization of Robust Regions of Attraction for Discrete-Time Systems Based on Bellman Equations

Bai Xue and Naijun Zhan and Yangjia Li

State Key Lab. of Computer Science, Institute of Software, CAS, China
University of Chinese Academy of Sciences, Beijing, China
(e-mail:{xuebai,znj,yangjia}@ios.ac.cn)

Abstract: In this paper we present a Bellman equation for computing robust regions of attraction for state-constrained perturbed discrete-time systems. The robust region of attraction of interest is a set of states such that every trajectory initialized in it will approach an equilibrium while never violating a specified state constraint, regardless of the actual perturbation. In this approach, the interior of the maximal robust region of attraction is characterized as the strict one sub-level set of the unique bounded and continuous solution to a Bellman equation.

Keywords: Robust Regions of Attraction; State-Constrained Perturbed Discrete-Time Systems; Bellman Equations.

1. INTRODUCTION

A robust region of attraction is a set of states such that every trajectory starting from it will approach an equilibrium while never leaving a specified state-constraint set irrespective of the actual perturbation. Computing it is a fundamental problem in control engineering. Its applications include, but are not limited to, biology systems (Merola et al., 2008) and ecology systems (Ludwig et al., 1997). Consequently, various computational approaches have been proposed over the past several decades, e.g., Lyapunov function-based methods (Salle and Lefschetz, 1961; Coutinho and de Souza, 2013; Chesi, 2004; Giesl, 2007; Valmorbida and Anderson, 2014; Giesl and Hafstein, 2014), trajectory reversing methods (Genesio et al., 1985) and moment-based methods (Korda et al., 2013).

Another attractive means in computing robust regions of attraction is by exploiting the link to optimal control. When the system is continuous-time, the link is established through viscosity solutions of Hamilton-Jacobi equations (Mitchell et al., 2005; Bokanowski et al., 2010; Margellos and Lygeros, 2011; Xue et al., 2019), which are widely used in continuous-time optimal control theory (Bardi and Capuzzo-Dolcettae, 1997). Zubov's equation, which is a type of Hamilton-Jacobi equations, was originally inferred to describe the maximal region of attraction for continuous-time dynamical systems free of state constraints and perturbation inputs. Recently, it was extended to perturbed systems in (Camilli et al., 2001) and further to state-constrained perturbed systems in (Grüne and Zidani, 2015). The appealing aspect of Hamilton-Jacobi equation-based methods is that it touches upon the problem of computing the maximal robust region of attraction. When the system is discrete-time, the Bellman equation is widely used in discrete-time optimal control (Bardi and Capuzzo-Dolcettae, 1997). It was investigated for computing the maximal robust invariant set of state-constrained perturbed discrete-time systems in (Xue and Zhan, 2018). The trajectories starting from the maximal robust invariant set in (Xue and Zhan, 2018) are not required to approach an equilibrium. However, to the best of our knowledge, there is no previous work on the use of Bellman equations to characterize the maximal robust region of attraction for state-constrained perturbed discrete-time systems. This motivates the study in this paper.

In this paper we present a modified Bellman equation for computing robust regions of attraction for state-constrained perturbed discrete-time systems with an equilibrium state, which is uniformly locally exponentially stable. The interior of the maximal robust region of attraction is characterized as the strict one sub-level set of the unique bounded and continuous solution to the modified Bellman equation. The derivation of the Bellman equation follows the reasoning in (Grüne and Zidani, 2015), which presented a modified Zubov's equation for computing robust regions of attraction of state-constrained perturbed continuous-time systems. One example is used to illustrate the computation of the interior of the maximal robust region of attraction via solving the Bellman equation.

The main contribution of this paper is summarized as follows. We for the first time infer a Bellman equation, to which the strict one sub-level set of the unique bounded and continuous solution characterizes the interior of the maximal robust region of attraction for state-constrained perturbed discrete-time systems. To the best of our knowledge, this is the first possibility to estimate the maximal robust region of attraction for state-constrained perturbed discrete-time systems.

This paper is structured as follows. In Section 2 basic notions and the problem of interest are introduced. After presenting the Bellman equation in Section 3, we estimate

⋆ Corresponding Authors: Bai Xue and Naijun Zhan

the maximal robust region of attraction for one example via solving the Bellman equation in Section 4. Finally, we conclude this paper in Section 5.

2. PRELIMINARIES

In this section we describe the system of interest and the concept of robust regions of attraction.

The notions will be used throughout this paper: $\mathbb{R}^n$ denotes the set of n-dimensional real vectors. Δ°, $\partial\Delta$, $\overline{\Delta}$ and Δ^c denote the interior, boundary, closure and complement of a set Δ, respectively. The space of continuous functions on a set Δ is denoted by $C(\Delta)$. The difference of two sets A and B is denoted by $A \setminus B$. $\mathbb{N}$ denotes the set of non-negative integers. $\|\boldsymbol{x}\|$ denotes the 2-norm, i.e., $\|\boldsymbol{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$, where $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$. $B(\boldsymbol{0}, r)$ denotes a ball of radius $r > 0$ and center $\boldsymbol{0}$, i.e., $B(\boldsymbol{0}, r) = \{\boldsymbol{x} \mid \|\boldsymbol{x}\|^2 \leq r\}$. Vectors are denoted by boldface letters.

The perturbed discrete-time system of interest in this paper is of the following form

$$\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{x}(k), \boldsymbol{d}(k)), k \in \mathbb{N}, \tag{1}$$

where $\boldsymbol{x}(\cdot) : \mathbb{N} \to \mathbb{R}^n$, $\boldsymbol{d}(\cdot) : \mathbb{N} \to D$, $D = \{\boldsymbol{d} \in \mathbb{R}^m \mid \wedge_{i=1}^{m_d} [h_i^D(\boldsymbol{d}) \leq 0]\}$ is a compact subset in $\mathbb{R}^m$ with $h_i^D \in C(\mathbb{R}^m)$ and $\boldsymbol{f} \in C(\mathbb{R}^n \times \mathbb{R}^m)$ is locally Lipschitz continuous over $\boldsymbol{x} \in \mathbb{R}^n$ uniformly over $\boldsymbol{d} \in D$, with $\boldsymbol{f}(\boldsymbol{0}, \boldsymbol{d}) = \boldsymbol{0}$ for $\boldsymbol{d} \in D$.

In order to define our problem succinctly, we present the definition of a perturbation input policy π.

Definition 1. A perturbation input policy, denoted by π, refers to a function $\pi(k) : \mathbb{N} \to D$. In addition, we denote the set of all perturbation input policies by $\mathcal{D}$.

Given a perturbation input policy π, a trajectory to system (1) is presented in Definition 2.

Definition 2. Given a perturbation input policy $\pi \in \mathcal{D}$, a trajectory of system (1) initialized in $\boldsymbol{x}_0 \in \mathbb{R}^n$ is defined as $\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(\cdot) : \mathbb{N} \to \mathbb{R}^n$, where $\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(0) = \boldsymbol{x}_0$, and

$$\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k+1) = \boldsymbol{f}(\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k), \pi(k)), \forall k \in \mathbb{N}.$$

We assume that $\boldsymbol{0}$ is uniformly locally exponentially stable for system (1).

Assumption 1. The equilibrium state $\boldsymbol{0}$ is uniformly locally exponentially stable for (1), i.e., there exist positive constants $M > 0$, $r > 0$ and $0 < \lambda < 1$ such that

$$\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k)\| \leq \lambda^k M \|\boldsymbol{x}_0\|, \forall \boldsymbol{x}_0 \in B(\boldsymbol{0}, r), \forall \pi \in \mathcal{D}, \forall k \in \mathbb{N}, \tag{2}$$

where $B(\boldsymbol{0}, r) \subset X$ and X is a state constraint set in $\mathbb{R}^n$, which will be defined explicitly later.

Assumption 1 implies the existence of a positive constant $\bar{\epsilon}$ such that $B(\boldsymbol{0}, \bar{\epsilon}) \subseteq X$ and

$$\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k) \in B(\boldsymbol{0}, \frac{r}{2}), \forall \boldsymbol{x}_0 \in B(\boldsymbol{0}, \bar{\epsilon}), \forall k \in \mathbb{N}, \forall \pi \in \mathcal{D}. \tag{3}$$

Since $0 < \lambda < 1$ in Assumption 1, $\bar{\epsilon}$ in (3) exists and can take the value of $\min\{\frac{r}{2M}, \frac{r}{2}\}$.

Suppose that the state constraint set

$$X = \{\boldsymbol{x} \in \mathbb{R}^n \mid \wedge_{i=1}^{n_X} [h_i^X(\boldsymbol{x}) < 1]\}$$

is a bounded and open set with $h_i^X(\boldsymbol{x}) \in C(\mathbb{R}^n)$ being locally Lipschitz continuous over $\boldsymbol{x}$. Also, $h_i^X(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$ and $h_i^X(\boldsymbol{0}) = 0$, $i = 1, \ldots, n_X$. Now, we present the concept of robust regions of attraction formally.

Definition 3. (Robust Regions of Attraction). The maximal robust region of attraction $\mathcal{R}$ is the set of states such that every possible trajectory of system (1) starting from it will approach the equilibrium state $\boldsymbol{0}$ while never leaving the state constraint set X, i.e.,

$$\mathcal{R} = \left\{ \boldsymbol{x}_0 \;\middle|\; \begin{array}{l} \boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k) \in X, \forall k \in \mathbb{N}, \forall \pi \in \mathcal{D}, \\ \text{and } \lim_{k \to \infty} \boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k) = \boldsymbol{0}, \forall \pi \in \mathcal{D} \end{array} \right\}.$$

Correspondingly, a robust region of attraction is a subset of the maximal robust region of attraction $\mathcal{R}$.

3. BELLMAN EQUATIONS

In this section we characterize the interior of the maximal region of attraction $\mathcal{R}$ as the strict one sub-level set of the unique bounded and continuous solution to a modified Bellman equation. In Subsection 3.1 we introduce the maximal robust region of uniform attraction, which is equal to the interior of the maximal robust region of attraction. In Subsection 3.2 we reduce the maximal robust region of uniform attraction to the strict one sub-level set of the unique bounded and continuous solution to a Bellman equation. The derivation of the Bellman equation follows the reasoning in Grüne and Zidani (2015).

3.1 Robust Regions of Uniform Attraction

In this subsection we introduce the maximal robust region of uniform attraction, which is equal to the interior of the maximal robust region of attraction. The maximal robust region of uniform attraction was first proposed in (Grüne and Zidani, 2015) for state-constrained perturbed continuous-time systems.

Denote the first hitting time $k'(\boldsymbol{x}_0, \pi)$, induced by the initial state $\boldsymbol{x}_0$ and the input policy π, of $B(\boldsymbol{0}, \bar{\epsilon})$ as

$$k'(\boldsymbol{x}_0, \pi) := \inf\{k > 0 \mid \boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k) \in B(\boldsymbol{0}, \bar{\epsilon})\}, \tag{4}$$

where $B(\boldsymbol{0}, \bar{\epsilon})$ is defined in (3). Also, let the Euclidean distance between a point $\boldsymbol{x} \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$ be $\mathtt{dist}(\boldsymbol{x}, A) := \mathtt{inf}_{\boldsymbol{y} \in A} \|\boldsymbol{x} - \boldsymbol{y}\|$, and the set of δ-admissible perturbation input policies be

$$\mathcal{D}_{ad,\delta}(\boldsymbol{x}_0) := \{\pi \mid \mathtt{dist}(\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k), X^c) > \delta \text{ for } k \in \mathbb{N}\},$$

where $\delta > 0$ and X^c is the complement of the set X. The maximal robust region of uniform attraction $\mathcal{R}_0$ is then defined by

$$\mathcal{R}_0 := \left\{ \boldsymbol{x}_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{there exists } \delta > 0 \text{ s.t. } \mathcal{D}_{ad,\delta}(\boldsymbol{x}_0) \\ = \mathcal{D} \text{ and } \sup_{\pi \in \mathcal{D}} k'(\boldsymbol{x}_0, \pi) < \infty \end{array} \right\}.$$

Lemma 1 presents the openness property of the region $\mathcal{R}_0$ as well as the relationship between $\mathcal{R}_0$ and $\mathcal{R}$.

Lemma 1. Under Assumption 1, then

(a) $\mathcal{R}_0 = \mathcal{R}_0'$, where

$$\mathcal{R}_0' = \left\{ \boldsymbol{x}_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{there exists } \delta > 0 \text{ s.t. } \mathcal{D}_{ad,\delta}(\boldsymbol{x}_0) \\ = \mathcal{D} \text{ and there exists a function} \\ \beta(k) : \mathbb{N} \to [0, \infty) \text{ satisfying} \\ \lim_{k \to \infty} \beta(k) = 0 \text{ and} \\ \|\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k)\| \leq \beta(k) \text{ for } k \in \mathbb{N} \\ \text{and } \pi \in \mathcal{D}. \end{array} \right\}.$$

(b) $\mathcal{R}_0$ is open.
(c) $\mathcal{R}_0 = \mathcal{R}^\circ$.

Proof. (a). Let $\boldsymbol{x}_0 \in \mathcal{R}_0$ and $K = \sup_{\pi\in\mathcal{D}} k'(\boldsymbol{x}_0,\pi) < \infty$. Then, for $k \geq K$ we have

$$\|\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k)\| \leq \beta(r,k) = \lambda^k M r,$$

where r is defined in (2). Hence, for $k \geq K$ we can choose $\beta(k) = \beta(r,k)$. Since $\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k) \in X$ for $k \in [0,K]$ and $\pi \in \mathcal{D}$, and X is bounded, there exists $M' \geq 0$ such that

$$\|\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k)\| \leq M', \forall k \in [0,K], \forall \pi \in \mathcal{D}.$$

Choosing $\beta(k) = M'$ for $k \in [0,K]$ then yields the function $\beta(k)$ with the desired properties. Thus,

$$\boldsymbol{x}_0 \in \mathcal{R}'_0,$$

implying that $\mathcal{R}_0 \subseteq \mathcal{R}'_0$.

Conversely, let $\boldsymbol{x}_0 \in \mathcal{R}'_0$ and pick the corresponding $\delta > 0$ and $\beta(k)$. Then there exists $K > 0$ such that

$$\beta(k) < \overline{\epsilon}, \forall k \geq K$$

(K exists since $\lim_{k\to\infty}\beta(k) = 0$), where $\overline{\epsilon}$ is defined in (3). Then we have

$$\|\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k)\| \leq \beta(k) < \overline{\epsilon}, \forall k \geq K, \forall \pi \in \mathcal{D},$$

which implies

$$\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k) \in B(\boldsymbol{0},\overline{\epsilon}), \forall k \geq K, \forall \pi \in \mathcal{D}.$$

Hence,

$$k'(\boldsymbol{x}_0,\pi) \leq K, \forall \pi \in \mathcal{D}$$

and thus

$$\sup_{\pi\in\mathcal{D}} k'(\boldsymbol{x}_0,\pi) \leq K < \infty.$$

Also, since $\mathcal{D}_{ad,\delta} = \mathcal{D}$, we have that $\boldsymbol{x}_0 \in \mathcal{R}_0$, implying that $\mathcal{R}'_0 \subseteq \mathcal{R}_0$.

(b). Since $\mathcal{R}_0 = \mathcal{R}'_0$, we prove the openness of $\mathcal{R}'_0$ instead. Let $\boldsymbol{x}_0 \in \mathcal{R}'_0$ with corresponding $\delta > 0$ and $\beta(\cdot): \mathbb{N} \to [0,\infty)$, and $K > 0$ be such that $\beta(k) < \frac{\overline{\epsilon}}{2}$ for $k \geq K$, where $\overline{\epsilon}$ is defined in (4).

Since $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{d})$ is Lipschitz continuous over $\boldsymbol{x} \in X$ uniformly over $\boldsymbol{d} \in D$, implying that there exists $B(\boldsymbol{x}_0,\epsilon)$ such that for $\boldsymbol{y}_0 \in B(\boldsymbol{x}_0,\epsilon), \pi \in \mathcal{D}$ and $k \in [0,K]$,

$$\|\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k) - \boldsymbol{\phi}^{\pi}_{\boldsymbol{y}_0}(k)\| < \min\{\frac{\delta}{2},\frac{\overline{\epsilon}}{2}\}.$$

This further implies that for $\boldsymbol{y}_0 \in B(\boldsymbol{x}_0,\epsilon)$, $\pi \in \mathcal{D}$ and $k \in [0,K]$,

$$\texttt{dist}(\boldsymbol{\phi}^{\pi}_{\boldsymbol{y}_0}(k), X^c) > \frac{\delta}{2}$$

holds. Thus, $\boldsymbol{\phi}^{\pi}_{\boldsymbol{y}_0}(K) \in B(\boldsymbol{0},\overline{\epsilon}), \forall \pi \in \mathcal{D}$. Hence

$$\sup_{\pi\in\mathcal{D}} k'(\boldsymbol{y}_0,\pi) \leq K.$$

Together with (3) this implies

$$\mathcal{D}_{ad,\min\{\frac{\delta}{2},\frac{r}{2}\}}(\boldsymbol{y}_0) = \mathcal{D},$$

hence we conclude that $\boldsymbol{y}_0 \in \mathcal{R}'_0$. Thus, $B(\boldsymbol{x}_0,\epsilon) \subset \mathcal{R}'_0$ and consequently $\mathcal{R}'_0$ is open.

(c). Obviously, $\mathcal{R}_0 \subseteq \mathcal{R}$. Therefore, $\mathcal{R}_0^\circ \subseteq \mathcal{R}^\circ$ and by (b) it implies $\mathcal{R}_0 \subseteq \mathcal{R}^\circ$.

Next we just prove that $\mathcal{R}^\circ \subseteq \mathcal{R}_0$. Let $\boldsymbol{x}_0 \in \mathcal{R}^\circ \setminus \mathcal{R}_0$. Since $\boldsymbol{x}_0 \notin \mathcal{R}_0$, either

$$\sup_{\pi\in\mathcal{D}} k'(\boldsymbol{x}_0,\pi) = \infty \tag{5}$$

or

$$\mathcal{D}_{ad,\delta}(\boldsymbol{x}_0) \neq \mathcal{D}, \forall \delta > 0 \tag{6}$$

must hold. If (5) holds, then we obtain $\boldsymbol{x}_0 \in \partial\mathcal{R}$ since in every neighborhood of $\boldsymbol{x}_0$ there exist $\boldsymbol{x}'_0$ and a perturbation input policy π such that $k'(\boldsymbol{x}'_0,\pi) = \infty$, contradicting $\boldsymbol{x}_0 \in \mathcal{R}^\circ$.

Hence assume

$$K = \sup_{\pi\in\mathcal{D}} k'(\boldsymbol{x}_0,\pi) < \infty.$$

Then we have the conclusion that (6) holds and thus there exists a sequence $(\pi_i,k_i)_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \texttt{dist}(\boldsymbol{\phi}^{\pi_i}_{\boldsymbol{x}_0}(k_i), X^c) = 0.$$

Since (3) and $k'(\boldsymbol{x}_0,\pi_i) \leq K$, we have that

$$k_i \leq K, \forall i \in \mathbb{N}.$$

Also, since $\boldsymbol{x}_0 \in \mathcal{R}$, we have that $\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(j) \in X$ for $j \in \mathbb{N}$ and $\pi \in \mathcal{D}$. Thus, $\boldsymbol{x}_i = \boldsymbol{\phi}^{\pi_i}_{\boldsymbol{x}_0}(k_i)$ is bounded. The fact that $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{d})$ is locally Lipschitz continuous over $\mathbb{R}^n$ yields that for every $\epsilon > 0$, the set

$$\{\boldsymbol{\phi}^{\pi_i}_{\boldsymbol{y}}(k_i) \mid \boldsymbol{y} \in B(\boldsymbol{x}_0,\epsilon)\}$$

contains a ball $B(\boldsymbol{x}_i,\rho)$ with $\rho > 0$ independent of i(since $k_i \leq K$, $\forall i \in \mathbb{N}$). For sufficiently large i this implies $B(\boldsymbol{x}_i,\rho) \nsubseteq X$. This means that

$$\pi_i \notin \mathcal{D}_{ad,0}(\boldsymbol{z}_i)$$

for some $\boldsymbol{z}_i \in B(\boldsymbol{x}_0,\epsilon)$ and consequently $\boldsymbol{z}_i \notin \mathcal{R}$. Since $\epsilon > 0$ is arbitrary, this implies $\boldsymbol{x}_0 \in \partial\mathcal{R}$, again contradicting $\boldsymbol{x}_0 \in \mathcal{R}^\circ$. Hence, $\mathcal{R}^\circ \setminus \mathcal{R}_0 = \emptyset$, implying $\mathcal{R}^\circ \subset \mathcal{R}_0$. □

3.2 Bellman Equations

In this section we mainly present a modified Bellman equation, to which the strict one sub-level set of the unique bounded and continuous solution is equal to the maximal robust region of uniform attraction $\mathcal{R}_0$. For this sake we first introduce a value function, whose strict one sub-level set is equal to the maximal robust region of uniform attraction $\mathcal{R}_0$. Then we reduce this value function to the unique continuous and bounded solution to a modified Bellman equation.

We first introduce a semi-definite positive polynomial cost $g: \mathbb{R}^n \to \mathbb{R}$ satisfying that $g(\boldsymbol{x}) = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$. For the sake of simplicity, we denote $\ln(g(\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}}(i)) + 1)$ and $\ln(l(1 - h^X_j(\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}}(i))))$ as $g_i(\boldsymbol{x},\pi)$ and $h_{j,i}(\boldsymbol{x},\pi)$ respectively, i.e.,

$$g_i(\boldsymbol{x},\pi) = \ln(g(\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}}(i)) + 1)$$

and

$$h_{j,i}(\boldsymbol{x},\pi) = \ln(l(1 - h^X_j(\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}}(i)))), \tag{7}$$

where

$$l(x) = \begin{cases} x, \text{ if } x \geq 0, \\ 0, \text{ otherwise.} \end{cases}$$

Besides, we define $\ln 0 := -\infty$.

We define the value function $V: \mathbb{R}^n \to \mathbb{R}^+ \cup \{\infty\}$ as

$$V(\boldsymbol{x}) := \sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} \Big\{ \sum_{i=1}^{k} g_{i-1}(\boldsymbol{x},\pi) - \min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{x},\pi) \Big\} \tag{8}$$

and consider the Kruzhkov transformed optimal value function $v: \mathbb{R}^n \to [0,1]$ given by

$$v(\boldsymbol{x}) := 1 - e^{-V(\boldsymbol{x})} = \sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} \big\{1 - e^{\tilde{V}}\big\}, \tag{9}$$

where

$$\tilde{V} = -\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x},\pi) + \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x},\pi). \tag{10}$$

Theorem 1. Under Assumption 1, then

(a) $\mathcal{R}_0 = \{\boldsymbol{x}\in\mathbb{R}^n \mid V(\boldsymbol{x})<\infty\} = \{\boldsymbol{x}\in\mathbb{R}^n \mid v(\boldsymbol{x})<1\}$.
(b) $V(\boldsymbol{x})$ is continuous over $\mathcal{R}_0$. Also, $V(\boldsymbol{x})=\infty$ for $\boldsymbol{x}\notin\mathcal{R}_0$.
(c) $v(\boldsymbol{x})$ is continuous over $\mathbb{R}^n$.

Proof. In these proofs, $\Omega(\boldsymbol{x}_0,k)$ denotes the set of states visited by system (1) initialized at $\boldsymbol{x}_0$ within $k\geq 1$ steps, i.e., $\Omega(\boldsymbol{x}_0,k) = \{\boldsymbol{y}\in\mathbb{R}^n \mid \boldsymbol{y}=\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(i), \forall i\in[0,k]\cap\mathbb{N}, \forall\pi\in\mathcal{D}\}$.

(a). Firstly, by (9), we obtain immediately the equality between the two sets $\{\boldsymbol{x}\in\mathbb{R}^n \mid V(\boldsymbol{x})<\infty\}$ and $\{\boldsymbol{x}\in\mathbb{R}^n \mid v(\boldsymbol{x})<1\}$. It remains to prove the first identity that $\mathcal{R}_0=\{\boldsymbol{x}\in\mathbb{R}^n \mid V(\boldsymbol{x})<\infty\}$.

Let $\boldsymbol{x}_0\in\mathcal{R}_0$. We first prove that

$$\sup_{\pi\in\mathcal{D}}\sum_{i=1}^{\infty} g_{i-1}(\boldsymbol{x}_0,\pi)<\infty.$$

Let $W(\boldsymbol{x}_0)=\sup_{\pi\in\mathcal{D}}\sum_{i=1}^{\infty}g_{i-1}(\boldsymbol{x}_0,\pi)$. According to Assumption 1 and the definition of $\mathcal{R}_0$, there exists $K>0$ such that $\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k)\in B(\boldsymbol{0},r)$ for $k\geq K$ and $\pi\in\mathcal{D}$. Also, the closure of the reachable set $\overline{\Omega(\boldsymbol{x}_0,K)}$ is compact. Thus for $\pi\in\mathcal{D}$,

$$\begin{aligned} W(\boldsymbol{x}_0)\leq & K\sup_{\pi\in\mathcal{D},\boldsymbol{x}\in\overline{\Omega(\boldsymbol{x}_0,K)}}\ln(g(\boldsymbol{x})+1)\\ &+\sum_{i=K+1}^{\infty}L_rMr\lambda^{i-K-1}\leq C,\end{aligned}$$

where L_r is the Lipschitz constant of $\ln(g(\boldsymbol{x})+1)$ over $\boldsymbol{x}\in B(\boldsymbol{0},r)$. Therefore $W(\boldsymbol{x}_0)<\infty$. Next we prove that

$$-\sup_{\pi\in\mathcal{D},k\in\mathbb{N}}\min_{j\in\{1,\dots,n_X\}}h_{j,k}(\boldsymbol{x}_0,\pi)<\infty.$$

Since $\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k)\|\leq\beta(k)$ for $\pi\in\mathcal{D}$, the reachable set $\Omega(\boldsymbol{x}_0,\infty)$ is bounded, hence $\overline{\Omega(\boldsymbol{x}_0,\infty)}$ is compact. Moreover, since $\mathcal{D}=\mathcal{D}_{ad,\delta}(\boldsymbol{x}_0)$ for some $\delta>0$, we have that $\overline{\Omega(\boldsymbol{x}_0,\infty)}\subset X$. Also, since each h_j^X, $j=1,\dots,n_X$, is continuous over X, it will attain a finite maximum being less than 1 on $\overline{\Omega(\boldsymbol{x}_0,\infty)}$ and thus

$$\sup_{\pi\in D,k\in\mathbb{N}}\min_{j\in\{1,\dots,n_X\}}h_{j,k}(\boldsymbol{x}_0,\pi)$$

will attain a finite minimum over $\overline{\Omega(\boldsymbol{x}_0,\infty)}$ according to (7). We prove the claim.

Let $\boldsymbol{x}_0\notin\mathcal{R}_0$. Then either $\sup_{\pi\in\mathcal{D}}k'(\boldsymbol{x}_0,\pi)=\infty$ or the existence of δ in the definition of $\mathcal{R}_0$ is not satisfied, where $k'(\boldsymbol{x}_0,\pi)$ is defined in (4).

For the first case, there exists a sequence $(\pi_{j'}\in\mathcal{D})_{j'\in\mathbb{N}}$ such that $\lim_{j'\to\infty}k'(\boldsymbol{x}_0,\pi_{j'})=\infty$. Then for any $j'\in\mathbb{N}$,

$$\begin{aligned}\sum_{i=1}^{\infty}g_{i-1}(\boldsymbol{x}_0,\pi_{j'})\geq&\sum_{i=1}^{k'(\boldsymbol{x}_0,\pi_{j'})}g_{i-1}(\boldsymbol{x}_0,\pi_{j'})\\ \geq&\ln(c_0+1)k'(\boldsymbol{x}_0,\pi_{j'}),\end{aligned}$$

where c_0 is a constant such that $\inf_{\boldsymbol{x}\notin B(\boldsymbol{0},r)}g(\boldsymbol{x})\geq c_0$ (Such c_0 exists since $g(\boldsymbol{x})$ is a polynomial function over $\boldsymbol{x}$ and $g(\boldsymbol{x})>0$ for $\boldsymbol{x}\neq\boldsymbol{0}$). It follows that $W(\boldsymbol{x}_0)\geq\lim_{j'\to\infty}\sum_{i=1}^{\infty}g_{i-1}(\boldsymbol{x}_0,\pi_{j'})=\infty$. Therefore, $V(\boldsymbol{x}_0)=\infty$ since $V(\boldsymbol{x}_0)\geq W(\boldsymbol{x}_0)$. In the second case, the non-existence of δ implies the existence of a sequence $(\pi_{j'},k_{j'})_{j'\in\mathbb{N}}$ with $\lim_{j'\to\infty}\mathtt{dist}(\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi_{j'}}(k_{j'}),X^c)=0$. Then either there exists $l_0\in\mathbb{N}$ such that $\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi_{l_0}}(k_{l_0})\in X^c$ or there exists a subsequence $(\boldsymbol{x}_{k_{j'_l}})_{l\in\mathbb{N}}$ converging to some $\boldsymbol{x}\notin X$ (This is due to the fact that the sequence $(\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi_{j'}}(k_{j'}))_{j'\in\mathbb{N}}$ lies in the bounded set X.), where $\boldsymbol{x}_{k_{j'_l}}=\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi_{j'_l}}(k_{j'_l})$. Both cases imply that

$$\lim_{j'\to\infty}\sup_{\pi\in\mathcal{D}}\big(-\min_{j\in\{1,\dots,n_X\}}h_{j,k_{j'}}(\boldsymbol{x}_0,\pi)\big)=\infty.$$

Also, since

$$V(\boldsymbol{x}_0)\geq\sup_{\pi\in\mathcal{D}}\sup_{j'\in\mathbb{N}}(-\min_{j\in\{1,\dots,n_X\}}h_{j,k_{j'}}(\boldsymbol{x}_0,\pi)),$$

we obtain $V(\boldsymbol{x}_0)=\infty$.

(b). Let $\boldsymbol{x}_0,\boldsymbol{y}_0\in\mathcal{R}_0$,

$$|V(\boldsymbol{x}_0)-V(\boldsymbol{y}_0)|\leq|W(\boldsymbol{x}_0)-W(\boldsymbol{y}_0)|+|W'(\boldsymbol{x}_0)-W'(\boldsymbol{y}_0)|,$$

where $W(\boldsymbol{x}_0)=\sup_{\pi\in\mathcal{D}}\sum_{i=1}^{\infty}g_{i-1}(\boldsymbol{x}_0,\pi)$ and $W'(\boldsymbol{x}_0)=\sup_{\pi\in\mathcal{D}}\sup_{k\in\mathbb{N}}\min_{j\in\{1,\dots,n_X\}}h_{j,k}(\boldsymbol{x}_0,\pi)$. In the following we separately prove the continuity of $W(\boldsymbol{x}_0)$ and $W'(\boldsymbol{x}_0)$. Firstly, we prove that W is continuous on $B(\boldsymbol{0},\frac{r}{M})$. Assume that $\boldsymbol{x}_0\in B(\boldsymbol{0},\frac{r}{M})$. Then

$$\begin{aligned}\sum_{i=0}^{\infty}|\ln(g(\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(i))+1)|\leq& L_r\sum_{i=0}^{\infty}\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(i)\|\\ \leq& L_rM\sum_{i=0}^{\infty}\lambda^i\|\boldsymbol{x}_0\|\leq M_1\|\boldsymbol{x}_0\|,\end{aligned}$$

where L_r is the Lipschitz constant of $\ln(g(\boldsymbol{x})+1)$ over $\boldsymbol{x}\in B(\boldsymbol{0},r)$, r, λ and M are defined in (2).

For arbitrary but fixed $\epsilon>0$, we can conclude from Assumption 1 that there exists $K>0$ such that $M_1\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k)\|\leq\frac{\epsilon}{3}$ for $k\geq K$ and $\boldsymbol{x}_0\in B(\boldsymbol{0},\frac{r}{M})$. In addition, by Lipschitz continuity of $\boldsymbol{f}$ there exists $\delta'>0$ such that

$$\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k)-\boldsymbol{\phi}_{\boldsymbol{y}_0}^{\pi}(k)\|\leq\frac{\epsilon}{3L_r(K+1)}$$

for $k\in[0,K]$ and $\boldsymbol{y}_0\in\{\boldsymbol{x}\in B(\boldsymbol{0},\frac{r}{M})\mid\|\boldsymbol{x}-\boldsymbol{x}_0\|<\delta'\}$. Then, we have

$$\begin{aligned}&|W(\boldsymbol{x}_0)-W(\boldsymbol{y}_0)|\\ &\leq\sup_{\pi\in\mathcal{D}}\sum_{i=1}^{\infty}|\ln(g(\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(i-1))+1)-\ln(g(\boldsymbol{\phi}_{\boldsymbol{y}_0}^{\pi}(i-1))+1)|\\ &\leq\sup_{\pi\in\mathcal{D}}\big(\sum_{i=0}^{K}L_r\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(i)-\boldsymbol{\phi}_{\boldsymbol{y}_0}^{\pi}(i)\|+\\ &\qquad M_1\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k)\|_{k>K}+M_1\|\boldsymbol{\phi}_{\boldsymbol{y}_0}^{\pi}(k)\|_{k>K}\big)\\ &\leq\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\leq\epsilon.\end{aligned} \tag{11}$$

Therefore, $W(\boldsymbol{x})$ is continuous over $B(\boldsymbol{0},\frac{r}{M})$.

For $\boldsymbol{x}_0\in\mathcal{R}_0$, let L be the Lipschitz constant of $\ln(g(\boldsymbol{x})+1)$ over $\boldsymbol{x}\in\overline{X}$. Since $\mathcal{R}_0$ is open and $\boldsymbol{f}$ is Lipschitz continuous over $\boldsymbol{x}\in\mathbb{R}^n$ uniformly over $\boldsymbol{d}\in D$, we have that for ϵ satisfying $0<\epsilon<LK\delta'$, there exists an open neighborhood O in $\mathcal{R}_0$ of $\boldsymbol{x}_0$ and $K>0$ such that

$$\boldsymbol{\phi}_{\boldsymbol{y}_0}^{\pi}(k) \in B(\boldsymbol{0}, \frac{r}{M}), \forall \boldsymbol{y}_0 \in O, \forall \pi \in \mathcal{D}, \forall k \geq K$$

and

$$\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k) - \boldsymbol{\phi}_{\boldsymbol{y}_0}^{\pi}(k)\| \leq \frac{\epsilon}{LK}, \forall k \in [0, K],$$

which implies that

$$\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(K) - \boldsymbol{\phi}_{\boldsymbol{y}_0}^{\pi}(K)\| \leq \frac{\epsilon}{LK} < \delta'.$$

Therefore, similar to the deduction in (11), we have

$$|W(\boldsymbol{x}_0) - W(\boldsymbol{y}_0)| \leq 2\epsilon.$$

In conclusion, $W(\boldsymbol{x}_0)$ is continuous over $\mathcal{R}_0$.

Next, we prove the continuity of $W'(\boldsymbol{x}_0)$. It is obvious that

$$|W'(\boldsymbol{x}_0) - W'(\boldsymbol{y}_0)| \leq \sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} |\min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x}_0, \pi) - \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{y}_0, \pi)|.$$

As $\boldsymbol{x}_0 \in \mathcal{R}_0$, $\lim_{k\to\infty} \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x}_0, \pi) = 0$. Observing that $h_{j,k}$ is Lipschitz continuous over $\mathcal{R}_0$ and there exists $\beta(k) : \mathbb{N} \to [0, \infty)$, which is independent of π, such that $\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k)\| \leq \beta(k)$ for $k \in \mathbb{N}, \pi \in \mathcal{D}$ and $\boldsymbol{x}_0 \in \mathcal{R}_0$, we can find a neighborhood $B(\boldsymbol{x}_0, \delta)$ and a function $\gamma(k) : \mathbb{N} \to [0, \infty)$ with $\lim_{k\to\infty} \gamma(k) = 0$ such that $|\min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{y}_0, \pi)| \leq \gamma(k)$ holds for $\boldsymbol{y}_0 \in B(\boldsymbol{x}_0, \delta)$. This implies that the supremum

$$\sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} |\min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x}_0, \pi) - \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{y}_0, \pi)|$$

is attained on a finite interval $[0, K] \cap \mathbb{N}$. On a compact time interval, the map $\boldsymbol{x} \to \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x}, \pi)$ is Lipschitz continuous over $\boldsymbol{x} \in \mathcal{R}_0$ uniformly over $\pi \in \mathcal{D}$ since $h_j^X(\boldsymbol{x})$ and $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d})$ are Lipschitz continuous over $\boldsymbol{x} \in \mathcal{R}_0$ uniformly over $\boldsymbol{d} \in D$, implying that

$$\lim_{\boldsymbol{y}_0\to\boldsymbol{x}_0} \sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} |\min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x}_0, \pi) - \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{y}_0, \pi)| = 0.$$

This shows the desired continuity.

The second assertion that $V(\boldsymbol{x}) = \infty$ if $\boldsymbol{x} \notin \mathcal{R}_0$, can be proved by following the proof when $\boldsymbol{x} \notin \mathcal{R}_0$ in (a).

(c). From (b) we have that $V(\boldsymbol{x}) = \infty$ for $\boldsymbol{x} \in \mathbb{R}^n \setminus \mathcal{R}_0$. Therefore, $v(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \mathbb{R}^n \setminus \mathcal{R}_0$ due to the fact that $v(\boldsymbol{x}) = 1 - e^{-V(\boldsymbol{x})}$ over $\mathbb{R}^n$. Therefore, $v(\boldsymbol{x})$ is continuous over $\mathbb{R}^n \setminus \mathcal{R}_0$.

Also since $V(\boldsymbol{x})$ is continuous over $\mathcal{R}_0$, we have that $v(\boldsymbol{x})$ is continuous over $\mathcal{R}_0$.

We just prove that if $\lim_{\boldsymbol{x}\to\boldsymbol{y}} v(\boldsymbol{x}) = v(\boldsymbol{y})$ for $\boldsymbol{x} \in \mathcal{R}_0$ and $\boldsymbol{y} \in \mathbb{R}^n \setminus \mathcal{R}_0$. According to (b) we have $\lim_{\boldsymbol{x}\to\boldsymbol{y}} V(\boldsymbol{x}) = \infty$ and consequently $\lim_{\boldsymbol{x}\to\boldsymbol{y}} v(\boldsymbol{x}) = 1 = v(\boldsymbol{y})$.

Above all, we have that $v(\boldsymbol{x})$ is continuous over $\mathbb{R}^n$. □

Theorem 1 indicates that the interior of the maximal robust region of attraction can be obtained by computing either the value function $V(\boldsymbol{x})$ in (8) or the value function $v(\boldsymbol{x})$ in (9). Below we show that they can be computed by solving modified Bellman equations. For this sake, we first show that $V(\boldsymbol{x})$ and $v(\boldsymbol{x})$ satisfy the dynamic programming principle.

Lemma 2. Under Assumption 1, the following assertions are satisfied:

(a) For $\boldsymbol{x} \in \mathbb{R}^n$ and $k \in \mathbb{N}$, we have:

$$\begin{aligned} V(\boldsymbol{x}) = \sup_{\pi\in\mathcal{D}} \max \{ & \\ & \sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}, \pi) + V(\boldsymbol{\phi}_{\boldsymbol{x}}^{\pi}(k)), \\ & \sup_{i\in[0,k-1]\cap\mathbb{N}} \{ \sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}, \pi) - \min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}, \pi) \} \\ \}. \end{aligned} \tag{12}$$

(b) For $\boldsymbol{x} \in \mathbb{R}^n$ and $k \in \mathbb{N}$, we have:

$$\begin{aligned} v(\boldsymbol{x}) = \sup_{\pi\in\mathcal{D}} \max \{ & \\ & 1 - \frac{1 - v(\boldsymbol{\phi}_{\boldsymbol{x}}^{\pi}(k))}{\prod_{i=1}^{k} e^{g_{i-1}(\boldsymbol{x},\pi)}}, \sup_{i\in[0,k-1]\cap\mathbb{N}} \{1 - e^{-\overline{V}}\} \\ \}, \end{aligned} \tag{13}$$

where

$$\overline{V} = \sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}, \pi) - \min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}, \pi).$$

Proof. (a). Let

$$\begin{aligned} W(\boldsymbol{x}_0, k) = \sup_{\pi\in\mathcal{D}} \max \{ & \\ & \sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0, \pi) + V(\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k)), \\ & \sup_{i\in[0,k-1]\cap\mathbb{N}} \{ \sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0, \pi) - \min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0, \pi) \} \\ \}. \end{aligned} \tag{14}$$

We will prove that $|W(\boldsymbol{x}_0, k) - V(\boldsymbol{x}_0)| \leq \epsilon$, $\forall \epsilon > 0$.

From (8), for any $\epsilon_1 > 0$, there exists $\pi \in \mathcal{D}$ such that

$$V(\boldsymbol{x}_0) \leq \epsilon_1 + \sup_{k\in\mathbb{N}} \{ \sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0, \pi) - \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x}_0, \pi) \} \}.$$

We respectively define $\pi_1 \in \mathcal{D}$ and $\pi_2 \in \mathcal{D}$ as follows: $\pi_1(i) = \pi(i)$ for $i = 0, \dots, k-1$, and $\pi_2(i) = \pi(i+k)$ for $i \in \mathbb{N}$, and $\boldsymbol{y} = \boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi_1}(k)$, then obtain that

$$\begin{aligned}
&W(\boldsymbol{x}_0,k)\geq \max\big\{\\
&\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0,\pi)+V(\boldsymbol{y}),\\
&\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi)\}\\
&\big\}\\
&\geq \max\big\{\\
&\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0,\pi_1)+\\
&\sup_{l\in[k,\infty)\cap\mathbb{N}}\{\sum_{i'=1}^{l-k} g_{i'-1}(\boldsymbol{y},\pi_2)-\min_{j\in\{1,\dots,n_X\}} h_{j,l-k}(\boldsymbol{y},\pi_2)\},\\
&\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi_1)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi_1)\}\\
&\big\}\\
&\geq \max\big\{\\
&\sup_{l\in[k,\infty)\cap\mathbb{N}}\{\sum_{j_1=1}^{l} g_{j_1-1}(\boldsymbol{x}_0,\pi)-\min_{j\in\{1,\dots,n_X\}} h_{j,l}(\boldsymbol{x}_0,\pi)\},\\
&\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi)\}\\
&\big\}\\
&\geq V(\boldsymbol{x}_0)-\epsilon_1.
\end{aligned}$$

Therefore, $V(\boldsymbol{x}_0)\leq W(\boldsymbol{x}_0,k)+\epsilon_1$.

According to (14), for any $\epsilon_1>0$, there exists a perturbation input policy $\pi_1\in\mathcal{D}$ such that

$$\begin{aligned}
&W(\boldsymbol{x}_0,k)\leq \epsilon_1+\max\big\{\\
&\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0,\pi_1)+V(\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi_1}(k)),\\
&\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi_1)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi_1)\}\\
&\big\}.
\end{aligned}$$

Also, by the definition of V, i.e. (8), for any ϵ_1, there exists an input policy $\pi_2\in\mathcal{D}$ such that

$$\begin{aligned}
V(\boldsymbol{y})\leq \epsilon_1+&\\
&\sup_{l\in\mathbb{N}}\big\{\sum_{i=1}^{l} g_{i-1}(\boldsymbol{y},\pi_2)-\min_{j\in\{1,\dots,n_X\}} h_{j,l}(\boldsymbol{y},\pi_2)\big\},
\end{aligned}$$

where $\boldsymbol{y}=\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi_1}(k)$. We define π:

$$\pi(i)=\begin{cases}\pi_1(i), & i\in[0,k)\cap\mathbb{N}\\ \pi_2(i-k), & i\in[k,\infty)\cap\mathbb{N}\end{cases}.$$

Therefore, we infer that

$$\begin{aligned}
&W(\boldsymbol{x}_0,k)\leq \epsilon_1+\max\big\{\\
&\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0,\pi_1)+V(\boldsymbol{y}),\\
&\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi_1)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi_1)\}\\
&\big\}\\
&\leq 2\epsilon_1+\max\big\{\\
&\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0,\pi_1)+\\
&\sup_{l\in[k,\infty)\cap\mathbb{N}}\{\sum_{i'=1}^{l-k} g_{i'-1}(\boldsymbol{y},\pi_2)-\min_{j\in\{1,\dots,n_X\}} h_{j,l-k}(\boldsymbol{y},\pi_2),\\
&\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi_1)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi_1)\}\\
&\big\}\\
&\leq 2\epsilon_1+\max\big\{\\
&\sup_{l\in[k,\infty)\cap\mathbb{N}}\{\sum_{j_1=1}^{l} g_{j_1-1}(\boldsymbol{x}_0,\pi)-\min_{j\in\{1,\dots,n_X\}} h_{j,l}(\boldsymbol{x}_0,\pi)\},\\
&\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi)\}\\
&\big\}\\
&\leq V(\boldsymbol{x}_0)+2\epsilon_1.
\end{aligned}$$

Therefore, we finally have $|W-V|\leq\epsilon=2\epsilon_1$, implying that $V=W$ since ϵ_1 is arbitrary.

(b). (13) can be obtained using $v(\boldsymbol{x}_0)=1-e^{-V(\boldsymbol{x}_0)}$. □

Based on Lemma 2 we can infer that the value functions $V(\boldsymbol{x}_0)$ and $v(\boldsymbol{x}_0)$ are solutions to the two modified Bellman equations (15) and (16), respectively.

Theorem 2. Under Assumption 1, the value function V is the unique continuous solution to the modified Bellman equation

$$\begin{aligned}
&\min\big\{\inf_{\boldsymbol{d}\in D}\{V-V(\boldsymbol{f})-\ln(g+1)\},\\
&\quad V+\min_{j\in\{1,\dots,n_X\}}\ln(l(1-h_j^X))\big\}=0,\forall\boldsymbol{x}\in\mathcal{R}_0, \quad (15)\\
&V(\mathbf{0})=0.
\end{aligned}$$

The value function v is the unique bounded and continuous solution to the modified Bellman equation

$$\begin{aligned}
&\min\big\{\inf_{\boldsymbol{d}\in D}\{v-v(\boldsymbol{f})-g\cdot(1-v)\},\\
&\quad v-1+\min_{j\in\{1,\dots,n_X\}} l(1-h_j^X)\big\}=0,\forall\boldsymbol{x}\in\mathbb{R}^n, \quad (16)\\
&v(\mathbf{0})=0.
\end{aligned}$$

Proof. The fact that the value functions $V(\boldsymbol{x})$ in (8) and $v(\boldsymbol{x})$ in (9) are solutions to (15) and (16) respectively can be verified when $k=1$ in (12) and (13).

Here, we just prove the uniqueness of solutions to (16). The uniqueness of solution to (15) can be guaranteed by the relationship $v(\boldsymbol{x})=1-e^{-V(\boldsymbol{x})}$ for $\boldsymbol{x}\in\mathbb{R}^n$.

Assume that $\tilde{v}$ is a bounded and continuous solution to (16) as well, we need to prove that $v = \tilde{v}$ over $\boldsymbol{x} \in \mathbb{R}^n$, where $v < 1$ over $\mathcal{R}_0$ and $v = 1$ over $\mathbb{R}^n \setminus \mathcal{R}_0$. Assume that there exists $\boldsymbol{y}_0$ such that $\tilde{v}(\boldsymbol{y}_0) \neq v(\boldsymbol{y}_0)$. First let's assume $v(\boldsymbol{y}_0) > \tilde{v}(\boldsymbol{y}_0)$ and $v(\boldsymbol{y}_0) \geq 1$. Obviously, $\boldsymbol{y}_0 \neq \boldsymbol{0}$ and consequently $g(\boldsymbol{y}_0) > 0$. Since both v and $\tilde{v}$ satisfy (16), we have that

$$\inf_{\boldsymbol{d}\in D}\{v(\boldsymbol{y}_0) - v(\boldsymbol{f}(\boldsymbol{y}_0,\boldsymbol{d})) - g(\boldsymbol{y}_0)(1 - v(\boldsymbol{y}_0))\} = 0.$$

Since v is continuous over $\mathbb{R}^n$ and $\boldsymbol{f}$ is continuous over $\mathbb{R}^n \times D$, there exists $\boldsymbol{d}'_1 \in D$ such that $v(\boldsymbol{y}_0) - v(\boldsymbol{f}(\boldsymbol{y}_0,\boldsymbol{d}'_1)) - g(\boldsymbol{y}_0)(1-v(\boldsymbol{y}_0)) = 0$. Since $\tilde{v}(\boldsymbol{y}_0)-\tilde{v}(\boldsymbol{f}(\boldsymbol{y}_0,\boldsymbol{d}'_1))-g(\boldsymbol{y}_0)(1-\tilde{v}(\boldsymbol{y}_0)) \geq 0$, we obtain that

$$\begin{aligned} v(\boldsymbol{f}(\boldsymbol{y}_0,\boldsymbol{d}'_1))-&\tilde{v}(\boldsymbol{f}(\boldsymbol{y}_0,\boldsymbol{d}'_1)) \\ &\geq (v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0))(1 + g(\boldsymbol{y}_0)). \end{aligned}$$

Let $\boldsymbol{y}_1 = \boldsymbol{\phi}^{\pi_1}_{\boldsymbol{y}_0}(1)$, where $\pi_1(0) = \boldsymbol{d}'_1$, then $v(\boldsymbol{y}_1) > \tilde{v}(\boldsymbol{y}_1)$. Also, we have $v(\boldsymbol{y}_0) \leq v(\boldsymbol{y}_1)$. Moreover, $\boldsymbol{y}_1 \neq \boldsymbol{0}$, $g(\boldsymbol{y}_1) > 0$. We continue the above deduction from $\boldsymbol{y}_0$ to $\boldsymbol{y}_1$, and obtain that there exists $\boldsymbol{d}'_2 \in D$ such that

$$\begin{aligned} v(\boldsymbol{f}(\boldsymbol{y}_1,\boldsymbol{d}'_2))-&\tilde{v}(\boldsymbol{f}(\boldsymbol{y}_1,\boldsymbol{d}'_2)) \\ &\geq (v(\boldsymbol{y}_1) - \tilde{v}(\boldsymbol{y}_1))(1 + g(\boldsymbol{y}_1)). \end{aligned}$$

Thus, we have

$$\begin{aligned} &v(\boldsymbol{f}(\boldsymbol{y}_1,\boldsymbol{d}'_2)) - \tilde{v}(\boldsymbol{f}(\boldsymbol{y}_1,\boldsymbol{d}'_2)) \geq \\ &(v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0))(1 + g(\boldsymbol{y}_1))(1 + g(\boldsymbol{y}_0)). \end{aligned}$$

Let $\boldsymbol{y}_2 = \boldsymbol{\phi}^{\pi_2}_{\boldsymbol{y}_1}(1)$, where $\pi_2(0) = \boldsymbol{d}'_2$, then $v(\boldsymbol{y}_2) > \tilde{v}(\boldsymbol{y}_2)$. Also, $v(\boldsymbol{y}_1) \leq v(\boldsymbol{y}_2)$.

Analogously, we deduce that for $k \in \mathbb{N}$,

$$\begin{aligned} &v(\boldsymbol{f}(\boldsymbol{y}_k,\boldsymbol{d}'_{k+1})) - \tilde{v}(\boldsymbol{f}(\boldsymbol{y}_k,\boldsymbol{d}'_{k+1})) \geq \\ &\quad (v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0))(1 + g(\boldsymbol{y}_k)) \cdots (1 + g(\boldsymbol{y}_0)). \end{aligned}$$

Moreover, let $\boldsymbol{y}_{k+1} = \boldsymbol{\phi}^{\pi_{k+1}}_{\boldsymbol{y}_k}(1)$, then $v(\boldsymbol{y}_k) \leq v(\boldsymbol{y}_{k+1})$, where $\pi_{k+1}(0) = \boldsymbol{d}'_{k+1}$. This implies that $\lim_{k\to\infty} \boldsymbol{y}_k \neq \boldsymbol{0}$ and thus $\boldsymbol{y}_k \notin B(\boldsymbol{0},\overline{\epsilon})$ for $k \in \mathbb{N}$, where $B(\boldsymbol{0},\overline{\epsilon})$ is defined in (3). Assume that $c_0 = \inf\{g(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}^n \setminus B(\boldsymbol{0},\overline{\epsilon})\}$. Obviously, $c_0 > 0$. Therefore,

$$\begin{aligned} v(\boldsymbol{f}(\boldsymbol{y}_k,\boldsymbol{d}'_{k+1}))-&\tilde{v}(\boldsymbol{f}(\boldsymbol{y}_k,\boldsymbol{d}'_{k+1})) \\ &\geq (v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0))(1 + c_0)^{k+1}, \end{aligned}$$

implying that $\lim_{k\to\infty} v(\boldsymbol{y}_k) = \infty$, which contradicts the fact that v is bounded over $\mathbb{R}^n$.

Next, assume $v(\boldsymbol{y}_0) > \tilde{v}(\boldsymbol{y}_0)$ and $v(\boldsymbol{y}_0) < 1$. According to Theorem 1, every possible trajectory starting from $\boldsymbol{y}_0$ will eventually approach $\boldsymbol{0}$. Also, we have

$$\inf_{\boldsymbol{d}\in D}\{v(\boldsymbol{y}_0,\boldsymbol{d}) - v(\boldsymbol{f}(\boldsymbol{y}_0,\boldsymbol{d})) - g(\boldsymbol{y}_0)(1 - v(\boldsymbol{y}_0))\} = 0.$$

Following the deduction mentioned above, we have

$$v(\boldsymbol{y}_k) - \tilde{v}(\boldsymbol{y}_k) \geq v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0), \forall k \in \mathbb{N}.$$

Since $\lim_{k\to\infty} \tilde{v}(\boldsymbol{y}_k) = 0$, $\lim_{k\to\infty} v(\boldsymbol{y}_k) \geq v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0)$ holds, contradicting $\lim_{k\to\infty} v(\boldsymbol{y}_k) = 0$.

For the case that $\tilde{v}(\boldsymbol{y}_0) > v(\boldsymbol{y}_0)$, we can obtain similar contradictions by following the proof procedure mentioned above with v and $\tilde{v}$ reversed. □

4. ILLUSTRATIVE EXAMPLES

In this section we apply the equation (16) to the computation of robust regions of attraction on one example.

Example 1. We consider a computer-based model of the following perturbed ordinary differential equation,

$$\begin{cases} \dot{x}_1 = \dfrac{x_2}{2} + x_1x_2 + (\dfrac{1}{2} + d)x_1^2 + \dfrac{1}{2}x_1^2x_2, \\ \dot{x}_2 = -2x_1 - x_2 - 2x_1x_2 - x_1^2 - x_1^2x_2, \end{cases} \tag{17}$$

where $d \in [-0.1, 0.1]$. Its unperturbed version is used to describe a chemical oscillator, and is obtained by transforming the equilibrium (1,0.5) and making $x_1 = x, x_2 = 2y$ of the following system from (Papachristodoulou and Prajna, 2002):

$$\begin{cases} \dot{x} = a - x + x^2y, \\ \dot{x}_2 = b - x^2y, \end{cases} \tag{18}$$

where $a = b = 0.5$.

When performing computer simulations, the Euler's method is often used to analyze an ordinary differential equation, which uses the idea of local linearity or linear approximation. When the simulation time step is 0.2, the resulting discrete-time system is of the following form:

$$\begin{cases} x_1(k+1) = x_1(k) + 0.2\big(\dfrac{x_2(k)}{2} + x_1(k)x_2(k) \\ \qquad + (\dfrac{1}{2} + d(k))x_1^2(k) + \dfrac{1}{2}x_1^2(k)x_2(k)\big), \\ x_2(k) = -x_2(k) + 0.2\big(-2x_1(k) - x_2(k) \\ \qquad - 2x_1(k)x_2(k) - x_1^2(k) - x_1^2(k)x_2(k)\big), \end{cases} \tag{19}$$

where $d(\cdot) : \mathbb{N} \to D$ with $D = [-0.1, 0.1]$.

The equilibrium (0,0) for system (19) is uniformly locally exponentially stable. In this example we take the state constraint set $X = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$. In the computation, $g(x_1, x_2) = 0.01(x_1^2 + x_2^2)$ is used for solving (16) based on the well-known value iteration method together with the regularization technique in (Grüne and Zidani, 2015). The estimation is illustrated in Fig. 1, which also showcases the computed $v(\boldsymbol{x})$. Four trajectories, where two trajectories stay inside the state constraint set for all time and two trajectories leave the state constraint set in finite time, are illustrated in Fig. 2. The trajectories are generated by extracting the perturbation input $d(j)$ from D randomly for $j \in \mathbb{N}$.

5. CONCLUSION

In this paper we presented a Bellman equation for computing robust regions of attraction for state-constrained perturbed discrete-time systems. The interior of the maximal robust region of attraction is characterized as the strict one sub-level set of the unique bounded and continuous solution to the derived Bellman equation. One example demonstrated the robust regions of attraction generation based on solving the derived Bellman equation.

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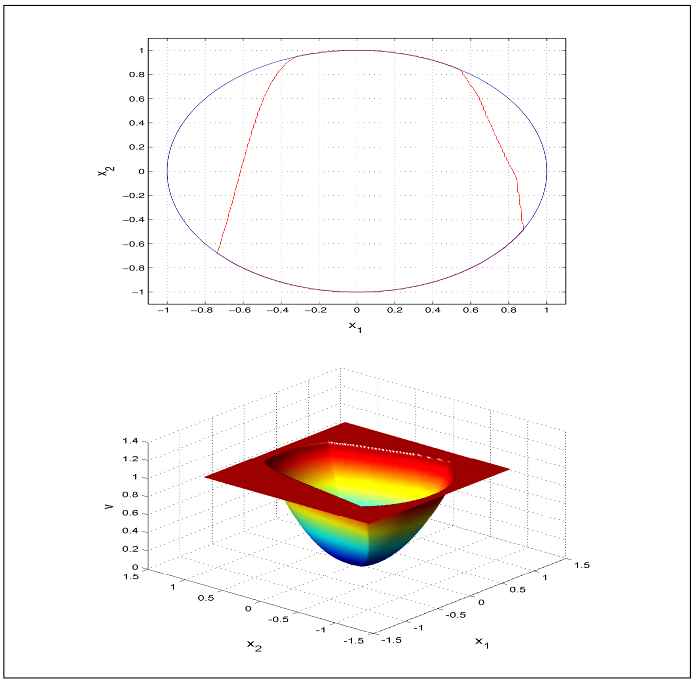

Fig. 1: An illustration of the interior of the maximal region of attraction computed via solving (16) for Example 1. Above: Blue and red curves denote the boundaries of the state constraint set X and the estimation of the interior of the maximal robust region of attraction, respectively. Below: $v(\boldsymbol{x})$ computed via solving (16).

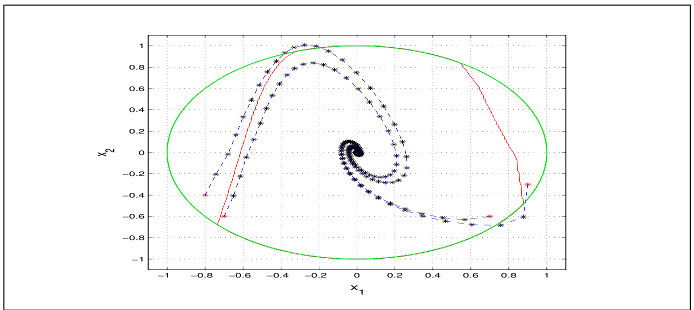

Fig. 2: An illustration of trajectories for Example 1. Green and red curves denote the boundaries of the state constraint set X and the estimation of the interior of the maximal robust region of attraction, respectively. Red stars and black stars denote the initial states and subsequent states, respectively. The dash blue line denotes the transition between states.

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