Computing invariant sets of discrete-time nonlinear systems via state immersion

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Abstract: In this paper, we propose a method for computing invariant sets of discrete-time nonlinear systems by lifting the nonlinear dynamics into a higher dimensional linear model. In particular, we will focus on the maximal invariant set. Some special types of nonlinear systems can be considered as the projection of a higher dimensional linear system with a state immersion transformation. For such systems, the equivalence between invariant sets of the nonlinear system and its linear equivalent can be also established, which allows to characterize the maximal invariant set of the nonlinear system using a lifted linear model. For general nonlinear systems, we will use linear approximations because equivalent linear models cannot be achieved exactly. To handle mismatch errors, we tighten the constraint set of the lifted linear model, which will lead to an invariant inner approximation of the maximal invariant set.

Keywords: Invariant set, nonlinear systems, state immersion

1. INTRODUCTION

Set invariance theory plays an important role in systems and control for stability analysis and control design, see, for instance (Blanchini, 1999; Blanchini and Miani, 2008) and the references therein. In particular, it is widely used in Model Predictive Control (MPC) (Mayne et al., 2000) for constrained systems.

The problem of computing invariant sets has been studied extensively for different types of systems. The early literature has been devoted to linear systems with polyhedral constraints, see, e.g., (Gilbert and Tan, 1991) and the references therein. In the presence of bounded disturbances in linear systems, robust invariant sets were studied (see, e.g., (Kolmanovsky and Gilbert, 1998; Rakovic et al., 2005; Ong and Gilbert, 2006; Trodden, 2016)). Recently, the authors in (Wang et al., 2019) have proposed an algorithm to deal with nonlinear constraints. However, for nonlinear systems, the computation of invariant sets is even more difficult and complicated. Although some existing algorithms for computing invariant sets of different types of nonlinear systems are available, see, e.g., (Bravo et al., 2005; Alamo et al., 2009; Fiacchini et al., 2010; Sassi and Girard, 2012; Henrion and Korda, 2014; Korda et al., 2014), obtaining an exact invariant set is still a difficult problem for general nonlinear systems. The aforementioned algorithms for nonlinear systems are focused on inner or outer approximations of invariant sets. However, these approximations are not necessarily invariant sets. In this paper, we attempt to get an inner approximation of the maximal invariant set by using a lifted linear model of the nonlinear system. Although this inner approximation is a subset of the maximal invariant set in general, it is in fact also an invariant set. For special classes of nonlinear systems, the technique allows to obtain the exact maximal invariant set.

In order to obtain a lifted linear model, we need first to perform linearization of the nonlinear system, which is one of the most well-known research topics in systems and control. Two classic linearization methods are Jacobian linearization and feedback linearization, see, (Khalil, 2002) for a comprehensive view. Another linearization method is the state immersion method, which allows to immerse a nonlinear system into a linear system in a higher dimension, see, e.g., (Monaco and Normand-Cyrot, 1983; Lee and Marcus, 1988; Menini and Tornambè, 2009). Recently, a new immersion technique has been proposed in (Jungers and Tabuada, 2019) for continuous-time systems by the use of polyflows. This technique often outperforms the Taylor approximation in practice. Inspired by the polyflows approximation, we have developed a similar immersion method (Wang and Jungers, 2020) for discrete-time systems. In this paper, we use such a method to design an algorithm for computing invariant sets of discrete-time nonlinear systems.

The rest of the paper is organized as follows. This section ends with the notation, followed by the next section on the review of preliminary results on invariant sets. Section 3 presents the proposed immersion-based method for computing the maximal invariant of nonlinear systems. In Section 4, we will discuss some computational issues of the proposed method. Numerical examples are provided Section 5. The last section concludes the work.

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Notation. The non-negative integer set is indicated by $\mathbb{Z}^+$. $I_n$ is the $n \times n$ identity matrix and $0_{n \times m}$ is the $n \times m$ matrix of all zeros (subscript omitted when the dimension is clear). For two set $X$ and $Y$, $X \subseteq Y$ denotes the Minkowski difference. Given a map $T$, let $T(X)$ denote $\{T(x) : x \in X\}$ and $T^{-1}(Y)$ denote the preimage of the set $Y$ under the map $T$, i.e., $T^{-1}(Y) := \{x : T(x) \in Y\}$ ($T$ is not necessarily invertible).

2. PRELIMINARIES

We consider discrete-time dynamical systems of the form

$$x(t + 1) = f(x(t)), \quad t \in \mathbb{Z}^+$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function over $X$. The system is subject to state constraints:

$$x(t) \in X, \quad t \in \mathbb{Z}^+, \quad t \text{ times}$$

(2)

Let $f^t(x) = f \circ \cdots \circ f(x)$ with $f^0(x) = x$ for all $t \in \mathbb{Z}^+$.

The following assumptions are made. (A1) The function $f(x)$ is continuous with $f(0) = 0$. (A2) The set $X$ is compact and contains the origin in its interior. (A3) The system (1) is asymptotically stable at the origin in $X$, i.e., $\lim_{t \to \infty} \|f^t(x)\| = 0, \forall x \in X$.

3. MAIN RESULTS

3.1 State immersion

First, we give a definition of immersibility of nonlinear systems, see, e.g., (Monaco and Normand-Cyrot, 1983; Lee and Marcus, 1988).

Definition 3. The system (1) is immersible into a linear system in the form of

$$\xi(t + 1) = A_2 \xi(t), \quad y(t) = C_2 \xi(t), \quad t \in \mathbb{Z}^+,$$

(6)

where $\xi \in \mathbb{R}^{n_{\xi}}$, $y(t) \in \mathbb{R}^n$, $A_2 \in \mathbb{R}^{n_{\xi} \times n_{\xi}}$ and $C_2 \in \mathbb{R}^{n \times n_{\xi}}$, if there exists a map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $f^t(x) = C_2 A_2^t T(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{Z}^+$. For notational simplicity, let us denote the linear system in (6) by $P(\xi_2, C_2)$.

A necessary and sufficient condition for immersibility is given in the following proposition.

Proposition 1. The system (1) is immersible into a linear system in the form of (6) if and only if there exist $M$ and a sequence of matrices $\{\gamma_\ell \in \mathbb{R}^{n_{\xi} \times n}\}^M_{\ell=0}$ satisfying

$$f^{M+1}(x) = \sum_{\ell=0}^{M} \gamma_\ell f^\ell(x), \forall x \in \mathbb{R}^n$$

(7)

Similar arguments can also be found in (Monaco and Normand-Cyrot, 1983; Lee and Marcus, 1988), although the proof is slightly different, see Theorem 1 in (Wang and Jungers, 2020). With the integer $M$ and the matrices $\gamma_M := \{\gamma_\ell \in \mathbb{R}^{n_{\xi} \times n}\}^M_{\ell=0}$ satisfying (7), we can immediately construct a linear system $P(\Gamma(\gamma_M), [I_n, 0_{n \times (M+1)n}])$, where

$$\Gamma(\gamma_M) := \begin{pmatrix} 0 & I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \gamma_0 & \gamma_1 & \cdots & \gamma_{M-1} & \gamma_M \end{pmatrix}.$$  

(8)

The condition in (7) implies that the system (1) is immersible into $P(\Gamma(\gamma_M), [I_n, 0_{n \times (M+1)n}])$ with the transformation map $F_M(x)$ defined by

$$F_M(x) := \begin{pmatrix} x \\ \vdots \\ f^M(x) \end{pmatrix}.$$  

(9)

However, there may exist redundancy in such a transformation. To remove redundancy, we will use linearly independent transformations. A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is called linearly independent if its components $\{T_1(x), \ldots, T_m(x)\}$ are linearly independent. From this definition, a linearly independent map $T : \mathbb{R}^n \to \mathbb{R}^m$ implies $\text{span}\{T(x) : x \in \mathbb{R}^n\} = \mathbb{R}^m$. Otherwise, there exists a vector $c \in \mathbb{R}^m$ such that $c^TT(x) = 0, \forall x \in \mathbb{R}^n$.

With a linearly independent transformation, a tight linear model can be obtained, as stated in the following lemma.
Proposition 2. Suppose A1 holds and the system (1) is immersible into a linear system in the form of (6), there always exist a continuous linearly independent map $T: \mathbb{R}^n \to \mathbb{R}^m$ and an observable pair $(C, A)$ such that $AT(x) = T(f(x))$ and $CT(x) = x$ for all $x \in \mathbb{R}^n$. In addition, $A$ is Schur stable if A2 & A3 hold.

Proof: From Proposition 1, when the system (1) is immersible to $\Pi(A_c, C_\xi)$, there exist $M$ and matrices $\gamma_M := \{ \gamma \in \mathbb{R}^{n \times (M+1)n} \}_{t=0}^M$ such that (7) is satisfied. This implies that

$$F_M(x) = \Gamma(\gamma_M)F_M(x)$$

where $\Gamma(\gamma_M)$ is defined in (8). Suppose there are $m$ linearly independent functions that form a basis for the spanning set $\{ x_1, \ldots, x_m, f_1^m(x), \ldots, f^M_n(x) \}$, let $T(x)$ be the stacked vector of these functions. As all the functions $\{ x_1, \ldots, x_m, f_1^m(x), \ldots, f^M_n(x) \}$ can be expressed as linear combinations of $T(x)$, there exists a full column rank matrix $P \in \mathbb{R}^{(M+1)n \times m}$ such that

$$F_M(x) = PT(x), F_M(f(x)) = PT(f(x))$$

Hence, from (10), $T(f(x)) = P^T\Gamma(\gamma_M)PT(x)$, where $P^T$ denotes the pseudo inverse. Letting $A = P^T\Gamma(\gamma_M)P$ and $C = [I_n \ 0_{n \times (M+1)n}]P$, we can get $AT(x) = T(f(x))$ and $CT(x) = x$. When $m = (M + 1)n$, $A = \Gamma(\gamma_M)$ and $C = [I_n \ 0_{n \times (M+1)n}]$. From the definition of $\Gamma(\gamma_M)$, $[I_n \ 0_{n \times (M+1)n}]\Gamma(\gamma_M)$ is observable. To show that $(C, A)$ is observable, we need to show that $O(A, C) := \begin{pmatrix} C \\ \vdots \\ CA^{m-1} \end{pmatrix}$ is full column rank.

Since $T(x)$ is linearly independent, there exist $N$ points $\{ \tilde{x}_1, \ldots, \tilde{x}_N \}$ such that $\text{span}(T(\tilde{x}_1), \ldots, T(\tilde{x}_N)) = \mathbb{R}^m$. Thus, for any $z \in \mathbb{R}^m$, there exist $\{ a_1, \ldots, a_N \}$ such that $z = \sum_{i=1}^N a_i T(\tilde{x}_i)$, which implies that $Pz = \sum_{i=1}^N a_i PT(\tilde{x}_i) = \sum_{i=1}^N a_i F_M(x_i)$, where the second equality follows from (11). Since $AT(x) = T(f(x))$ and

$$CT(x) = x, \begin{pmatrix} C \\ \vdots \\ CA^M \end{pmatrix} = \begin{pmatrix} C \\ \vdots \\ CA^M \end{pmatrix} = \sum_{i=1}^N a_i T(\tilde{x}_i) = Pz, \forall z \in \mathbb{R}^m,$$

Hence, $\text{rank}(O(A, C)) = m$ and $(C, A)$ is observable. Finally, we show that $A$ is Schur stable under A2 & A3. As $T(x)$ is linearly independent and $T(X)$ is compact and contains the origin in the interior, we can choose $N$ points $\{ \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N \}$ inside $X$ such that $\text{span}(T(\tilde{x}_1), T(\tilde{x}_2), \ldots, T(\tilde{x}_N)) = \mathbb{R}^m$. Hence, for any $z \in \mathbb{R}^m$, there exist $\{ a_1, a_2, \ldots, a_N \}$ such that $z = \sum_{i=1}^N a_i T(\tilde{x}_i)$, which leads to $A^k z = \sum_{i=1}^N a_i A^k T(\tilde{x}_i) = \sum_{i=1}^N a_i T(f^k(\tilde{x}_i)), \forall k \in \mathbb{Z}^+$. The asymptotic stability of the system (1) implies that $A^k z \to 0$ as $k \to \infty$ for any $z \in \mathbb{R}^m$. Therefore, $A$ is asymptotically stable and thus Schur stable. □

3.2 Set invariance under immersion

For systems that are immersible into a linear system (see Definition 3), we can also establish the immersion on invariant sets of the nonlinear system and its linear equivalent, as shown in the following proposition.

Proposition 3. Suppose there exist a continuous map $T: \mathbb{R}^n \to \mathbb{R}^m$ and matrices $A \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times m}$ such that $AT(x) = T(f(x))$ and $CT(x) = x$ for all $x \in \mathbb{R}^n$. Given the constraint set $X$, let $Z \subseteq X$ be an invariant set for the system (1) and $\Xi \subseteq \{ \xi \in \mathbb{R}^m : \xi \in X \}$ be an invariant set for $\Pi(A, C)$. Then, (i) $T^{-1}(\Xi) := \{ x \in \mathbb{R}^n : T(x) \in \Xi \} \subseteq \{ x \in \mathbb{R}^n : CT(x) \in X \} \subseteq X$ is invariant for the system (1); (ii) $T(Z) \subseteq \{ \xi \in \mathbb{R}^m : C\xi \in X \}$ is invariant for $\Pi(A, C)$.

Proof: (i) As $CT(x) = x$ and $\Xi \subseteq \{ \xi \in \mathbb{R}^m : \xi \in X \}$, we can show that $T^{-1}(\Xi) = \{ x \in \mathbb{R}^n : T(x) \in \Xi \} \subseteq \{ x \in \mathbb{R}^n : CT(x) \in X \} \subseteq X$. Hence, we only need to show the invariance of $T^{-1}(\Xi)$. For any $x \in T^{-1}(\Xi)$, we want to show that $f(x) \in T^{-1}(\Xi)$, i.e., $T(x) \in \Xi$ implies $T(f(x)) \in \Xi$. From the invariance of $\Xi$, we know that $AT(x) \in \Xi$. This, together with the fact that $AT(x) = T(f(x))$, proves the invariance of $T^{-1}(\Xi)$. (ii) Similarly, we will first show that $T(Z) \subseteq \{ \xi \in \mathbb{R}^m : \xi \in X \}$, which implies $T(Z) \subseteq \{ \xi \in \mathbb{R}^m : C\xi \in X \}$. For any $\xi \in T(Z)$, there exists an $x \in \mathbb{Z}^+ \subseteq X$ such that $\xi = T(x)$. From the fact that $CT(x) = x$, we can see that $C\xi = CT(x) = x \in X$, which implies $T(Z) \subseteq \{ \xi \in \mathbb{R}^m : C\xi \in X \}$. Then, we will show the invariance of $T(Z)$. As $Z$ is invariant for system (1), we know that $f(x) \in Z$. Hence, $A_T = AT(x) = T(f(x)) \in T(Z)$. This completes the proof. □

The results in Proposition 3 allow us to use the lifted linear system to compute the maximal invariant set $O_\infty(\Pi)$ of the system (1). Given any pair $(A, C)$ with $A \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{m \times m}$, let us define

$$O_\infty(A, C) := \{ x \in \mathbb{R}^m : CA^k x = 0, \forall k \in \mathbb{Z}^+ \}$$

From Theorem 4.1 in (Gilbert and Tan, 1991), $O_\infty(A, C)$ exists and can be finitely determined when $(\tilde{C}, \tilde{A})$ is observable and $\tilde{A}$ is Schur stable. Based on the discussions above, the following theorem can be obtained.

Theorem 1. Suppose A1-A3 hold, let $O_\infty$ be defined as in (5) for the system (1) with the constraint set $X$. Assume that the system (1) is immersible into a linear system $\Pi(A_c, C_\xi)$ in (6). Then, there exist a continuous linearly independent map $T: \mathbb{R}^n \to \mathbb{R}^m$ and an observable pair $(C, A)$ with $A$ being Schur stable such that $O_\infty = T^{-1}(O_\infty(A, C))$, where $O_\infty(A, C)$ is defined in (12).

Proof: This result can be derived from Propositions 2 & 3 and hence the proof is omitted due to page limitation. □

3.3 Approximate immersion and robustness

Since linear equivalents exist only for very particular classes of systems, we will discuss approximate state immersion and robustness issues. In general cases, we want to find a transformation map $T: \mathbb{R}^n \to \mathbb{R}^m$ such that $T(f^{t+1}(x)) - AT(f^t(x))$ is within some given tolerance for all $x \in X$ and $t \in \mathbb{Z}^+$. With A1-A3, the system (1) can be arbitrarily close to the projection of a linear system by increasing the dimension of the lifted space, which is formally stated in the following lemma.

Lemma 1. Suppose A1-A3 hold, for any given $\delta > 0$, there exist a continuous linearly independent map $T: \mathbb{R}^n \to \mathbb{R}^m$, an observable pair $(C, A)$ with $A$ being Schur stable and a matrix $B \in \mathbb{R}^{m \times m}$ such that

$$CT(x) = x, T(f^{t+1}(x)) - AT(f^t(x)) \in BW_\delta$$

(13)
for all \( x \in X \) and \( t \in \mathbb{Z}^+ \), where
\[
W_\delta := \{ w \in \mathbb{R}^n : \| w \|_\infty \leq \delta \}. \tag{14}
\]
Proof: From A3, there always exist \( M \) and matrices \( \{ \gamma_\ell \in \mathbb{R}^{n \times n} \}_{\ell = 0}^M \) such that
\[
\| f^{M+1+k}(x) - \sum_{\ell=0}^M \gamma_\ell f^{k+\ell}(x) \|_\infty \leq \delta \tag{15}
\]
for all \( x \in X \) and \( k \in \mathbb{Z}^+ \). Hence, \( F_M(f^{k+1}(x)) - \Gamma(\gamma_M)F_M(f^k(x)) \in \left( \begin{array}{c} 0_{M \times n} \\ I_n \end{array} \right) W_\delta \) for all \( x \in X \) and \( k \in \mathbb{Z}^+ \). Let \( T(x) \) be the \( m \) linearly independent functions that form a basis for the spanning set of \( \{ x_1, x_2, \ldots, f^M(x) \} \). We can find a full column rank matrix \( P \in \mathbb{R}^{(M+1) \times n} \) such that \( T(f^{k+1}(x)) \in P^+(0_{M \times n}) W_\delta, \forall x \in \mathbb{R}^n, k \in \mathbb{Z}^+ \). Letting \( A = P^+ \Gamma(\gamma_M)P, C = [I_n \ 0_{n \times (M+1)n}] \) and \( B = P^+ \left( \begin{array}{c} 0_{M \times n} \\ I_n \end{array} \right) \), we can get (13). Then, we will show that we can always find an observable (\( C, A \)) with \( A \) being Schur stable. Since \( T(x) \) is linearly independent, there exist \( N \) points \( \{ \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N \} \) inside \( X \) such that \( \text{span}\{ T(\bar{x}_1), T(\bar{x}_2), \ldots, T(\bar{x}_N) \} = \mathbb{R}^n \). Hence, for any \( z \in \mathbb{R}^n \), there exist \( \alpha_1, \alpha_2, \ldots, \alpha_N \) such that \( z = \sum_{i=1}^N \alpha_i T(\bar{x}_i) \). By increasing \( M, AT(x) - T(f(x)) \) can be made arbitrary small for any \( x \in X \). Following similar arguments in Proposition 2, we can see that
\[
\left( \begin{array}{c} C \\ \vdots \\ C A^M \end{array} \right) z - P z \text{ can be made arbitrarily small for any } z. \text{ Hence, we can find } M \text{ such that } \left( \begin{array}{c} C \\ \vdots \\ C A^M \end{array} \right) \text{ is full column rank, which implies } (C, A) \text{ is observable. Similarly, we can also find a Schur stable } A \text{ by increasing the dimension.} \]

To account for the mismatch between the system (1) and the linear system \( \Pi(A, C) \) that satisfies (13) in Lemma 1, we compute a tightened subset of \( O_L^\delta(A, C) \), instead of \( O_L^\infty(A, C) \). Given \( A, B, C, D \) and \( \delta > 0 \), let us define
\[
O_L^\delta(A, B, C) := \{ x \in \mathbb{R}^n : CA^k x \in X, \forall k \leq M \} \tag{16}
\]
where \( W_\delta \) is given in (14). From (Kolmanovsky and Gilbert, 1998), the set \( O_L^\delta(A, B, C) \) is nonempty when \( \text{disturb}(\delta) \subseteq X \) and it is the maximal robust invariant set, that is, the maximal invariant set for the disturbed system \( x' = Ax + Bu \) where the disturbance \( w \) is constrained in \( W_\delta \).

Theorem 2. Suppose A1-A3 hold, let \( O_\infty \) be defined as in (5) for the system (1). For any given \( \delta > 0 \), let \( W_\delta \) be defined in (14). Consider a continuous linearly independent map \( T : \mathbb{R}^n \to \mathbb{R}^n \), an observable pair \( (A, C) \) with \( A \) being Schur stable and a matrix \( B \in \mathbb{R}^{m \times n} \) that satisfy (13), the following results hold: (i) \( T^{-1}(O_L^\infty(A, B, C)) \subseteq O_\infty \); (ii) \( T^{-1}(O_{L}^\infty(A, B, C)) \) is invariant for the system (1), where \( O_{L}^\infty(A, B, C) \) is defined as in (16).

Proof: First, we show that \( T^{-1}(O_L^\infty(A, B, C)) \subseteq O_\infty \). For any \( x \in T^{-1}(O_L^\infty(A, B, C)) \subseteq O_\infty, \) we know that \( CA^T x \in X \). From the fact that \( T(f^{k+1}(x)) - AT(f^k(x)) \in BW_\delta \) for all \( k \in \mathbb{Z}^+ \), we know that \( T(f^k(x)) \in A^T T(x) + \sum_{\ell=0}^{k-1} A^\ell B W_\delta \). The proof continues as in Theorem 2 in (Kolmanovsky and Gilbert, 1998). Therefore, \( T^{-1}(O_L^\infty(A, B, C)) \subseteq O_\infty \). To prove the invariance of \( T^{-1}(O_L^\infty(A, B, C)) \), we need to show that \( f(x) \in T^{-1}(O_L^\infty(A, B, C)) \), which means that \( CA^T f(x) \in X \). Since \( T(f(x)) \in AT(x) + BW_\delta \), any \( k \in \mathbb{Z}^+ \), \( CA^T f(x) \in CA^T + T(x) + \sum_{\ell=0}^{k-1} CA^\ell B W_\delta \). The last inclusion again follows from Theorem 2.1 in (Kolmanovsky and Gilbert, 1998). \( \square \)

4. COMPUTATIONAL ISSUES

From the discussions in the previous section, to obtain a linear approximation and the associated transformation, for the given \( \delta > 0 \), we need to find \( M \in \mathbb{Z}^+ \) and \( \gamma_M := \{ \gamma_\ell \in \mathbb{R}^{n \times n} \}_{\ell = 0}^M \) such that
\[
\| g_M^k(x, \gamma_M) \|_\infty \leq \delta, \forall x \in X, k \in \mathbb{Z}^+ \tag{17}
\]
where
\[
g_M^k(x, \gamma_M) = f^{k+M+1}(x) - \sum_{\ell=0}^M \gamma_\ell f^{k+\ell}(x). \tag{18}
\]
We will solve this problem numerically by griding or random sampling. Suppose we take \( N \) initial states inside \( X \), denoted by \( \omega_N := \{ x_1, x_2, \ldots, x_N \} \), and generate the trajectory \( \{ x(0), x(t), \ldots, f^k(x) \} \) with a sufficiently long horizon \( L \in \mathbb{Z}^+ \) for each \( x \in \omega_N \), the following least squares regression problem is formulated for any \( M \in \mathbb{Z}^+ \),
\[
\min_{\gamma_M} \sum_{x \in \omega_N} \sum_{k=0}^{L} \| g_M^k(x, \gamma_M) \|_2^2 + \rho \| \gamma_M \|_F^2 \tag{19}
\]
where \( g_M^k(x, \gamma_M) \) is given in (18) and \( \rho > 0 \). Let \( \rho \) be small. From Problem (19) be denoted by \( \gamma_M^* \). With this solution, we can compute
\[
\delta_M = \max_{x \in \omega_N} \min_{0 \leq t \leq M-1} \| g_M^k(x, \gamma_M^*) \|_\infty \tag{20}
\]
From \( \gamma_M^* \), a linear system \( \Pi(\Gamma(\gamma_M^*), [I_n \ 0_{n \times (M+1)n}]P) \) can be obtained with the transformation \( F_M(x) \). By checking and removing the redundancy, we get a linearly independent transformation map \( T : \mathbb{R}^n \to \mathbb{R}^n \) and a full column rank matrix \( P \in \mathbb{R}^{(M+1)n \times n} \) such that \( F_M(x) = PT(x), \forall x \), which implies that
\[
T(x) = P^+ F_M(x) \tag{21}
\]
where \( P^+ \) denotes the pseudo inverse. Then, we can get a linear system \( \Pi(A_M, C_M) \) with \( A_M = P^+ \Gamma(\gamma_M^*)P \) and \( C_M = [I_n \ 0_{n \times (M+1)n}]P \), and a matrix \( B_M = P^+ \left( \begin{array}{c} 0_{M \times n} \\ I_n \end{array} \right) \). Note that \( F_M(x) \) is already linearly independent in many real applications. We will increase \( M \) and repeat the procedure above until \( \delta_M \) is smaller than some
given $\delta > 0$. Let the solution from the computations above be denoted by $(\hat{M}, \hat{\delta}, A\hat{M}, B\hat{M}, C\hat{M})$. As $N$ increases, it can be made arbitrarily close to the actual solution.

For a given $\delta > 0$, we can obtain $\Pi(A\hat{M}, B\hat{M}, C\hat{M})$ using the standard algorithm (Gilbert and Tan, 1991). For notational simplicity, let

$$\Omega_{\delta} := O^{L, \delta, \delta} \Pi(A\hat{M}, B\hat{M}, C\hat{M}).$$

(22)

If $\Omega_{\delta}$ is empty, we will have to reduce $\delta$ and repeat the computations above again.

5. SIMULATION RESULTS

**Example 1.** We consider the following nonlinear system:

$$x^{+} = f(x) := [2x_{1}^{2} + x_{2} - 2 (x_{1}^{2} + x_{2})^{2} - 0.8x_{1}]^{T},$$

where $x = [x_{1}, x_{2}]^{T}$. This system is globally stable at the origin. With the transformation $T(x) = \mathcal{F}_{1}(x)$ it is globally immersible to $\Pi(A, C)$ with $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let us consider the state constraint set $X := \{x \in \mathbb{R}^{2} : \|x\|_{\infty} \leq 1\}$. We can easily compute $O_{\infty}^{L}(A, C)$, expressed by $\|x\|_{\infty} \leq 1$, with $C = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and obtain $\Omega_{\delta}^{1}$.

**Fig. 2.** Mismatch errors for different values of $M$: those with empty $O^{L, \delta, \delta} \Pi(A\hat{M}, B\hat{M}, C\hat{M})$ are labeled by $\times$.

**Fig. 3.** Trajectories for different values of $M$ with the initial state $[-25.8 - 5.6 29.7]^{T}$.

From Figure 2, to get a non-empty set, $\delta$ has to be smaller than $\hat{\delta} = 5.34 \times 10^{-2}$. Without the preprocessing in Figure 2, the value of $\delta$ can be obtained by bisection. In the simulation, we set $\delta = 5 \times 10^{-2}$ and obtain $\Omega_{\delta} = O_{\infty}^{\hat{z}, \hat{\delta}, \delta}(A_{18}, B_{18}, C_{18})$, which is a 57-dimensional polytope expressed as $\xi \in \mathbb{R}^{57}$.

Then, the invariant set $O_{\infty}$ of the original system can be expressed as $T^{-1}(O_{\infty}^{\hat{z}, \hat{\delta}, \delta}(A, C)) = \{x \in \mathbb{R}^{2} : GT(x) \leq 1, \|T(x)\|_{\infty} \leq 1\}$, which is shown in Figure 1.

**Example 2.** Consider a controlled Lorenz system: $\dot{x}_{1} = 10(x_{2} - x_{1}), \dot{x}_{2} = 28x_{1} - x_{2} - x_{1}x_{3} + u, \dot{x}_{3} = x_{1}x_{2} - \frac{8}{3}x_{3},$ where $x = [x_{1}, x_{2}, x_{3}]^{T}$. As shown in (Wan and Bernstein, 1995), this system can be globally saturated at the origin with the linear control law $u = -38x_{1}$. The closed-loop system is given by $\dot{x}_{1} = 10(x_{2} - x_{1}), \dot{x}_{2} = -10x_{1} - x_{2} - x_{1}x_{3}, \dot{x}_{3} = x_{1}x_{2} - \frac{8}{3}x_{3}$. We consider the maximal invariant set in $X := \{x \in \mathbb{R}^{3} : \|x\|_{\infty} \leq 30\}$. The closed-loop system is then discretized by the Runge–Kutta–Fehlberg method with discretization period 0.03. It can be verified that the discretized system does not satisfy the condition in Proposition 1 and thus is not exactly immersible into any linear system. We obtain $\nu_{M}$ by gridding over $X$ with $N = 6.4 \times 10^{4}$ and set the trajectory horizon to be $t_{f} = 100$ and the regularization weight to be $\rho = 10^{-4}$. For different values of $M$, we compute $\delta_{M}$ and $O^{L, \delta, \delta}(A_{M}, B_{M}, C_{M})$. It turns out the set $O^{L, \delta, \delta}(A_{M}, B_{M}, C_{M})$ is non-empty only when $M \geq 18$. The curve of $\delta_{M}$ is shown in Figure 2. As we can see from this figure, for small values of $M$, $\delta_{M}$ is large, which leads to an empty robust invariant set $O^{L, \delta, \delta}(A_{M}, B_{M}, C_{M})$ as defined in (16). To illustrate the effect of $M$ on the accuracy of the lifted linear model, trajectories are also shown in Figure 2 for several choices of $M$ starting from the initial state $[-25.8 - 5.6 29.7]^{T}$. 

![Fig. 1. The maximal invariant set $O_{\infty}$ of Example 1.](image1)

![Fig. 2.](image2)

![Fig. 3.](image3)
an invariant set of the discretized system but may not be invariant for the original continuous system.

Fig. 4. The set $T^{-1}(\Omega_k)$.

Fig. 5. Monte Carlo verification for the invariance of $T^{-1}(\Omega_k)$.

6. CONCLUSIONS

We propose an immersion-based method for computing the maximal invariant set of discrete-time nonlinear systems. It characterizes the maximal invariant set using a lifted linear model of the nonlinear system. For certain nonlinear systems, the set computed from the linear model is the exact maximal invariant set. For general cases, the lifted linear system is not exactly equivalent to the nonlinear system and the computed set is can be only considered as an inner approximation of the actual maximal invariant set. In spite of that, this inner approximation is in fact an invariant set that can be made arbitrary close to the actual maximal invariant set by increasing the dimension of the lifted system. Finally, the proposed method is demonstrated on two nonlinear examples.

REFERENCES


