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Robust Point-to-Set Control of Hybrid Systems with Uncertainties Using Constraint Tightening

Zonglin Liu * **Olaf Stursberg** *

* *Control and System Theory, Dept. of Electrical Engineering and Computer Science, University of Kassel, Germany. Email: {Z.Liu,stursberg}@uni-kassel.de.*

Abstract: This paper proposes a solution to the problem of computing optimized point-to-set control strategies for hybrid systems with mixed inputs and uncertainties in the continuous-valued dynamics as well as the reset functions. The solution is based on the idea to account for the uncertainties by tightening the guard and invariant sets, and thus construct a substitute hybrid automaton with deterministic transition dynamics. It can be ensured that a solution for this new model also solves the problem for the original system, i.e. the latter is robustly transferred between initial point and target set.

Keywords: Hybrid systems, uncertain systems, constrained control, robust control, optimization.

1. INTRODUCTION

The optimal control of hybrid systems with transitions conditioned upon both guard sets and external triggers has attracted interest recently (Liu and Stursberg (2018, 2019); Adamek et al. (2008)). In contrast to the relative popular class of piecewise affine (PWA) systems, where the change of the discrete state is only an immediate consequence of the continuous state crossing hyperplanes which partition the state space, the class of hybrid systems considered here is more general: As in the definition of hybrid automata in Henzinger (1996); Lynch et al. (2003), for this type of system the change of the discrete state is enabled by the fact that the continuous state lies inside of a *guard set*. Whether and when the transition indeed takes place is decided by a discrete input. In addition, the transition may cause an instantaneous change of the continuous state through *reset functions*. The optimal control problem of such hybrid systems requires not only to determine the optimal continuous controls, but also to determine the optimal times to trigger enabled transitions by discrete inputs. The work in Liu and Stursberg (2019) has proposed a method to cast the transition logic into a set of linear constraints among binary and real variables, and thus to solve the optimal control problem quite efficiently (by solvers like CPLEX (2009)).

The important extension considered in this paper is that the continuous as well as discrete dynamics (more precisely the reset functions) are subject to uncertainties. So far robust control of hybrid systems with focus on optimization-based approaches has only considered simpler system classes: In Kerrigan and Mayne (2002), PWA systems with additive disturbances were under study, and robust controllers were obtained by backward computation of robust controllable sets. The work in Lin and Antsaklis (2003) extended the idea of backward computation for the case of parametric model uncertainties. For the same class of PWA systems, the authors in Necoara et al. (2004) and Silva et al. (2003) aimed at setting up min-max problems, or to employ multi-parametric linear programming (see Borrelli et al. (2003)), or to use a branch-and-bound strategy tailored to worst-case uncertainty and thus ensure robustness. Specific to the considered class of hybrid systems with transitions conditioned by guard sets, the work in Moor and Davoren (2001) proposed a robust switching law for the case when no continuous input is applied. The work in Adamek et al. (2008) focused on another class of uncertainty, where the discrete transition structure is probabilistic.

In this work, however, both parametric uncertainties and additive disturbances affecting the the continuous dynamics are considered, as well as uncertain reset functions assigned to transitions. For the task of computing robust control trajectories, we employ and extend ideas that were used for simpler systems in Langson et al. (2004); Mayne et al. (2005); Ghasemi and Afzalian (2017), namely to construct reachable tubes around nominal trajectories. Through tightening of the invariants and guard sets of the hybrid systems by use of such tubes, and by optimizing the nominal trajectories for a modified hybrid automaton, we show that optimal point-to set control tasks can be solved reliably despite the presence of the uncertainties.

This paper is organized as follows: in Sec. 2, the class of uncertain hybrid system and the robust controller synthesis problem are defined. Sec. 3 explains the procedure of set tightening to obtain robustness by consideration of a single transitions. Then, based on the findings of Sec. 3, a deterministic hybrid system is constructed in Sec. 4, and we prove that the robust controller synthesis problem of the uncertain hybrid system can be reduced to an optimal control problem of a deterministic substitute model. A

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numerical example is presented in Sec. 5, followed by conclusions.

2. PROBLEM FORMULATION

This section describes the type of system and the control task under consideration. First, let $\mathcal{P}$ denote the set of all polytopes P in $\mathbb{R}^{n_x}$ with $P = \{x \mid C \in \mathbb{R}^{n_p \times n_x}, d \in \mathbb{R}^{n_p \times 1} : C \cdot x \leq d\}$.

Now consider a class of uncertain hybrid systems formulated in discrete time and with mixed inputs according to $HA = (T, U, X, Z, I, \mathcal{T}, G, V, r, f)$ containing:

- the discrete time domain $T = \{t_k \mid k \in \mathbb{N} \cup \{0\}, \Delta \in \mathbb{R}^{>0} : t_k := k \cdot \Delta\}$;
- the continuous inputs $u(t_k) \in U \subseteq \mathbb{R}^{n_u}$,
- the continuous states $x(t_k) \in X \subseteq \mathbb{R}^{n_x}$;
- the finite set of discrete states $Z = \{z_{(1)}, \cdots, z_{(n_z)}\}$;
- a set $I = \{I_{(1)}, \ldots, I_{(n_z)}\}$ of invariants $I_{(i)} \in \mathcal{P}$;
- the finite set of transitions $\mathcal{T} \subseteq Z \times Z$, where the transition from $z_{(i)}$ to $z_{(j)}$ is denoted by $\tau_{(i,j)} \in \mathcal{T}$;
- the set G of guard sets, of which any element $G_{(i,j)} \in \mathcal{P}$ is assigned to $\tau_{(i,j)} \in \mathcal{T}$, and we require that $G_{(i,j)} \subseteq I_{(i)}$ and that for no pairs of transitions from $z_{(i)}$ the guard sets intersect;
- the finite set V of discrete inputs, where any element $v_{(i,j)} \in \{0, 1\}$ in V refers to one transition $\tau_{(i,j)} \in \mathcal{T}$; we use a vector $v_k := v(t_k)$ to denote the choice of discrete inputs at time t_k;
- an uncertain reset function $r : \mathcal{T} \times X \to X$ to update the continuous states $\tilde{x}$ upon a transition $\tau_{(i,j)} \in \mathcal{T}$ according to the following scheme:
$$x = F_{(i,j)} \cdot \tilde{x} + e_{(i,j)}. \tag{1}$$
with a matrix $F_{(i,j)}$ and a vector $e_{(i,j)}$, both being uncertain: $F_{(i,j)}$ is randomly selected from a given polytope $\mathcal{F}_{(i,j)} \in \mathcal{P}$ defined by:
$$\mathcal{F}_{(i,j)} = \left\{ F_{(i,j)} = \sum_{l=1}^{\rho_{(i,j)}} \gamma_l \cdot \mathcal{F}_{(i,j)}^{(l)}, \sum_{l=1}^{\rho_{(i,j)}} \gamma_l^{\geq 0} = 1 \right\}. \tag{2}$$
with $\mathcal{F}_{(i,j)}^{(l)}$ denoting the vertices of $\mathcal{F}_{(i,j)}$, and $\rho_{(i,j)}$ is the number of these vertices, and $\gamma_l \in [0, 1]$. The vector $e_{(i,j)}$ is randomly taken from a given polytope $\mathcal{E}_{(i,j)} \in \mathcal{P}$;
- an uncertain flow function $f : X \times U \times Z \to X$ defining the discrete-time continuous-valued dynamics to:
$$x_{k+1} = A_{(i)} \cdot x_k + B_{(i)} \cdot u_k + w_{(i),k} \tag{3}$$
with $x_{k+1} := x(t_{k+1})$ and $z_{(i)} \in Z$. The matrices $A_{(i)}$ and $B_{(i)}$ are uncertain, but are randomly selected from given polytopic sets $\mathcal{A}_{(i)}$ and $\mathcal{B}_{(i)}$:
$$\mathcal{A}_{(i)} = \left\{ A_{(i)} = \sum_{l=1}^{\rho_{(i,A)}} \gamma_l \cdot \mathcal{A}_{(i)}^{(l)}, \sum_{l=1}^{\rho_{(i,A)}} \gamma_l^{\geq 0} = 1 \right\},$$
$$\mathcal{B}_{(i)} = \left\{ B_{(i)} = \sum_{l=1}^{\rho_{(i,B)}} \gamma_l \cdot \mathcal{B}_{(i)}^{(l)}, \sum_{l=1}^{\rho_{(i,B)}} \gamma_l^{\geq 0} = 1 \right\}, \tag{4}$$
with coefficients $\gamma_l \in [0, 1]$. The additive disturbances $w_{(i),k}$ can assume arbitrary values from a set $\mathcal{W}_{(i)} \in \mathcal{P}$ containing the origin.

The set of admissible executions of the model HA considering the uncertain components of the continuous dynamics and reset functions is defined as follows:

Definition 1. (**Admissible execution of the uncertain model** HA) Let a finite time domain $T_N = \{0, 1, \ldots, N\} \subset T$ and an initial hybrid state (x_0, z_0) with $z_0 := z_{(s)} \in Z$, $x_0 \in I_{(s)}$, and $x_0 \notin G_{(s,j)}$ for all $\tau_{(s,j)} \in \mathcal{T}$ be given. Then, for given input sequences $\phi_u = \{u_0, u_1, \cdots, u_{N-1}\}$ and $\phi_v = \{v_0, v_1, \cdots, v_{N-1}\}$, an admissible execution is a pair of state sequences $\phi_x = \{x_0, x_1, \cdots, x_N\}$ and $\phi_z = \{z_0, z_1, \cdots, z_N\}$ complying to the following rules: for $k \in \{0, \ldots, N-1\}$, $x_k \in \phi_x$, and $z_k = z_{(i)} \in \phi_z$, the successor states $x_{k+1} \in \phi_x$ and $z_{k+1} \in \phi_z$ must satisfy:

- if $v_k = 0$, then there must exist $A_{(i)} \in \mathcal{A}_{(i)}$, $B_{(i)} \in \mathcal{B}_{(i)}$, and $w_{(i),k} \in \mathcal{W}_{(i)}$ to obtain $x_{k+1} \in I_{(i)}$ according to (3) and $z_{k+1} = z_{(i)}$;
- if $v_k = v_{(i,j)} = 1$ and if $A_{(i)} \in \mathcal{A}_{(i)}$, $B_{(i)} \in \mathcal{B}_{(i)}$, and $w_{(i),k} \in \mathcal{W}_{(i)}$ exist to obtain an intermediate state $\tilde{x}_{k+1} \in G_{(i,j)}$ according to (3), then there must exist $F_{(i,j)} \in \mathcal{F}_{(i,j)}$ and $e_{(i,j)} \in \mathcal{E}_{(i,j)}$ to have $x_{k+1} = F_{i,j} \cdot \tilde{x}_{k+1} + e_{i,j} \in I_{(j)}$ according to (1), and $z_{k+1} = z_{(j)}$.

Obviously, for given inputs at time k, any value of the uncertain components $A_{(i)}$, $B_{(i)}$, $w_{(i)}$, $F_{(i,j)}$, and $e_{(i,j)}$ may contribute to and determine the hybrid successor state, as long as the necessary containment in the invariants and guard sets are observed – thus, any controller designed for a model of type HA must consider the complete set of possible executions according to Def. 1.

Turning to the control task to be addressed within this paper, now consider the optimal transfer of HA from an initial hybrid state (x_0, z_0) into a set of goal states while taking the uncertainties into account. For the hybrid goal states, assume the pair (X_g, z_g) with $z_g \in Z$ and $X_g \in \mathcal{P}$ as well as $X_g \subseteq I_g$. Furthermore, let a continuous state $x_g \in X$ denote the volumetric center of X_g.

Furthermore, let a cost functional $\mathcal{J}(x_0, x_g, N)$ be specified to quantify the performance of transferring the system into the goal within a finite time domain of T_N.

Definition 2. (**Point-to-set control task for** HA) For an initial hybrid state (x_0, z_0) of HA, a time domain T_N, as well as a set of goal states (X_g, z_g), find the pair of sequences of continuous inputs $\phi_u = \{u_0, u_1, \cdots, u_{N-1}\}$ and discrete inputs $\phi_v = \{v_0, v_1, \cdots, v_{N-1}\}$ such that

- the resulting pair of state sequences $\phi_x = \{x_0, x_1, \cdots, x_N\}$ and $\phi_z = \{z_0, z_1, \cdots, z_N\}$ satisfy Def. 1,
- the terminal states satisfy $x_N \in X_g$ and $z_N = z_g$,
- and the cost functional $\mathcal{J}(x_0, x_g, N)$ is minimized for a nominal evolution.

Note that an equivalent problem for the case without uncertainties was already addressed in Liu and Stursberg (2018, 2019)), proposing a particular structure to take care of the theoretically exponential increase of the number of possible ϕ_z and ϕ_v over N. For the variant of hybrid systems with uncertainties considered in this paper, the additional challenge is to guarantee that the selected pair of ϕ_u and ϕ_v realizes the path into (X_g, z_g) for all possible realizations of the uncertainties. To succeed in this task, we employ principles inspired the so-called *tube-based predictive control* for continuous-valued dynamics, see Langson et al. (2004); Mayne et al. (2005). Along this line, we show that, based on tubes of reachable sets, the control problem in Def. 2 can be cast into a problem for a modified model $\overline{HA}$ with deterministic sequence of discrete states, to which the method in Liu and Stursberg (2019) can be applied.

3. REACHABILITY TUBES TO HANDLE UNCERTAIN TRANSITIONS

In order to explain the principle of using reachability tubes to robustly control HA, this section focuses first on a single transition as part of the sequence ϕ_z which solves the problem of Def. 2. More precisely, we consider a transition $\tau_{(i,j)} \in \mathcal{T}$, and show how to ensure the semantics in Def. 1 in terms of: 1.) the continuous state evolving inside of $z_{(i)}$ (called the *pre-transition phase*), 2.) transitioning from $z_{(i)}$ to $z_{(j)}$, and 3.) further evolving inside of $z_{(j)}$ (the *post-transition phase*). If this procedure is later applied to all transitions in $\mathcal{T}$, this will lead to a substitute hybrid system $\overline{HA}$ with deterministic sequence ϕ_z.

3.1 Pre-Transition Phase

First, let the polytopic sets leading to the uncertainty in both, reset function and flow function, be decomposed into two parts, namely a nominal part (indicated by $\bar{\bullet}$), and a disturbance set containing the origin:

$$\mathcal{F}_{(i,j)} = \bar{F}_{(i,j)} \oplus \mathbb{F}_{(i,j)},\ \mathcal{E}_{(i,j)} = \bar{e}_{(i,j)} \oplus \mathbb{E}_{(i,j)};$$
$$\mathcal{A}_{(i)} = \bar{A}_{(i)} \oplus \mathbb{A}_{(i)},\ \mathcal{B}_{(i)} = \bar{B}_{(i)} \oplus \mathbb{B}_{(i)}. \quad (5)$$

Here, the symbol $\oplus$ denotes the Minkowski addition. Note that the set $\mathcal{W}_{(i)}$ contains the origin by definition and needs not to be decomposed. Next, a nominal flow function as well as a nominal reset function is defined (excluding all uncertainties in f and r) to obtain the nominal continuous state and input:

$$\bar{x}_{k+1} := \bar{A}_{(i)} \cdot \bar{x}_k + \bar{B}_{(i)} \cdot \bar{u}_k; \quad (6)$$
$$\bar{x} = \bar{F}_{(i,j)} \cdot \bar{\bar{x}} + \bar{e}_{(i,j)}. \quad (7)$$

Now assume for step k that the state x_k from (3) and the nominal state $\bar{x}_k$ from (6) are located inside of invariant $I_{(i)}$. Then, by applying a continuous input u_k and a nominal $\bar{u}_k$ in (3) and (6) respectively, the difference between x_{k+1} and $\bar{x}_{k+1}$ can be determined according to:

$$x_{k+1} - \bar{x}_{k+1} = \bar{A}_{(i)}(x_k - \bar{x}_k) + \bar{B}_{(i)}(u_k - \bar{u}_k) + w_{(i),k} + \Delta_{(A,i),k} \cdot x_k + \Delta_{(B,i),k} \cdot u_k \quad (8)$$

with $\Delta_{(A,i),k} \in \mathbb{A}_{(i)}$ and $\Delta_{(B,i),k} \in \mathbb{B}_{(i)}$.

Suppose further that a closed-loop controller $K_i \in \mathbb{R}^{n_u \times n_x}$ is defined such that $\bar{A}_{K,(i)} := \bar{A}_{(i)} + \bar{B}_{(i)} \cdot K_i$ is stable. If x_k is measurable and u_k chosen to:

$$u_k = \bar{u}_k + K_i \cdot (x_k - \bar{x}_k), \quad (9)$$

then the difference between x_{k+1} and $\bar{x}_{k+1}$ in (8) can be written as:

$$x_{k+1} - \bar{x}_{k+1} = \bar{A}_{K,(i)}(x_k - \bar{x}_k) + w_{(i),k} + \Delta_{(A,i),k} \cdot x_k + \Delta_{(B,i),k} \cdot u_k. \quad (10)$$

Furthermore, as $x_k \in I_{(i)}$ and $\Delta_{(A,i),k} \in \mathbb{A}_{(i)}$, and both $I_{(i)}$ and $\mathbb{A}_{(i)}$ are polytopic, the following applies according to Bünger (2014):

$$\Delta_{(A,i),k} \cdot x_k \in Conv\left(\left\{\mathbb{A}^{(l)}_{(i)} \cdot I^{(q)}_{(i)} \mid l \in \{1, \dots, \rho_{(\mathbb{A}_{(i)})}\}\ ,\ q \in \{1, \dots, \rho_{(I_{(i)})}\}\right\}\right), \quad (11)$$

where l and q are the indices running over the $\rho_{(\mathbb{A}_{(i)})}$ vertices, or respectively $\rho_{(I_{(i)})}$ vertices of the respective polytopes, and $Conv$ is the function to determine the convex hull over the combinations of vertices. Similarly, the value of $\Delta_{(B,i),k} \cdot u_k$ in (10) can be bounded by:

$$\Delta_{(B,i),k} \cdot u_k \in Conv\left(\left\{\mathbb{B}^{(l)}_{(i)} \cdot U^{(q)} \mid l \in \{1, \dots, \rho_{(\mathbb{B}_{(i)})}\}\ ,\ q \in \{1, \dots, \rho_{(U)}\}\right\}\right). \quad (12)$$

For abbreviation, we write $\mathrm{Conv}(\mathbb{A}_{(i)} I_{(i)})$ and $\mathrm{Conv}(\mathbb{B}_{(i)} U)$ to denote the convex set on the right hand side of (11), and (12) respectively. With it, a disturbance invariant set $\mathcal{D}_i$ can be determined according to (10) by satisfying:

$$\bar{A}_{K,(i)}\mathcal{D}_i \oplus (W_{(i)} \oplus \mathrm{Conv}(\mathbb{A}_{(i)} I_{(i)}) \oplus \mathrm{Conv}(\mathbb{B}_{(i)} U)) \subseteq \mathcal{D}_i. \quad (13)$$

The set $\mathcal{D}_i$ exists, since $\bar{A}_{K,(i)}$ is stable and contains the origin, see Mayne et al. (2005). The relation (13) means that, if $x_k - \bar{x}_k \in \mathcal{D}_i$ applies, then $x_{k+1} - \bar{x}_{k+1} \in \mathcal{D}_i$ also follows from using the control law (9) despite all the uncertainties in the flow function, and the following holds:

Lemma 1. If for two given states x_k and $\bar{x}_k \in I_{(i)}$ applies that $x_k \in \bar{x}_k \oplus \mathcal{D}_i$, then using the control law (9) implies that the relation $x_{k+1} \in \bar{x}_{k+1} \oplus \mathcal{D}_i$ holds for all $\Delta_{(A,i),k} \in \mathbb{A}_{(i)}$, $\Delta_{(B,i),k} \in \mathbb{B}_{(i)}$ and $w_{(i),k} \in \mathbb{W}_{(i)}$. □

By use of Lemma 1 and when denoting the Pontryagin difference by $\ominus$, the following fact can also be established:

Proposition 1. If the nominal state satisfies $\bar{x}_k \in I_{(i)} \ominus \mathcal{D}_i$ and if the nominal input satisfies $\bar{u}_k \in U \ominus K_i\mathcal{D}_i$, then there always exists $u_k := \bar{u}_k + K_i(x_k - \bar{x}_k) \in U$ such that $x_{k+1} \in I_{(i)}$ for all $\Delta_{(A,i),k} \in \mathbb{A}_{(i)}$, $\Delta_{(B,i),k} \in \mathbb{B}_{(i)}$ and $w_{(i),k} \in \mathbb{W}_{(i)}$. □

The proofs of Lemma 1 and Proposition 1 follow the lines in Mayne et al. (2005) and the fact that $(I_{(i)} \ominus \mathcal{D}_i) \oplus \mathcal{D}_i \subseteq I_{(i)}$. The proposition means that, as long as the evolution of the nominal $\bar{x}_k$ lies inside of $I_{(i)} \ominus \mathcal{D}_i$ (i.e. inside of a tightened invariant), and if the nominal continuous input $\bar{u}_k$ is selected from set $U \ominus K_i\mathcal{D}_i$ (i.e. from a tightened input set), then we can always find $u_k \in U$ to obtain the next continuous state x_{k+1} inside of $I_{(i)}$ despite all the uncertainties. This is also illustrated in Fig. 1.

In addition, as all the sets in (13) are in $\mathcal{P}$, the computation of $\mathcal{D}_i$ can be assumed to be tractable according to Blanchini (1999). Furthermore, it is desired that $\mathcal{D}_i$ is as small as possible in order to reduce conservatism.

With respect to the semantics of HA in Def. 1, the state x can be kept inside of $I_{(i)}$ during the evolution in $z_{(i)}$ by forcing the nominal state $\bar{x}$ to be inside of $I_{(i)} \ominus \mathcal{D}_i$.

3.2 The Transition

Next, if the transition $\tau_{(i,j)}$ is triggered in step k, it must hold that $v_{(i,j),k} = 1$ and the following two conditions must be satisfied:

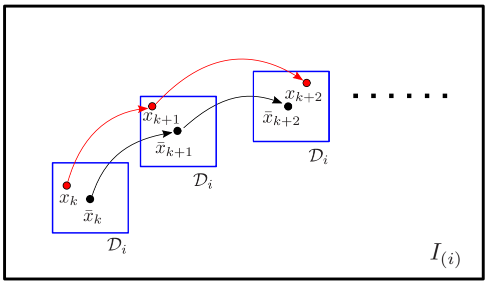

Fig. 1. By applying (9), the difference between x and $\bar{x}$ is recursively lying inside of $\mathcal{D}_i$ (denoted by the blue polytope) despite all the uncertainties.

(1) An intermediate state $\tilde{x}$ obtained from (3) must lie inside of $G_{(i,j)}$;
(2) the state x_{k+1} resulting from (1) must be inside of $I_{(j)}$.

For the first condition, it is known from Proposition 1 that by applying the control law (9) the state x_k lies inside of a tube $\mathcal{D}_i$ around the nominal state $\bar{x}_k$ during evolution in $I_{(i)}$. As the guard $G_{(i,j)}$ is fully contained in $I_{(i)}$, the tube around the nominal state must also exist when $\bar{x}_k$ evolves inside of $G_{(i,j)}$. The Proposition 1 can thus be extended to:

Lemma 2. By applying (9), if $\bar{x}_k \in G_{(i,j)} \ominus \mathcal{D}_i$ is satisfied, then $x_k \in G_{(i,j)}$ holds. □

Proof. According to Lemma 1, the relation $x_k \in \bar{x}_k \oplus \mathcal{D}_i$ holds when $\bar{x}_k \in I_{(i)} \ominus \mathcal{D}_i$. Thus, as $\bar{x}_k \in G_{(i,j)} \ominus \mathcal{D}_i$, and $\{G_{(i,j)} \ominus \mathcal{D}_i\} \subseteq \{I_{(i)} \ominus \mathcal{D}_i\}$, the relation $x_k \in G_{(i,j)}$ must apply. □

The lemma implies that $\tilde{x} \in G_{(i,j)}$ exists despite the uncertainties, if a nominal intermediate state $\bar{\tilde{x}}$, as obtained from (7), is inside of $G_{(i,j)} \ominus \mathcal{D}_i$. Thus, the first condition is guaranteed to be satisfied by restricting the position of the nominal intermediate state $\bar{\tilde{x}}$.

Next, due to the uncertainties on the reset function (1), a difference between the states $\bar{x}_{k+1}$ and x_{k+1} may be obtained according to:

$$x_{k+1} - \bar{x}_{k+1} = \bar{F}_{(i,j)}(\tilde{x} - \bar{\tilde{x}}) + \Delta_{F_{(i,j)}} \cdot \tilde{x} + \Delta_{e_{(i,j)}}, \quad (14)$$

where $\Delta_{F_{(i,j)}} \in \mathbb{F}_{(i,j)}$ and $\Delta_{e_{(i,j)}} \in \mathbb{E}_{(i,j)}$. By Lemma 2, the difference between $\tilde{x}$ and $\bar{\tilde{x}}$ is bounded to $\mathcal{D}_i$. As $\tilde{x}$ must be inside of $G_{(i,j)}$, the value of $\Delta_{F_{(i,j)}} \cdot \tilde{x}$ must be bounded by $Conv(\mathbb{F}_{(i,j)} G_{(i,j)})$. Hence, the difference of $\bar{x}_{k+1}$ and x_{k+1} in (14) is bounded by:

$$x_{k+1} - \bar{x}_{k+1} \in \bar{F}_{(i,j)}\mathcal{D}_i \oplus Conv(\mathbb{F}_{(i,j)} G_{(i,j)}) \oplus \mathbb{E}_{(i,j)}. \quad (15)$$

With the help of this bound, Lemma 2 can be extended to:

Lemma 3. If the nominal state after the transition $\tau_{(i,j)}$ satisfies $\bar{x}_{k+1} \in I_{(j)} \ominus (\bar{F}_{(i,j)}\mathcal{D}_i \oplus Conv(\mathbb{F}_{(i,j)} G_{(i,j)}) \oplus \mathbb{E}_{(i,j)})$, then the relation $x_{k+1} \in I_{(j)}$ holds for the state obtained from $\tau_{(i,j)}$ despite all the uncertainties in (1). □

The proof to Lemma 3 is similar to that of Lemma 2. According to Lemma 3, the second condition listed at the beginning of this section is satisfied by restricting the position of the nominal state $\bar{x}_{k+1}$.

3.3 The Post-Transition Phase

After reaching the discrete state $z_{(j)}$, the further evolution inside of $I_{(j)}$ has to be ensured. Obviously, this can be again achieved by applying a similarly control law as in (9), considering a disturbance invariant set $\mathcal{D}_j$ for $z_{(j)}$. However, such a recursive robustness guaranty relies on the condition that the difference of the states x_{k+1} and $\bar{x}_{k+1}$ is in $\mathcal{D}_j$ according to Lemma 1. But according to (15), the difference is known to be inside of $\bar{F}_{(i,j)}\mathcal{D}_i \oplus \text{Conv}(\mathbb{F}_{(i,j)} G_{(i,j)}) \oplus \mathbb{E}_{(i,j)}$, which is not depending on $\mathcal{D}_j$, see Fig. 3.

To avoid that robustness is lost at this point, the following criterion is formulated.

Definition 3. (**Robust Transition**) Any transition $\tau_{(i,j)} \in \mathcal{T}$ of HA may only be executed, if the condition:

$$\bar{F}_{(i,j)}\mathcal{D}_i \oplus Conv(\mathbb{F}_{(i,j)} G_{(i,j)}) \oplus \mathbb{E}_{(i,j)} \subseteq \mathcal{D}_j \quad (16)$$

is satisfied, otherwise it is prohibited.

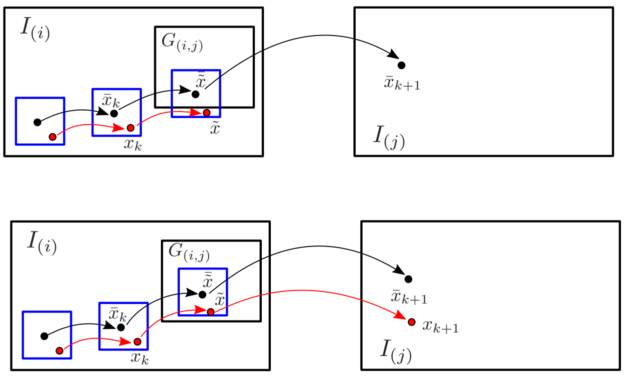

Fig. 2. For the case that $\bar{\tilde{x}} \oplus \mathcal{D}_i$ is not fully contained in $G_{(i,j)}$ (upper part), the transition $\tau_{(i,j)}$ may not be correctly triggered due to the deviation between $\bar{\tilde{x}}$ and $\tilde{x}$; If $\bar{\tilde{x}} \oplus \mathcal{D}_i \subseteq G_{(i,j)}$ (lower part), then the transition $\tau_{(i,j)}$ is guaranteed to be triggered correctly. The blue polytope represents the disturbance invariant set $\mathcal{D}_i$.

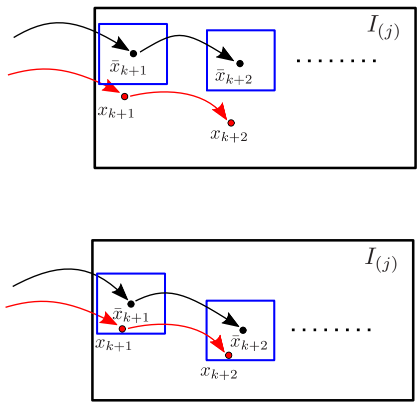

Fig. 3. For the case $x_{k+1} \notin \bar{x}_{k+1} \oplus \mathcal{D}_j$ due to (15), the further evolution of the real continuous state in $z_{(j)}$ may not be bounded by the $\mathcal{D}_j$ around the nominal state (upper part); for $x_{k+1} \in \bar{x}_{k+1} \oplus \mathcal{D}_j$ (lower part), such a bound applies. The blue polytope represents the disturbance invariant set $\mathcal{D}_j$.

Lemma 4. Let the transition $\tau_{(i,j)}$ satisfy (16) and let the control law (9) be applied in both $z_{(i)}$ and $z_{(j)}$. If $x - \bar{x}$ is then bounded to $\mathcal{D}_i$ before the transition $\tau_{(i,j)}$, then $x - \bar{x}$ is also bounded to the set $\mathcal{D}_j$ after the transition $\tau_{(i,j)}$. □

Proof. If $x_k - \bar{x}_k \in \mathcal{D}_i$ applies before the transition $\tau_{(i,j)}$, then $x_{k+1} - \bar{x}_{k+1} \in \bar{F}_{(i,j)}\mathcal{D}_i \oplus Conv(\mathbb{F}_{(i,j)}G_{(i,j)}) \oplus \mathbb{E}_{(i,j)} \subseteq \mathcal{D}_j$ holds true after the transition according to (15) and (16). Then, using Lemma 1, it follows that the difference will be recursively bounded to $\mathcal{D}_j$ by applying the control law (9) in the further steps. □

Eventually at this stage, the sequence of system evolution in $z_{(i)}$, of the transition $\tau_{(i,j)}$, and of the further evolution in $z_{(j)}$ was successfully cast into a set of constraints on the nominal state and inputs as well as the condition (16). In the next section, a substitute hybrid automaton $\overline{HA}$ with deterministic behavior for the nominal trajectory is constructed based on these constraints for all transitions in $\mathcal{T}$. It is then shown that the control task for HA in Def. 2 can be transformed into a corresponding problem for $\overline{HA}$.

4. HYBRID AUTOMATON WITH DETERMINISTIC BEHAVIOR

We construct a modified hybrid automaton $\overline{HA} = (T, \bar{U}, X, Z, \bar{I}, \bar{\mathcal{T}}, \bar{G}, \bar{V}, \bar{r}, \bar{f})$ based on the original one HA. The sets T, X and Z are identical to those in HA, while the other components of $\overline{HA}$ are determined according to the following rules:

- a set $\bar{U} = \{\bar{U}_{(1)}, \ldots, \bar{U}_{(n_z)}\}$ of continuous input sets, where for any $z_{(i)} \in Z$, the continuous input set is $\bar{U}_{(i)} := U \ominus K_i\mathcal{D}_i$;
- a set $\bar{I} = \{\bar{I}_{(1)}, \ldots, \bar{I}_{(n_z)}\}$ of invariants, where the invariant of any discrete state $z_{(i)} \in Z$ is obtained to $\bar{I}_{(i)} := I_{(i)} \ominus \mathcal{D}_i$;
- the finite set of transitions $\bar{\mathcal{T}}$, obtained from deleting those of $\mathcal{T}$, which do not satisfy (16);
- the set $\bar{G}$ of guard sets contains one polytopic set $\bar{G}_{(i,j)} := G_{(i,j)} \ominus \mathcal{D}_i$ assigned to any transition $\bar{\tau}_{(i,j)} \in \bar{\mathcal{T}}$;
- the finite set $\bar{V}$ of discrete input variables, where any element $\bar{v}_{(i,j)} \in \{0, 1\}$ in $\bar{V}$ refers to one transition $\bar{\tau}_{(i,j)} \in \bar{\mathcal{T}}$;
- a deterministic reset function $\bar{r}$, which updates the continuous state $\bar{\bar{x}}$ according to (7);
- a deterministic flow function $\bar{f}$ defines the continuous dynamics according to (6).

Note that all the uncertain components of HA are excluded here, and an admissible execution of the deterministic automaton $\overline{HA}$ is defined as:

Definition 4. (**Admissible Execution of** $\overline{HA}$) For $\overline{HA}$, let a finite time set $T_N = \{0, 1, \ldots, N\} \subset T$ and an initial hybrid state $(\bar{x}_0, z_0)$ satisfying $z_0 := z_{(s)} \in Z$, $\bar{x}_0 \in \bar{I}_{(s)}$, and $\bar{x}_0 \notin \bar{G}_{(s,j)}$ for all $\bar{\tau}_{(s,j)} \in \bar{\mathcal{T}}$ be given. For selected input sequences $\phi_{\bar{u}} = \{\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1}\}$ and $\phi_{\bar{v}} = \{\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{N-1}\}$, the pair of state sequences $\phi_{\bar{x}} = \{\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_N\}$ and $\phi_z = \{z_0, z_1, \cdots, z_N\}$ is *admissible*, if and only if for any $k \in \{0, \ldots, N\}$ the pair $(\bar{x}_{k+1}, z_{k+1})$ follows from $(\bar{x}_k, z_k)$, $\bar{x}_k \in \bar{I}_{(i)}$, $z_k := z_{(i)}$ according to the following semantics:

1.) $\bar{\bar{x}} := \bar{A}_{(i)} \cdot \bar{x}_k + \bar{B}_{(i)} \cdot \bar{u}_k \in \bar{I}_{(i)}$,
2.) if $\bar{G}_{(i,j)} \in \bar{G}$ exists such that $\bar{\bar{x}} \in \bar{G}_{(i,j)}$ and if $\bar{v}_{(i,j),k} = 1$ applies, then $\bar{x}_{k+1} := \bar{F}_{(i,j)} \cdot \bar{\bar{x}} + \bar{e}_{(i,j)} \in \bar{I}_{(j)}$ and $z_{k+1} := z_{(j)}$; otherwise, $\bar{x}_{k+1} := \bar{\bar{x}}$ and $z_{k+1} := z_{(i)}$ is assigned.

Then, by defining the new hybrid goal state $(\bar{X}_g,\ z_g)$, where $\bar{X}_g := X_g \ominus \mathcal{D}_g$ (x_g is still the volumetric center of $\bar{X}_g$), as well as assigning $\bar{x}_0 := x_0$, the following cost functional $\mathcal{J}(x_0, x_g, N)$ is selected:

$$\mathcal{J}(x_0, x_g, N) = \sum_{k=1}^{N}\{(\bar{x}_k - x_g)^{\mathrm{T}}Q(\bar{x}_k - x_g) + (\bar{u}_{k-1} - u_g)^{\mathrm{T}}R(\bar{u}_{k-1} - u_g)\} + q_g \cdot N_g, \quad (17)$$

where Q and R are semi-positive-definite weighting matrices, and $q_g \in \mathbb{R}^{\geq 0}$. The variable $N_g := \min\{k \in \{1, \ldots, N\} \mid \bar{x}_k \in \bar{X}_g, z_k = z_{(g)}\}$ encodes the first point of time at which the continuous state has reached the goal set. Note if $Q = R = 0$ and $q_g \neq 0$, such cost functional matches the concept of time optimal control.

The control problem of the deterministic $\overline{HA}$ can then be defined as:

Definition 5. For $\overline{HA}$ initialized to $(\bar{x}_0, z_0)$, $z_0 := z_{(s)}$, let a time set T_N and a goal $(\bar{X}_g, z_{(g)})$ be given. Then, determine input sequences $\phi^*_{\bar{u}}$ and $\phi^*_{\bar{v}}$ as the solution of:

$$\begin{aligned}
&\min_{\phi_{\bar{u}},\phi_{\bar{v}}} \mathcal{J}(x_0, x_g, N)\\
\text{s.t.: }&\text{For all } k \in \{0, \ldots, N-1\}:\\
&\phi_{\bar{u}} \text{ with } \bar{u}_k \in \bar{U}_{(i)}, \text{ when } \bar{x}_k \in \bar{I}_{(i)},\ \forall\, z_{(i)} \in Z,\\
&\phi_{\bar{v}} \text{ with } \bar{v}_{(i,j),k} \in \{0, 1\};\\
&\phi_{\bar{x}}, \phi_z \text{ admissible for } \overline{HA};\\
&\bar{x}_N \in \bar{X}_g, z_N = z_g.
\end{aligned}$$

If now the parameters of the initial and goal sets as well as that of the cost functional in Def. 5 are chosen identical to that of Def. 2, a solution of the previous one can be referred to the latter problem, see below. Note first that the problem in Def. 5 is of the same type as the one considered in Liu and Stursberg (2018, 2019), i.e., it can also be solved by using the methods proposed there (also briefly reviewed at the end of this section). The following result can be obtained:

Theorem 1. If, for the same parameterization of Def. 2 and Def. 5, a feasible solution is obtained for the problem in Def. 5, then this solution also solves the control task in Def. 2. □

Proof. For the initial continuous state, the difference $\bar{x}_0 - x_0 = 0 \in \mathcal{D}_s$ applies. The optimized inputs $\bar{u}^*_k$ in ϕ^*_u obtained from the solution of the problem in Def. 5 are all selected from the tightened input sets $\bar{U}$, The optimal continuous states $\bar{x}^*_k$ in ϕ^*_x are all located in the tightened invariant sets $\bar{I}$, as well as the transitions to be executed in ϕ^*_z are all satisfying the condition (16). Thus, according to Proposition 1 as well as Lemma 2, 3, 4, the satisfaction of the semantics in Def. 1 is always ensured during the evolution in each z_k^*, as well as for each transition in ϕ_z^* when using the through control law (9). In addition, for the last state $\bar{x}_N^*$ in $\phi_{\bar{x}}^*$, as $x_N - \bar{x}_N^* \in \mathcal{D}_g$ applies according to Lemma 1, and $\bar{x}_N^* \in X_g \ominus \mathcal{D}_g$ applies according the last constraint in the problem in Def. 2, it is ensured that $x_N \in X_g$. □

Accordingly, by controlling the original HA over the time domain T_N, we only have to: 1.) solve the problem in Def. 5; 2.) assign $v_{(i,j),k} := \bar{v}^*_{(i,j),k}$ and calculate u_k according to (9) at each step k. Then the goal states (X_g, z_g) are ensured to be reached at the end of the horizon, while satisfying the semantics in Def. 1. For the solution of the problem in Def. 5 according to the method in Liu and Stursberg (2018, 2019), we first decompose the semantics in Def. 4 into two parts, namely:

(1) for discrete state evolution:

$$\begin{aligned}&\text{If } z_k = z_{(i)}, \bar{\bar{x}} \notin \bar{G}_{(i,j)}, \text{ then } z_{k+1} := z_{(i)};\\&\text{If } z_k = z_{(i)}, \bar{\bar{x}} \in \bar{G}_{(i,j)}, \bar{v}_{(i,j),k} = 0, \text{then } z_{k+1} := z_{(i)};\\&\text{If } z_k = z_{(i)}, \bar{\bar{x}} \in \bar{G}_{(i,j)}, \bar{v}_{(i,j),k} = 1, \text{then } z_{k+1} := z_{(j)}.\end{aligned} \tag{18}$$

(2) for continuous state evolution:

$$\begin{aligned}&\text{If } z_k = z_{(i)}, \bar{\bar{x}} \notin \bar{G}_{(i,j)}, \text{ then dynamic (6) applies};\\&\text{If } z_k = z_{(i)}, \bar{\bar{x}} \in \bar{G}_{(i,j)}, \bar{v}_{(i,j),k} = 0, \text{ then (6) applies};\\&\text{If } z_k = z_{(i)}, \bar{\bar{x}} \in \bar{G}_{(i,j)}, \bar{v}_{(i,j),k} = 1, \text{ then (7) applies};.\end{aligned} \tag{19}$$

Next, by introducing one binary variable per time step for each invariant set, guard set, and the terminal set in $\overline{HA}$, both of the (18) and (19) can be equivalently cast into a set of linear constraints formulated for the binary and real variables, employing the modeling principles introduced in Williams (2013). The last term in the cost functional $\mathcal{J}(x_0, x_g, N)$ can also be reformulated into a sum over a set of binary variables without any approximation. See the specifications in Liu and Stursberg (2018, 2019) for details on this modeling procedure, and on the solution as mixed-integer programming problem with linear constraints.

5. NUMERICAL EXAMPLE

The uncertain HA considered here for illustration of the procedure consists of 4 discrete states with a set of possible transitions, as shown in the lower part of Fig. 4. In the lower part of this figure, the invariants of the model are marked in red, the guards are in green, and the selected terminal set $X_g \subseteq I_{(2)}$ is marked in yellow. The flow function f and reset function r of the states and transitions are parametrized with different bounds of uncertainties. The initial state is $x_0 = [10, 4.5]^{\mathrm{T}} \in I_{(1)}$ and a horizon of $N = 20$ is available to realize the transition from the initial state into the goal set.

The controller K_i for each discrete state is selected to be the minimum time controller according to Langson et al. (2004), such that $\bar{A}^{n_x}_{K,(i)} = 0$ applies. The minimal disturbance invariant set $\mathcal{D}_i$ is computed with the help of the Matlab Invariant Set Toolbox proposed in Kerrigan (2001). When constructing the deterministic model $\overline{HA}$, each invariant set, guard set, the terminal set as well as the

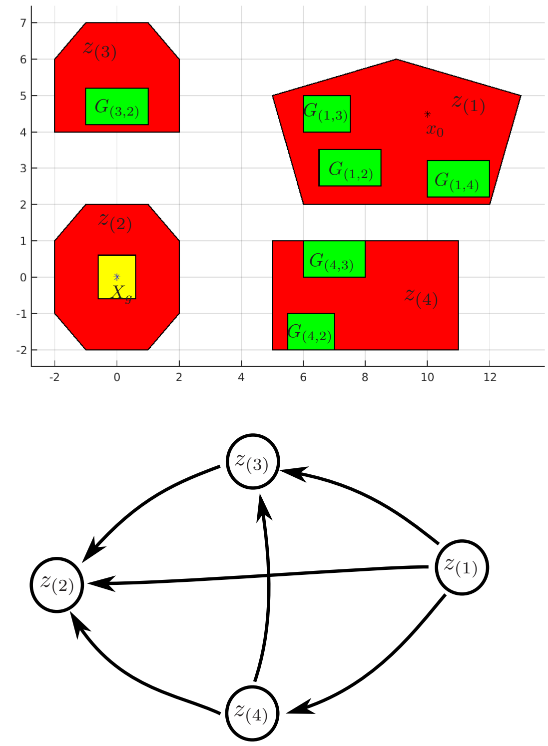

Fig. 4. Relevant sets and the transitions of the model HA.

continuous input set are tightened according to the disturbance invariant set, see Fig. 5. By evaluating the condition (16) for each transition in HA, only the transition $\tau_{(1,2)}$ fails to satisfy it, and is thus prohibited. Thereafter, by solving the problem in Def. 5 for the resulting deterministic hybrid automaton $\overline{HA}$, the optimal trajectory is found by first transferring from $z_{(1)}$ to $z_{(4)}$, and then to $z_{(2)}$, see Fig. 5. The time required for the computation is $0.66sec$ on a 3.4GHz processor using Matlab-2015a.

Based on the solution of the problem in Def. 5, the original automaton HA is controlled by the control law (9) in each step, and for 50 different realizations of the uncertainties the trajectories is shown in Fig. 6 are obtained. It can be

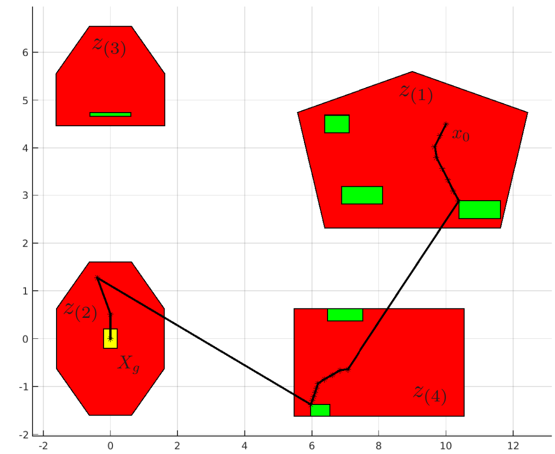

Fig. 5. Deterministic model $\overline{HA}$ obtained by tightening each invariant set, guard set, and the terminal set of HA; The trajectory in black is the optimal nominal state sequence of the problem in Def. 5.

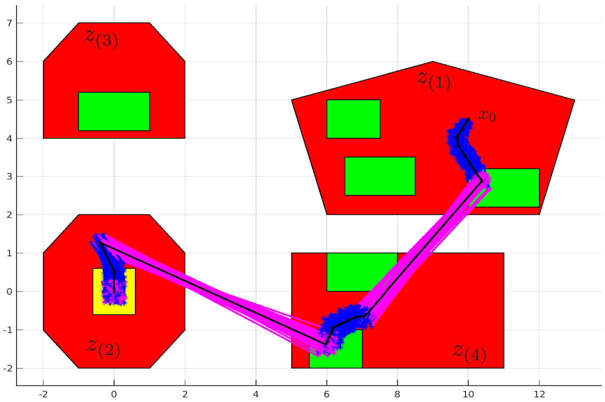

Fig. 6. The set of blue trajectories are the state sequences under different realizations of the uncertainties. The set of magenta lines represent the resets of the continuous states for the transitions; the set of magenta points contained in the terminal set X_g are the last states of the trajectories. The black trajectory is the solution of the problem in Def. 5.

seen in this figure that all continuous states along these trajectories evolve inside of the invariant and that the transitions are correctly triggered in the guard set, despite the uncertainties. In addition, all 50 trajectories reach the terminal set at the end of the horizon.

6. CONCLUSION

This work has extended principles of tube-based control and constraint tightening, which were established for purely continuous dynamics, to hybrid systems with guarded transitions and mixed inputs. As the main result, point-to-set transitions for this class of hybrid systems can be computed by the proposed techniques in an optimized fashion, while reaching the goal set is guaranteed despite uncertainties of the flow and reset functions. The key idea is to optimize the nominal trajectory, to compute control laws to let the continuous dynamics converge towards the nominal trajectory (thus counteracting the uncertainties), and to reduce the size of guards and invariant sets to ensure that the sequence of discrete states (corresponding to the nominal trajectory) is realized for any possible disturbance. The computational procedure involves to compute a modified hybrid automaton $\overline{HA}$ with tightened sets, for which an optimized trajectory is obtained, that in turn is feasible and goal-attaining for the original system.

In current work, we aim at extending the control technique to uncertain nonlinear flow functions, and to hybrid automata with invariants and guard sets which can vary over time.

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