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Topological entropy and minimal data rate for state observation of LTV systems ⋆

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Abstract: In this paper, we show that the topological entropy of linear time-varying systems coincide with their minimal data rate for state observation, thereby extending the well-known "observation data rate" theorem for LTI systems and time-invariant nonlinear systems with compact domain. This result is relevant for the problem of controlling and observing dynamical systems via limited-capacity communication networks, as it provides a tight bound on the data rate required for the state observation of these systems. This bound, which relies only on the topological properties of the system, can thus be used as a benchmark for the comparison of different implementations of coders–decoders observing the system.

Keywords: Networked control systems, topological entropy, linear time-varying systems

1. INTRODUCTION

Many modern control systems (such as cyber-physical systems, IoT, etc.) involve spatially distributed components that communicate through a shared, digital communication network. Due to the digital nature of the network, all data must be quantized before transmission, resulting in quantization error that can affect the performance of the observing/controlling scheme. Furthermore, in applications, the capacity of the network is often limited by cost, power, physical and/or security constraints. Consequently, a major challenge in the design of such networked systems is to determine the minimal communication data rate required to achieve a given control task. This fundamental question has attracted a lot of attention from the control community in recent years, with great theoretical and practical advances; as surveyed in Hespanha et al. (2007); Matveev and Savkin (2009)

In this paper, we study Linear Time-Varying (LTV) systems. Since these systems appear in the modeling of many cyber-physical systems, it is essential to understand the impact of data rate constraints on the observation and control of these systems. In particular, we are interested in determining the minimal data rate required for state observation of LTV systems, that is, the smallest data rate at which a coder needs to send information to a decoder to estimate the state of the system with arbitrary finite accuracy; see also Figure 1 for an illustration.

Now, we review the literature relevant to our problem. Inspired from the works of Shannon on information entropy and the minimal data rate to transmit information reliably, it was soon realized that the question of minimal data rate for state observation of dynamical systems has strong connections with the notion of *topological entropy*

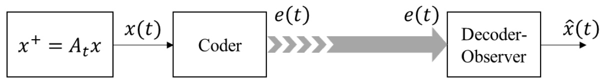

Fig. 1. State observation of linear time-varying systems with limited data rate. The coder measures the state $x(t)$ and sends a discrete-valued symbol $e(t)$ to the decoder, which constructs an estimate $\hat{x}(t)$ of the state of the system based on the past symbols.

of a dynamical system. This quantity, introduced in the late 60's (Adler et al., 1965; Bowen, 1971; Dinaburg, 1970) and now ubiquitous in dynamical system theory, measures the rate at which information about the initial condition is generated by the system, as time evolves. The study of topological entropy and its link with the minimal data rate for state observation of dynamical systems has attracted a lot of attention from the control community in the last decades; see, e.g., Savkin (2006); Matveev and Pogromsky (2016); Kawan (2017); Matveev and Pogromsky (2016). Most of the works available in the literature focus on LTI systems and time-invariant nonlinear systems with a compact domain. For these systems, it is shown that topological entropy and the minimal data rate for state observation coincide (Savkin, 2006; Matveev and Pogromsky, 2016). Beyond these specific classes of systems, the situation is unfortunately much more elusive. Topological entropy is only known to be a lower bound on the minimal data rate for state observation. In particular, it seems an open question whether topological entropy also provides an upper bound on the minimal data rate for state observation. As an evidence of this, we refer to Matveev and Pogromsky (2016); Kawan (2017), where the lower bound is discussed but no proof, or counter-example, for the upper bound is presented.

In this paper, we show that, for *LTV systems*, topological entropy and the minimal data rate for state observation are equal. The relevance of this result is first from the theoretical viewpoint, as it extends the "observation data

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rate" theorem for time-invariant systems, to LTV systems. Moreover, the proof of the theorem is constructive, in the sense that it provides a coder–decoder that observes the state of the system with arbitrary accuracy, and with data rate as close as desired to the topological entropy of the system. It is nonetheless important to mention that the implementation of the proposed coder–decoder requires an unbounded memory in general, which limits the practical usage of the coder–decoder. Nevertheless, we still believe that the result and its proof have a strong practical relevance. For instance, the topological entropy can be used as a benchmark to evaluate the performance of different implementations of coders–decoders. Furthermore, the ideas presented in the proof of the theorem can be used to obtain efficient coders–decoders satisfying memory limitations, though possibly operating at suboptimal data rates.

Compared to time-invariant systems, the following difficulty arises when we seek to relate the topological entropy with data rate bounds for state observation of time-varying systems. Since the coder–decoder observes the state of the system at periodic sampling times, data rate requirements are mainly driven by the amount of information generated by the system *between two sampling times*. By contrast, the topological entropy gives the growth rate of information generated by the system *since the beginning of time*. For time-invariant systems, this problem is overcome by using the fact that the amount of information generated on each sampling interval is the same, and thus it is possible to relate the minimal data rate with the topological entropy. However, this is not true in general for time-varying systems. Therefore, in our analysis of LTV systems, we have used a different approach exploiting the fact that from the growth rate of information generated by the system, i.e., the topological entropy, we can derive an upper bound on the amount of information generated during the sampling intervals. This requires that the trajectories of the system are spatially distributed in a "uniform way" (see Section 3 for details), and which is ensured by the linearity of the system. In particular, it seems that the proof argument presented in this paper does not extend straightforwardly to nonlinear time-varying systems.

The main content of the paper is divided into two sections. In Section 2, we introduce the problem of state observation with limited data rate, and the notions of topological entropy and LTV systems. We also discuss some previous results about the link between topological entropy and the minimal data rate for state observation. In Section 3, we present and prove the main result of the paper, namely the equivalence of topological entropy and minimal data rate for state observation of LTV systems. For the sake of simplicity and conciseness, the systems considered in this paper are in discrete time, but the main results and definitions extend naturally to the continuous-time case.

Notation. $\mathbb{N}$ is the set of nonnegative integers $\{0, 1, 2, \ldots\}$. d is a positive integer representing the dimension of the system. For vectors, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$, and for matrices, it denotes the associated matrix norm, $\|M\| = \sigma_{\max}(M)$. $\lceil\alpha\rceil$ is the smallest integer larger than or equal to $\alpha \in \mathbb{R}$ (aka. *ceil* of α). For $E, F \subseteq \mathbb{R}^d$, we let $E + F$ be their Minkowski sum. For the ease of notation, we also let $E + \xi = E + \{\xi\}$ ($\xi \in \mathbb{R}^d$). In this paper, we consider dynamical systems is discrete-time; therefore, if $[T_1, T_2]$ (resp. $[T_1, T_2)$) refers to an interval of *times* (in particular, $T_1, T_2 \in \mathbb{N}$), then it is understood to contain only the integers from T_1 to T_2 (resp. $T_2 - 1$) inclusive.

2. PRELIMINARIES

2.1 State observation with limited data rate

Consider a time-varying dynamical system

$$x(t+1) = f_t(x(t)), \quad x(0) \in K, \quad t \in \mathbb{N}, \tag{1}$$

where $K \subseteq \mathbb{R}^d$ is a nonempty compact set (called the set of initial states) and $f_t : \mathbb{R}^d \to \mathbb{R}^d$ is continuous for all $t \in \mathbb{N}$. We denote by $x(t, \xi)$ the solution, at time t, of (1) with initial state $\xi \in K$ (by extension, we also define $x(t, \xi)$ for every $\xi \in \mathbb{R}^d$). In formulas, it will be convenient to denote system (1) by the pair (f_t, K).

The aim of state observation is to produce an estimation of the state of the system when direct observation of the system is not possible. Information about the system will thus be delivered to the observer via a communication channel which can carry only a limited amount of information per unit of time. The observation procedure works as follows (see, e.g., Matveev and Pogromsky (2016)); see also Figure 1. At time t, a *coder* measures the state $x(t)$ of the system and is connected to a *decoder* via a discrete channel which carries one discrete-valued symbol $e(t)$ per unit of time, selected from a coding alphabet $\mathcal{E}_t$ of time-varying size. Each symbol takes one sample interval to be completely transmitted. Hence, at time $t + 1$, the decoder has the symbols $e(0), \ldots, e(t)$ available and generates an estimate $\hat{x}(t + 1)$ of the state $x(t + 1)$.

More precisely, the coder is a family of functions $\mathcal{C}_t$:

$$e(t) = \mathcal{C}_t(x(0), \ldots, x(t) \,|\, \hat{x}(0), \delta), \quad t \in \mathbb{N}, \tag{2}$$

where $\hat{x}(0)$ is an estimate of the initial state $x(0)$ satisfying $\|x(0) - \hat{x}(0)\| \leq \delta$. The output is $e(t) \in \mathcal{E}_t$ where $\mathcal{E}_t$ is a finite set with size depending on t. The symbol $e(t)$ will be transmitted to the decoder at most at $t + 1$. The decoder is a family of functions $\mathcal{D}_t$:

$$\hat{x}(t) = \mathcal{D}_t(e(0), \ldots, e(t-1) \,|\, \hat{x}(0), \delta), \quad t \in \mathbb{N}, \tag{3}$$

where $\hat{x}(t)$ is an estimate of $x(t)$ based on the past symbols and the initial estimate $\hat{x}(0)$.

Definition 1. The coder–decoder (2)–(3) is said to *observe* system (1) with accuracy $\varepsilon > 0$ if there is $\delta > 0$ such that for every trajectory $x(\cdot)$ of (1) and every $\hat{x}(0) \in K$ satisfying $\|x(0) - \hat{x}(0)\| \leq \delta$, it holds that

$$\|x(t) - \hat{x}(t)\| \leq \varepsilon, \quad \forall\, t \in \mathbb{N}.$$

The symbols $e(t)$ must be transmitted from the coder to the decoder via a limited channel. *For a given system, what is the minimal channel capacity needed by a coder–decoder to observe the system with accuracy $\varepsilon > 0$?* Several notions of "channel capacity" have been proposed in the literature, accounting for various non-idealities of communication channels (e.g., noise, delay, etc.). Borrowing the model [1] of Matveev and Savkin (2009), we let $b(r)$ be a lower bound on the number of bits of data that can

[1] This model is general enough to take into account non-idealities of the channel, like unsteady instant data rates, transmission delays and dropouts (Matveev and Savkin, 2009, §3.4).

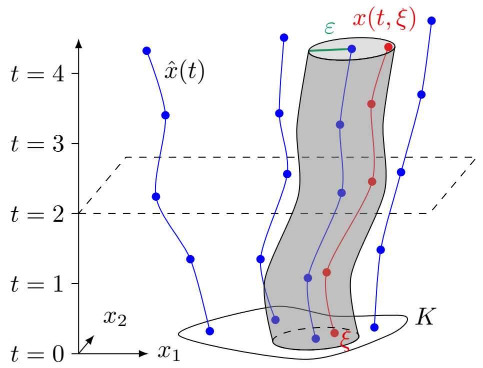

Fig. 2. The set of functions in blue is (ε, T)-spanning for (1) if every trajectory $x(\cdot)$ (e.g., the trajectory represented in red) is contained in the "ε-tube" around at least one of the functions in blue, for all $t \in [0, T]$.

be transmitted accurately via the channel during a time interval of duration $r \in \mathbb{N}_{>0}$. We assume that the average w.r.t. time of $b(r)$ converges to some value R called the *channel capacity*:

$$\lim_{r\to\infty} \frac{1}{r} b(r) = R. \tag{4}$$

Definition 2. System (1) is said to be *observable* over the communication channel $\mathfrak{C}$ if for every $\varepsilon > 0$, there is a coder–decoder (2)–(3) that observes (1) with accuracy $\varepsilon > 0$ and whose symbols can be transmitted through $\mathfrak{C}$.

We define the *minimal data rate* (or channel capacity) *for state observation* of system (1) as

$$\mathcal{R}_o(f_t, K) = \inf\{R : (f_t, K) \text{ is observable over any communication channel with capacity } R\}.$$

In the next section, we will see (Theorem 5) how $\mathcal{R}_o(f_t, K)$ can be related with the topological entropy of the system.

2.2 Topological entropy

We use the definition of topological entropy introduced by Bowen (1971) and Dinaburg (1970) for time-invariant systems on metric spaces, and extended to time-varying systems by Kolyada and Snoha (1996). The definition relies on the notion of minimal sets of functions necessary to approximate the state of the system with finite accuracy, on any interval $[0, T]$.

More precisely, for $\varepsilon > 0$ and $T \in \mathbb{N}$, a set $\mathcal{E}$ of functions from $[0, T]$ to $\mathbb{R}^d$ (i.e., $\mathcal{E} \subseteq (\mathbb{R}^d)^{[0,T]}$) is said to be (ε, T)-*spanning set* for (1) if for every trajectory $x(\cdot)$ of (1), there is $\hat{x}(\cdot) \in \mathcal{E}$ such that $\|x(t) - \hat{x}(t)\| \leq \varepsilon$ for all $t \in [0, T]$. See Figure 2 for an illustration. We let $s_{\text{span}}(\varepsilon, T; f_t, K)$ be the smallest cardinality of an (ε, T)-spanning set for (1).

Definition 3. The *topological entropy* of (1) is defined as

$$h(f_t, K) = \sup_{\varepsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log_2 s_{\text{span}}(\varepsilon, T; f_t, K). \tag{5}$$

Alternatively, topological entropy can be defined starting from maximal separated sets of trajectories: a set $F \subseteq K$ is (ε, T)-*separated* for (1) if for any two points $\xi, \eta \in F$, there is $t \in [0, T]$ such that $\|x(t, \xi) - x(t, \eta)\| > \varepsilon$. Letting $s_{\text{sep}}(\varepsilon, T; f_t, K)$ be the largest cardinality of an (ε, T)-separated set for (1), we obtain an equivalent definition of $h(f_t, K)$:

Proposition 4. The topological entropy of (1) satisfies

$$h(f_t, K) = \sup_{\varepsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log_2 s_{\text{sep}}(\varepsilon, T; f_t, K). \tag{6}$$

The proof is along the same lines as (Liberzon and Mitra, 2017, Theorem 1) and thus omitted here.

The following theorem, which can be found in (Matveev and Pogromsky, 2016, Theorem 8) [2], states that the topological entropy is a lower bound on the minimal data rate for state observation of dynamical systems. In some cases, it is also an upper bound:

Theorem 5. For any time-varying system (1), it holds that $h(f_t, K) \leq \mathcal{R}_o(f_t, K)$. Moreover, if (1) is time-invariant and K is positively invariant, then $h(f_t, K) = \mathcal{R}_o(f_t, K)$.

The equality $h(f_t, K) = \mathcal{R}_o(f_t, K)$ also holds for LTI systems with compact initial set K with nonempty interior, —*not necessarily positively invariant*; see, e.g., (Matveev and Savkin, 2009, §2.5–2.6). However, it seems an open question whether $h(f_t, K) = \mathcal{R}_o(f_t, K)$ in general for time-varying systems. In this paper, we show that $h(f_t, K) = \mathcal{R}_o(f_t, K)$ holds for *linear* time-varying systems.

2.3 Linear time-varying systems

Linear Time-Varying (LTV) systems are time-varying systems, like (1), where f_t is linear for each $t \in \mathbb{N}$. They are thus described by

$$x(t+1) = A_t x(t), \quad x(0) \in K, \quad t \in \mathbb{N}, \tag{7}$$

where $A_t \in \mathbb{R}^{d\times d}$ for all $t \in \mathbb{N}$. Following the notation of the previous subsections, in formulas, we will write (A_t, K) to denote system (7). For instance, the topological entropy of (7) is denoted by $h(A_t, K)$ and $s_{\text{span}}(\varepsilon, T; A_t, K)$ is the smallest cardinality of an (ε, T)-spanning set for (7).

Linearity of (7) implies that for every $\xi, \eta \in \mathbb{R}^d$, $k, \ell \in \mathbb{R}$, and $t \in \mathbb{N}$, $x(t, k\xi + \ell\eta) = kx(t, \xi) + \ell x(t, \eta)$. It follows from linearity that the topological entropy and the minimal data rate for state observation of LTV systems are *independent of the initial set* K, as long as it is compact with nonempty interior; see, e.g., (Yang et al., 2018, Proposition 2). For this reason, in the following, we will omit the initial set in the notation, and simply use $h(A_t)$ and $\mathcal{R}_o(A_t)$ to denote the respective quantities. Finally, let us mention that the estimation of topological entropy for LTV systems was addressed for instance in Yang et al. (2018).

In conclusion to this section, the example below illustrates the notions of spanning sets, separated sets, and topological entropy, with a simple LTV system.

Example 6. Consider the 1-dimensional LTV system (7) with $A_t = 1$ if t is even and $A_t = 2$ if t is odd. Trajectories of the system are thus described by $x(t, \xi) = 2^{t/2}\xi$ if t is even and $x(t, \xi) = 2^{(t-1)/2}\xi$ if t is odd. We will show that

[2] Theorem 8 in Matveev and Pogromsky (2016) is presented for time-invariant systems only. However, the time-varying case can be deduced, without loss of generality, by considering time as a state variable. (Note that the resulting system will not have a compact positively invariant initial set, since time goes to infinity.)

$h(A_t) = 1/2$. As explained above, the topological entropy of linear systems does not depend on the initial set; hence, we set $K = [0, 1]$.

For $\varepsilon > 0$ and $T \in \mathbb{N}$, let $n = \lceil 2^{-1}\varepsilon^{-1}2^{T/2} \rceil$ and define $E = \{i/n : 0 \le i \le n\}$. Let $\mathcal{E} \subseteq \mathbb{R}^{[0,T]}$ be the set of functions $\{x(\cdot, \eta) : \eta \in E\}$. We show that $\mathcal{E}$ is (ε, T)-spanning for (A_t, K). To do this, fix $\xi \in K$, and let $\eta \in E$ minimize the distance to ξ. Then, by definition of E, it holds that $|\xi - \eta| \le 1/(2n) \le \varepsilon 2^{-T/2}$. This implies that

$$|x(t,\xi) - x(t,\eta)| \le 2^{t/2}|\xi - \eta| \le \varepsilon, \quad \forall\, t \in [0,T].$$

Hence, E is (ε, T)-spanning for (A_t, K) and thus

$$s_{\mathrm{span}}(\varepsilon, T; A_t, K) \le |E| = n + 1 \le 2^{-1}\varepsilon^{-1}2^{T/2} + 2.$$

Injecting in (5), we get that $h(A_t)$ is upper-bounded by

$$\sup_{\varepsilon>0}\ \limsup_{T\to\infty} \frac{1}{T}\log_2\left(\varepsilon^{-1}2^{T/2-1} + 2\right) = \sup_{\varepsilon>0} \frac{1}{2} = \frac{1}{2}.$$

In order to show that $1/2$ is also a lower bound on $h(A_t)$, we rely on (ε, T)-separated sets. Let $m = \lceil \varepsilon^{-1}2^{T/2} \rceil - 1$. We may assume $m > 0$ since in (6) we take the limit when $\varepsilon \to 0$ and $T \to \infty$. Define $F = \{i/m : 0 \le i \le m\}$. Then, any two points $\xi, \eta \in F$ satisfy $|\xi - \eta| > \varepsilon 2^{-T/2}$. Hence, for $T \in \mathbb{N}$ even, $|x(T,\xi) - x(T,\eta)| > \varepsilon$, showing that F is (ε, T)-separated for (A_t, K). This implies that

$$s_{\mathrm{sep}}(\varepsilon, T; A_t, K) \ge |F| = m + 1 \ge \varepsilon^{-1}2^{T/2} - 1.$$

Injecting in (6), this finally gives that $h(A_t) \ge \log_2 2^{1/2} \ge 1/2$. Summarizing, we have shown that $h(A_t) = 1/2$. □

3. EQUIVALENCE OF TOPOLOGICAL ENTROPY AND MINIMAL DATA RATE FOR LTV SYSTEMS

The following theorem is the main result of the paper. It extends the second part of Theorem 5 to LTV systems.

Theorem 7. For any LTV system (7), it holds that $h(A_t) = \mathcal{R}_o(A_t)$.

The rest of this section is devoted to the proof of Theorem 7. We first introduce some notation: for a given LTV system (7) and $T \in \mathbb{N}$, we define the norm

$$\|\xi\|_{A_t,T} = \max_{0\le t\le T} \|x(t,\xi)\|, \quad \xi \in \mathbb{R}^d,$$

where $x(t,\xi)$ is the trajectory of (7) starting from ξ. From the linearity of $x(t,\xi)$ with respect to ξ, it is clear that $\|\cdot\|_{A_t,T}$ is a norm in $\mathbb{R}^d$. For $\xi_0 \in \mathbb{R}^d$ and $r \in \mathbb{R}_{\ge 0}$, we let

$$B_{A_t,T}(\xi_0, r) = \{\xi \in \mathbb{R}^d : \|\xi - \xi_0\|_{A_t,T} \le r\}.$$

Let $\varepsilon > 0$ and let $X \subseteq \mathbb{R}^d$. We will say that $E \subseteq \mathbb{R}^d$ is an $(\varepsilon, T; A_t)$-*cover* of X if

$$X \subseteq \textstyle\bigcup_{\xi\in E} B_{A_t,T}(\xi, \varepsilon).$$

Finally, an $(\varepsilon, T; A_t)$-cover E of X is said to be *minimal* if there is no $(\varepsilon, T; A_t)$-cover of X with cardinality strictly smaller than the cardinality of E. We let $s_{\mathrm{cov}}(\varepsilon, T; A_t, X)$ be the cardinality of a minimal $(\varepsilon, T; A_t)$-cover of X.

Basic idea. The intuition behind the proof of Theorem 7 is that for linear systems, any minimal $(\varepsilon, T; A_t)$-cover of X must be "uniformly distributed" over X. Based on this observation, we obtain that if E_1 is a minimal $(\varepsilon, T_1; A_t)$-cover of X, E_2 is an $(\varepsilon, T_2; A_t)$-cover of X, and $T_1 \le T_2$, then there is an $(\varepsilon, T_2; A_t)$-cover of $B_{A_t,T_1}(0, \varepsilon)$ with cardinality of the order of $|E_2|/|E_1|$ (this is where the "uniformly distributed" assumption is used). These claims are encapsulated in the following two lemmas, whose proofs can be found in Appendix A.

Lemma 8. Consider system (7), and let $\varepsilon > 0$ and $T \in \mathbb{N}$. Then, for every $\xi \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}_{\ge 0}$, there is an (ε, T, A_t)-cover E of $B_{A_t,T}(\xi, \alpha\varepsilon)$ with $|E| \le (2\alpha + 1)^d$.

Lemma 9. Consider system (7). Let $\varepsilon > 0$ and $T_1, T_2 \in \mathbb{N}$, $T_1 \le T_2$. Let E_1 be a minimal $(\varepsilon, T_1; A_t)$-cover of K and let E_2 be an $(\varepsilon, T_2; A_t)$-cover of $K + B_{A_t,T_1}(0, 2\varepsilon)$. Then, there exists an $(\varepsilon, T_2; A_t)$-cover E of $B_{A_t,T_1}(0, \varepsilon)$ with cardinality $|E| \le 11^d|E_2|/|E_1|$.

We are now able to prove Theorem 7.

Proof of Theorem 7. Since $h(A_t, K)$ and $\mathcal{R}_o(A_t, K)$ are independent from K as long as it is compact with nonempty interior (see Subsection 2.3), we fix K to be the closed unit Euclidean ball centered at 0 in $\mathbb{R}^d$. We will show that $h(A_t, K) = \mathcal{R}_o(A_t, K)$. From Theorem 5, we already know that $h(A_t, K) \le \mathcal{R}_o(A_t, K)$. Hence, it remains to show that $h(A_t, K)$ is an upper bound on $\mathcal{R}_o(A_t, K)$.

<u>Claim</u>: We claim that $s_{\mathrm{cov}}(\varepsilon, T; A_t, K) \le s_{\mathrm{sep}}(\varepsilon, T; A_t, K)$.

<u>Proof</u>: Let $F \subseteq K$ be an (ε, T)-separated set for (A_t, K) with maximal cardinality. Then, for every $\xi \in K$ there is $\eta \in F$ such that $\|x(t,\xi) - x(t,\eta)\| \le \varepsilon$ for all $t \in [0, T]$, as otherwise $F \cup \{\xi\}$ would be an (ε, T)-separated set for (A_t, K), contradicting the maximality of F. The above is equivalent to say that $\|\xi - \eta\|_{A_t,T} \le \varepsilon$ and thus it implies that F is an $(\varepsilon, T; A_t)$-cover of K, proving the claim. ⌟

From the above Claim and Proposition 4, we get that

$$\sup_{\varepsilon>0}\ \limsup_{T\to\infty} \frac{1}{T}\log_2 s_{\mathrm{cov}}(\varepsilon, T; A_t, K) \le h(A_t). \tag{8}$$

Now, fix a communication channel $\mathfrak{C}$ with capacity $R > h(A_t)$. We will show that (7) is observable over $\mathfrak{C}$. Therefore, we fix $\varepsilon \in (0, 1/2)$, and we will build a coder–decoder that observes (7) with accuracy ε and whose symbols can be transmitted through $\mathfrak{C}$.

<u>Preliminaries</u>: To do this, we fix $\alpha \in (h(A_t), R)$, and we let $r \in \mathbb{N}_{>0}$ be large enough such that (i) $b(r) \ge \alpha r + 9d$ where $b(r)$ is as in (4); and (ii) $s_{\mathrm{cov}}(\varepsilon, T; A_t, K) \le 2^{\alpha T}$ for all $T \in [r, \infty)$. Clauses (i) and (ii) are satisfiable because of (4), (8), and the definition of α. Then, in our construction of the coder–decoder, it is important that the norms of the matrices involved in the LTV system are uniformly bounded. Since this is not necessarily the case, we decompose the matrix sequence A_t as follows. For each $t \in \mathbb{N}$, let $n_t \in \mathbb{N}_{>0}$ and $B_{t,0}, \ldots, B_{t,n_t-1} \in \mathbb{R}^{d\times d}$ be a factorization of A_t, i.e., $A_t = B_{t,n_t-1}B_{t,n_t-2}\cdots B_{t,0}$, such that for every $i \in \{0, \ldots, n_t - 1\}$, $\|B_{t,i}\| \le 2$ and

$$\|B_{t,i-1}\cdots B_{t,0}\xi\| \le \max\{\|\xi\|, \|A_t\xi\|\}, \quad \forall\, \xi \in \mathbb{R}^d. \tag{9}$$

See Lemma 11 (in Appendix A) for the construction of such a factorization. Then, let $\{\Phi_s\}_{s\in\mathbb{N}}$ (we use the index s instead of t to distinguish Φ_s from A_t) be the matrix sequence obtained by concatenating the sequences of matrices $B_{t,0}, \ldots, B_{t,n_t-1}$. In other words, $\Phi_s = B_{\tau(s),s-\tau(s)}$ where $\tau(s) \in \mathbb{N}$ is defined by $\sum_{t=0}^{\tau(s)} n_t \le s < \sum_{t=0}^{\tau(s)+1} n_t$.

Using these preliminaries, we build a coder–decoder that observes (7) with accuracy ε and communicates by using packets of r symbols. Thus, at time $t = (j+1)r$, the decoder has received the last r symbols $e(jr), \ldots, e((j+1)r-1)$, which by assumption on $\mathfrak{C}$ contain together at least $b(r)$ bits of information, and the decoder uses this information to estimate the state $x(t)$ of the system during the ongoing "epoch" $[(j+1)r, (j+2)r)$.

The coder–decoder is built as follows. For every $j \in \mathbb{N}$, let $s_j \in \mathbb{N}$ be the largest integer such that $s_{\mathrm{cov}}(\varepsilon, s_j; \varPhi_s, K) \leq 2^{\alpha(j+1)r}$. Then, by (9) and the definition of r, we have that $\tau(s_j) \geq (j+1)r$. Moreover, by Lemma 12 (see Appendix A), it holds that $s_{\mathrm{cov}}(\varepsilon, s_j; \varPhi_s, K) \geq 5^{-d}2^{\alpha(j+1)r}$.

We proceed by induction on j (the index of the epoch) to build the coder–decoder. For the base case ($j = 0$), it is not difficult to see that since there is a finite $(\varepsilon, s_0; \varPhi_s)$-cover of K, there is $\delta > 0$ such that $\|x(0) - \hat{x}(0)\| \leq \delta$ implies that $x(0) \in B_{\varPhi_s, s_0}(\hat{x}(0), \varepsilon)$. Denote $\xi_0 = \hat{x}(0)$. Now, for the induction step, fix $j \in \mathbb{N}$ and assume that $x(0) \in B_{\varPhi_s, s_j}(\xi_j, \varepsilon)$. Let E_j be a minimal $(\varepsilon, s_{j+1}; \varPhi_s)$-cover of $B_{\varPhi_s, s_j}(\xi_j, \varepsilon)$.

Claim: We claim that $|E_j| \leq 275^d 2^{\alpha r}$.

Proof: Since $\varepsilon < 1/2$, it holds that $K + B_{\varPhi_s, s_j}(0, 2\varepsilon) \subseteq 2K$. By Lemma 8 (use $\varepsilon = 1$, $T = 0$ and $\alpha = 2$), it is possible to cover $2K$ with 5^d translated copies of K. Thus, we find that $s_{\mathrm{cov}}(\varepsilon, s_{j+1}; \varPhi_s, 2K) \leq 5^d s_{\mathrm{cov}}(\varepsilon, s_{j+1}; \varPhi_s, K)$. By Lemma 9, we have thus

$$|E_j| \leq 55^d s_{\mathrm{cov}}(\varepsilon, s_{j+1}; \varPhi_s, K) / s_{\mathrm{cov}}(\varepsilon, s_j; \varPhi_s, K).$$

Now, by the definition of s_j and s_{j+1}, we get

$$|E_j| \leq (55^d 2^{\alpha(j+2)r}) / (5^{-d} 2^{\alpha(j+1)r}) = 275^d 2^{\alpha r}. \quad \lrcorner$$

Because, $b(r) \geq \alpha r + 9d$, we may transmit via $\mathfrak{C}$ $b(r)$ bits of information during the interval $[jr, (j+1)r)$. This is sufficient to give a unique binary code to each of the $275^d 2^{\alpha r}$ points of E_j (see the Claim), since $\log_2(275^d 2^{\alpha r}) \leq \alpha r + 9d$.

Summarizing, the coder–decoder is defined as follows.

– *Coder.* At time $t = jr$, the coder can compute E_j since it knows the system and ξ_j (as either $\xi_0 = \hat{x}(0)$ or ξ_j was computed at the previous epoch). Hence, it can compute $\xi_{j+1} \in E_j$ such that $x(0) \in B_{\varPhi_s, s_{j+1}}(\xi_{j+1}, \varepsilon)$. Then, during the coming epoch $[jr, (j+1)r)$, the coders $\mathcal{C}_t$, $t \in [jr, (j+1)r)$, will send the symbols $e(jr), \ldots, e((j+1)r-1)$, such that these symbols define together uniquely the point ξ_{j+1} (assuming ξ_j is known). By the above, there is a set of symbols satisfying this property and that are transmissible via the communication network $\mathfrak{C}$.

– *Decoder.* At time $t = (j+1)r$, the decoder can compute E_j since it knows the system and also knows ξ_j (as either $\xi_0 = \hat{x}(0)$ or ξ_j was computed at the previous epoch). It has also received all the symbols $e(jr), \ldots, e((j+1)r-1)$ and thus it can compute the point ξ_{j+1}. Hence, during the coming epoch $[(j+1)r, (j+2)r)$, the decoders $\mathcal{D}_t$, $t \in [(j+1)r, (j+2)r)$, compute the estimates $\hat{x}(t) = x(t, \xi_{j+1})$. By definition of s_{j+1}, we have that $x(0) \in B_{A_t, \tau(s_{j+1})}(\xi_{j+1}, \varepsilon)$. Thus, since $\tau(s_{j+1}) \geq (j+2)r$, it holds that for all $t \in [(j+1)r, (j+2)r)$, $\|x(t) - \hat{x}(t)\| \leq \varepsilon$. This concludes the proof of the theorem. $\square$

4. CONCLUSIONS AND FURTHER WORKS

In this paper, we have shown that the "observation data rate" theorem, which is a well-known result for LTI systems and time-invariant nonlinear systems with invariant compact initial set, extends to LTV systems. This theorem is a cornerstone of networked control systems theory. It implies that the topological entropy can be used as a benchmark for the minimal channel capacity necessary for the state observation of the system, as it provides a tight bound on the optimal channel capacity.

For future work, we want to address the question of practical implementability of the coder–decoder. Indeed, in the proof of Theorem 7, the construction of the coder–decoder relies on the existence of minimal sets of functions necessary to approximate the state of the system, disregarding the computability of these functions. In fact, the same limitation holds for the proof of the "observation data rate" theorem for time-invariant systems with invariant compact initial set; see, e.g., Matveev and Pogromsky (2016). However, for our specific case, we plan to draw on the linearity of the system to show that "almost-minimal" $(\varepsilon, T; A_t)$-spanning sets can be computed numerically, thriving on the notion of $(\varepsilon, T; A_t)$-covers (defined in Section 3) which can be approximated with ellipsoids. Based on this idea, we plan to provide practical coders–decoders for the state observation of LTV systems, operating at data rates as close as desired to topological entropy of the system; thereby showing that topological entropy is not only a theoretical but also a practical upper bound on the minimal data for state observation of LTV systems.

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Appendix A

A.1 Proof of Lemma 8

The proof of the lemma follows from the fact that every maximal (ε, T)-separated for (A_t, K) is an $(\varepsilon, T; A_t)$-cover of K (see the first Claim in the proof of Theorem 7).

Let F be an (ε, T)-separated subset for (A_t, K). We will give an upper bound on the cardinality of F. Being (ε, T)-separated for (A_t, K) is equivalent to the property that for any two points $\eta, \zeta \in F$, $\|\eta - \zeta\|_{A_t,T} > \varepsilon$. This implies that the balls $B_{A_t,T}(\eta, \varepsilon/2)$, $\eta \in E$, are pairwise disjoint. Moreover, because $F \subseteq K$, these balls are contained in $K' = K + B_{A_t,T}(0, \varepsilon/2)$.

Consequently, the volume of the union of these balls is equal to $|F| \,\mathrm{vol}(B_{A_t,T}(0, \varepsilon/2))$ and cannot exceed $\mathrm{vol}(K')$. Hence, for $K = B_{A_t,T}(\xi, \alpha\varepsilon)$, this gives

$$|F| \leq \frac{\mathrm{vol}(B_{A_t,T}(\xi, (\alpha + 1/2)\varepsilon))}{\mathrm{vol}(B_{A_t,T}(0, \varepsilon/2))} = (2\alpha + 1)^d.$$

The latter comes from $\mathrm{vol}(B_{A_t,T}(\xi, r)) = r^d \mathrm{vol}(B_{A_t,T}(0, 1))$, which follows from $B_{A_t,T}(\xi, r) = \xi + rB_{A_t,T}(0, 1)$. This concludes the proof of the lemma.

A.2 Proof of Lemma 9

We will need the following result:

Lemma 10. Consider system (7) and let $\varepsilon > 0$ and $T \in \mathbb{N}$. Let E be a minimal $(\varepsilon, T; A_t)$-cover of K. Then, there exists a subset $F \subseteq E$ such that $|F| \geq 11^{-d}|E|$ and for every two points $\xi, \eta \in F$, we have that $\|\xi - \eta\|_{A_t,T} > 4\varepsilon$ (or said otherwise, F is $(4\varepsilon, T)$-separated for (A_t, E)).

Proof. Fix $\xi \in E$. We claim that $|B_{A_t,T}(\xi, 4\varepsilon) \cap E| \leq 11^d$. Assume the contrary, and let $E_\xi = B_{A_t,T}(\xi, 4\varepsilon) \cap E$. Then, E_ξ covers at most the region $B_{A_t,T}(\xi, 5\varepsilon)$, that is,

$$\bigcup_{\eta \in E_\xi} B_{A_t,T}(\eta, \varepsilon) \subseteq B_{A_t,T}(\xi, 5\varepsilon).$$

On the other hand, we have shown in Lemma 8 that there exists an $(\varepsilon, T; A_t)$-cover E' of $B_{A_t,T}(\xi, 5\varepsilon)$ with $|E'| \leq 11^d$. This implies that $E^* = (E \setminus E_\xi) \cup E'$ is an $(\varepsilon, T; A_t)$-cover of K with $|E^*| < |E|$, a contradiction with the minimality of E. This proves the claim.

Using the above, we build the set F inductively as follows. Let $G_0 = E$ and $F_0 = \varnothing$. Then, for $i = 0, 1, 2, \ldots$ and while $G_i \neq \varnothing$, pick $\xi_i \in G_i$ and let

$$G_{i+1} = G_i \setminus E_{\xi_i}, \quad \text{and} \quad F_{i+1} = F_i \cup \{\xi_i\}.$$

Let k be the first i such that $G_i = \varnothing$ and define $F = F_k$. Because $|E_{\xi_i}| \leq 11^d$ for each $0 \leq i \leq k - 1$, we have that $k \geq 11^{-d}|E|$. This concludes the proof of the lemma. □

We proceed with the proof of Lemma 9. Note that since E_1 is a minimal $(\varepsilon, T_1; A_t)$-cover of K, it holds that $B_{A_t,T_1}(\xi, \varepsilon) \cap K \neq \varnothing$ for all $\xi \in E_1$. Hence, $B_{A_t,T_1}(\xi, \varepsilon) \subseteq K + B_{A_t,T_1}(0, 2\varepsilon)$ for all $\xi \in E_1$.

By Lemma 10, we know that there is $F_1 \subseteq E_1$ such that $|F_1| \geq 11^{-d}|E_1|$ and for every two points $\xi, \eta \in F_1$, $\|\xi - \eta\|_{A_t,T_1} > 4\varepsilon$. For each $\xi \in F_1$, let $n^*(\xi)$ be the smallest cardinality of a subset $E_\xi \subseteq E_2$ such that E_ξ is an $(\varepsilon, T_2; A_t)$-cover of $B_{A_t,T_1}(\xi, \varepsilon)$. Such E_ξ alway exists since E_2 is an $(\varepsilon, T_2; A_t)$-cover of $K + B_{A_t,T_1}(0, 2\varepsilon)$.

Because $\|\cdot\|_{A_t,T_2} \geq \|\cdot\|_{A_t,T_1}$, a ball $B_{A_t,T_2}(\eta, \varepsilon)$ cannot intersect simultaneously $B_{A_t,T_1}(\zeta, \varepsilon)$ and $B_{A_t,T_1}(\theta, \varepsilon)$ if $\zeta, \theta \in F_1$ and $\zeta \neq \theta$. Thus, the subsets E_ξ, $\xi \in F_1$, are pairwise disjoint. This implies that $\sum_{\xi \in F_1} n^*(\xi) \leq |E_2|$, which in turn implies that

$$\min_{\xi \in F_1} n^*(\xi) \leq |E_2|/|F_1| \leq 11^d |E_2|/|E_1|.$$

Now, it is not difficult to see that if E_ξ is an $(\varepsilon, T_2; A_t)$-cover of $B_{A_t,T_1}(\xi, \varepsilon)$, then $E_\xi - \xi$ is an $(\varepsilon, T_2; A_t)$-cover of $B_{A_t,T_1}(0, \varepsilon)$. This concludes the proof of the lemma.

A.3 Two useful lemmas

Lemma 11. Every matrix $A \in \mathbb{R}^{d \times d}$ can be factorized as $A = B_n B_{n-1} \cdots B_1$ where $B_i \in \mathbb{R}^{d \times d}$, $\|B_i\| \leq 2$, and

$$\|B_{i-1} B_{i-2} \cdots B_1 \xi\| \leq \max\{\|\xi\|, \|A\xi\|\}, \tag{A.1}$$

for all $i \in \{1, \ldots, n\}$ and $\xi \in \mathbb{R}^d$.

Proof. A possible factorization is the following. Let $U\Sigma V$ be the Singular Value Decomposition of A and let σ be its maximal singular value. Let $n \in \mathbb{N}_{>0}$ be such that $\sigma^{1/n} \leq 2$. Let $\Pi = \Sigma^{1/n}$ (just take the nth root of the diagonal elements which are nonnegative). Then, define $B_1 = \Pi V$, $B_i = \Pi$ for $i \in \{2, \ldots, n-1\}$ and $B_n = U\Pi$. Clearly, $\|B_i\| \leq 2$ and $B_n \cdots B_1 = A$. Finally, (A.1) follows from the fact that $\max_{0 \leq j \leq n} \|\Pi^j \xi\|^2 = \max_{0 \leq j \leq n} \sum_{k=1}^d \xi_k^2 \Pi_{kk}^{2j}$ is reached either at $j = 0$ or $j = n - 1$ (because of the convexity of exponential functions). □

Lemma 12. Consider system (7) and let $\varepsilon > 0$ and $T \in \mathbb{N}$. Let $\alpha \geq 1$ such that $\|A_T\| \leq \alpha$. It holds that

$$s_{\mathrm{cov}}(\varepsilon, T + 1; A_t, K) \leq (2\alpha + 1)^d s_{\mathrm{cov}}(\varepsilon, T; A_t, K).$$

Proof. Since $\alpha \geq 1$ and $\|A_T\| \leq \alpha$, we have that for every $\xi \in \mathbb{R}^d$, $\|\xi\|_{A_t,T+1} \leq \alpha\|\xi\|_{A_t,T}$, and thus $B_{A_t,T}(0, \varepsilon) \subseteq B_{A_t,T+1}(0, \alpha\varepsilon)$. Thus, by Lemma 8, there is an $(\varepsilon, T + 1; A_t)$-cover E' of $B_{A_t,T}(0, \varepsilon)$ with $|E'| \leq (2\alpha+1)^d$. Hence, if E is an $(\varepsilon, T; A_t)$-cover of K, we may define

$$E^* = E + E' = \bigcup_{\xi \in E} (E' + \xi).$$

Clearly, E^* is an $(\varepsilon, T+1; A_t)$-cover of K and its cardinality satisfies $|E'| \leq (2\alpha + 1)^d |E|$. Since E is arbitrary, this concludes the proof of the lemma. □