Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Computing invariant sets of discrete-time nonlinear systems via state immersion

Zheming Wang *, **Raphaël M. Jungers** *, **Chong-Jin Ong** **

* *The ICTEAM Institute, UCLouvain, Louvain-la-Neuve,1348, Belgium (email: zheming.wang@uclouvain.be, raphael.jungers@uclouvain.be)*
** *The Department of Mechanical Engineering, National University of Singapore, 117576, Singapore (email:mpeongcj@nus.edu.sg)*

Abstract: In this paper, we propose a method for computing invariant sets of discrete-time nonlinear systems by lifting the nonlinear dynamics into a higher dimensional linear model. In particular, we will focus on the maximal invariant set. Some special types of nonlinear systems can be considered as the projection of a higher dimensional linear system with a state immersion transformation. For such systems, the equivalence between invariant sets of the nonlinear system and its linear equivalent can be also established, which allows to characterize the maximal invariant set of the nonlinear system using a lifted linear model. For general nonlinear systems, we will use linear approximations because equivalent linear models cannot be achieved exactly. To handle mismatch errors, we tighten the constraint set of the lifted linear model, which will lead to an invariant inner approximation of the maximal invariant set.

Keywords: Invariant set, nonlinear systems, state immersion

1. INTRODUCTION

Set invariance theory plays an important role in systems and control for stability analysis and control design, see, for instance (Blanchini, 1999; Blanchini and Miani, 2008) and the references therein. In particular, it is widely used in Model Predictive Control (MPC) (Mayne et al., 2000) for constrained systems.

The problem of computing invariant sets has been studied extensively for different types of systems. The early literature has been devoted to linear systems with polyhedral constraints, see, e.g., (Gilbert and Tan, 1991) and the references therein. In the presence of bounded disturbances in linear systems, robust invariant sets were studied (see, e.g., (Kolmanovsky and Gilbert, 1998; Rakovic et al., 2005; Ong and Gilbert, 2006; Trodden, 2016)). Recently, the authors in (Wang et al., 2019) have proposed an algorithm to deal with nonlinear constraints. However, for nonlinear systems, the computation of invariant sets is even more difficult and complicated. Although some existing algorithms for computing invariant sets of different types of nonlinear systems are available, see, e.g., (Bravo et al., 2005; Alamo et al., 2009; Fiacchini et al., 2010; Sassi and Girard, 2012; Henrion and Korda, 2014; Korda et al., 2014), obtaining an exact invariant set is still a difficult problem for general nonlinear systems. The aforementioned algorithms for nonlinear systems are focused on inner or outer approximations of invariant sets. However, these approximations are not necessarily invariant sets. In this paper, we attempt to get an inner approximation of the maximal invariant set by using a lifted linear model of the nonlinear system. Although this inner approximation is a subset of the maximal invariant set in general, it is in fact also an invariant set. For special classes of nonlinear systems, the technique allows to obtain the exact maximal invariant set.

In order to obtain a lifted linear model, we need first to perform linearization of the nonlinear system, which is one of the most well-known research topics in systems and control. Two classic linearization methods are Jacobian linearization and feedback linearization, see (Khalil, 2002) for a comprehensive view. Another linearization method is the state immersion method, which allows to immerse a nonlinear system into a linear system in a higher dimension, see, e.g.,(Monaco and Normand-Cyrot, 1983; Lee and Marcus, 1988; Menini and Tornambè, 2009). Recently, a new immersion technique has been proposed in (Jungers and Tabuada, 2019) for continuous-time systems by the use of polyflows. This technique often outperforms the Taylor approximation in practice. Inspired by the polyflows approximation, we have developed a similar immersion method (Wang and Jungers, 2020) for discrete-time systems. In this paper, we use such a method to design an algorithm for computing invariant sets of discrete-time nonlinear systems.

The rest of the paper is organized as follows. This section ends with the notation, followed by the next section on the review of preliminary results on invariant sets. Section 3 presents the proposed immersion-based method for computing the maximal invariant of nonlinear systems. In Section 4, we will discuss some computational issues of the proposed method. Numerical examples are provided Section 5. The last section concludes the work.

[1] Raphaël M. Jungers is a FNRS Research Associate. He is supported by the French Community of Belgium, the Walloon Region and the Innoviris Foundation.

Notation. The non-negative integer set is indicated by $\mathbb{Z}^+$. I_n is the $n \times n$ identity matrix and $\mathbf{0}_{n\times m}$ is the $n \times m$ matrix of all zeros (subscript omitted when the dimension is clear). For two set X and Y, $X \ominus Y$ denotes the Minkowski difference. Given a map T, let $T(X)$ denote $\{T(x) : x \in X\}$ and $T^{-1}(Y)$ denote the preimage of the set Y under the map T, i.e., $T^{-1}(Y) := \{x : T(x) \in Y\}$ (T is not necessarily invertible).

2. PRELIMINARIES

We consider discrete-time dynamical systems of the form

$$x(t+1) = f(x(t)), \quad t \in \mathbb{Z}^+ \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function over X. The system is subject to state constraints:

$$x(t) \in X, \quad t \in \mathbb{Z}^+, \tag{2}$$

Let $f^t(x) = \underbrace{f \circ \cdots \circ f}_{t\ times}(x)$ with $f^0(x) = x$ for all $t \in \mathbb{Z}^+$. The following assumptions are made. (**A1**) The function $f(x)$ is continuous with $f(0) = 0$. (**A2**) The set X is compact and contains the origin in its interior. (**A3**) The system (1) is asymptotically stable at the origin in X, i.e., $\lim_{t\to\infty} \|f^t(x)\| = 0, \forall x \in X$.

The definition of an invariant set is given below.

Definition 1. (Blanchini, 1999; Blanchini and Miani, 2008) A nonempty set $Z \subseteq \mathbb{R}^n$ is a positively invariant set for the system (1) if and only if $x \in Z$ implies $f(x) \in Z$.

Invariant sets throughout the paper are all positively invariant sets. Computing an invariant set is a difficult problem even for linear systems, depending on the constraint set. For nonlinear systems, the computation will be more difficult and complicated. For instance, let us consider the computation of the maximal invariant set (Gilbert and Tan, 1991; Kolmanovsky and Gilbert, 1998), which is defined below.

Definition 2. A nonempty set S is the maximal invariant set for the system (1) if and only if S is an invariant set and contains all the invariant sets in X.

A standard algorithm for computing the maximal invariant set was first introduced in (Gilbert and Tan, 1991):

$$O_0 := X, O_{k+1} := O_k \bigcap \{x : f(x) \in O_k\}, k \in \mathbb{Z}^+. \tag{3}$$

With these iterates, one can see that

$$O_k = \{x \in \mathbb{R}^n : f^\ell(x) \in X, 0 \le \ell \le k\}, \quad k \in \mathbb{Z}^+ \tag{4}$$

Thus, the maximal invariant set can be expressed as

$$O_\infty = \{x \in \mathbb{R}^n : f^k(x) \in X, \forall k \in \mathbb{Z}^+\}. \tag{5}$$

Under **A1-A3**, the existence of O_∞ can be guaranteed and the algorithm above terminates in a finite time. The proof for the linear case is given in Theorem 4.1 in (Gilbert and Tan, 1991) and it can be extended to nonlinear systems, see, e.g., Proposition 1 in (Wang and Jungers, 2019).

The key step in the standard algorithm is to check whether the condition $O_{k+1} = O_k$, is reached as k increases from 0. However, for general nonlinear systems, to check this condition, one needs to solve a number of nonconvex optimization problems and the complexity grows with the iteration. Such an issue also arises in the computation of other invariant sets in the the case of nonlinear systems. In this paper, we will attempt to solve the issue of nonlinearity via state immersion. More precisely, we propose to use a lifted linear model of the nonlinear system to compute invariant sets.

3. MAIN RESULTS

3.1 State immersion

First, we give a definition of immersibility of nonlinear systems, see, e.g., (Monaco and Normand-Cyrot, 1983; Lee and Marcus, 1988).

Definition 3. The system (1) is immersible into a linear system in the form of

$$\xi(t+1) = A_\xi \xi(t),\ y(t) = C_\xi \xi(t),\ t \in \mathbb{Z}^+, \tag{6}$$

where $\xi \in \mathbb{R}^{n_\xi}$, $y(t) \in \mathbb{R}^n$, $A_\xi \in \mathbb{R}^{n_\xi \times n_\xi}$ and $C_\xi \in \mathbb{R}^{n \times n_\xi}$, if there exists a map $T : \mathbb{R}^n \to \mathbb{R}^{n_\xi}$ such that $f^t(x) = C_\xi A_\xi^t T(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{Z}^+$. For notational simplicity, let us denote the linear system in (6) by $\Pi(A_\xi, C_\xi)$.

A necessary and sufficient condition for immersibility is given in the following proposition.

Proposition 1. The system (1) is immersible into a linear system in the form of (6) if and only if there exist M and a sequence of matrices $\{\gamma_\ell \in \mathbb{R}^{n\times n}\}_{\ell=0}^M$ such that

$$f^{M+1}(x) = \sum_{\ell=0}^{M} \gamma_\ell f^\ell(x), \forall x \in \mathbb{R}^n \tag{7}$$

Similar arguments can also be found in (Monaco and Normand-Cyrot, 1983; Lee and Marcus, 1988), although the proof is slightly different, see Theorem 1 in (Wang and Jungers, 2020). With the integer M and the matrices $\boldsymbol{\gamma}_M := \{\gamma_\ell \in \mathbb{R}^{n\times n}\}_{\ell=0}^M$ satisfying (7), we can immediately construct a linear system $\Pi(\Gamma(\boldsymbol{\gamma}_M), [I_n\ \mathbf{0}_{n\times(M+1)n}])$, where

$$\Gamma(\boldsymbol{\gamma}_M) := \begin{pmatrix} \mathbf{0} & I_n & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & I_n \\ \gamma_0 & \gamma_1 & \cdots & \gamma_{M-1} & \gamma_M \end{pmatrix}. \tag{8}$$

The condition in (7) implies that the system (1) is immersible into $\Pi(\Gamma(\boldsymbol{\gamma}_M), [I_n\ \mathbf{0}_{n\times(M+1)n}])$ with the transformation map $\mathcal{F}_M(x)$ defined by

$$\mathcal{F}_M(x) := \begin{pmatrix} x \\ \vdots \\ f^M(x) \end{pmatrix}. \tag{9}$$

However, there may exist redundancy in such a transformation. To remove redundancy, we will use *linearly independent* transformations. A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *linearly independent* if its components $\{T_1(x), \cdots, T_m(x)\}$ are *linearly independent.* From this definition, a *linearly independent* map $T : \mathbb{R}^n \to \mathbb{R}^m$ implies $\text{span}\{T(x) : x \in \mathbb{R}^n\} = \mathbb{R}^m$. Otherwise, there exists a vector $c \in \mathbb{R}^m$ such that $c^T T(x) = 0, \forall x \in \mathbb{R}^n$.

With a *linearly independent* transformation, a tight linear model can be obtained, as stated in the following lemma.

Proposition 2. Suppose **A1** holds and the system (1) is immersible into a linear system in the form of (6), there always exist a continuous *linearly independent* map $T : \mathbb{R}^n \to \mathbb{R}^m$ and an observable pair (C, A) such that $AT(x) = T(f(x))$ and $CT(x) = x$ for all $x \in \mathbb{R}^n$. In addition, A is Schur stable if **A2** & **A3** hold.

Proof: From Proposition 1, when the system (1) is immersible to $\Pi(A_\xi, C_\xi)$, there exist M and matrices $\boldsymbol{\gamma}_M := \{\gamma_\ell \in \mathbb{R}^{n\times n}\}_{\ell=0}^M$ such that (7) is satisfied. This implies that

$$\mathcal{F}_M(f(x)) = \Gamma(\boldsymbol{\gamma}_M)\mathcal{F}_M(x) \tag{10}$$

where $\Gamma(\boldsymbol{\gamma}_M)$ is defined in (8) . Suppose there are m *linearly independent* functions that form a basis for the spanning set $\{x_1, \cdots, x_n, \cdots, f_1^M(x), \cdots, f_n^M(x)\}$, let $T(x)$ be the stacked vector of these functions. As all the functions $\{x_1, \cdots, x_n, \cdots, f_1^M(x), \cdots, f_n^M(x)\}$ can be expressed as linear combinations of $T(x)$, there exists a full column rank matrix $P \in \mathbb{R}^{(M+1)n\times m}$ such that

$$\mathcal{F}_M(x) = PT(x), \mathcal{F}_M(f(x)) = PT(f(x)) \tag{11}$$

Hence, from (10), $T(f(x)) = P^+\Gamma(\boldsymbol{\gamma}_M)PT(x)$, where P^+ denotes the pseudo inverse. Letting $A = P^+\Gamma(\boldsymbol{\gamma}_M)P$ and $C = [I_n\ \mathbf{0}_{n\times(M+1)n}]P$, we can get $AT(x) = T(f(x))$ and $CT(x) = x$. When $m = (M+1)n$, $A = \Gamma(\boldsymbol{\gamma}_M)$ and $C = [I_n\ \mathbf{0}_{n\times(M+1)n}]$. From the definition of $\Gamma(\boldsymbol{\gamma}_M)$, $([I_n\ \mathbf{0}_{n\times(M+1)n}], \Gamma(\boldsymbol{\gamma}_M))$ is observable. To show that (C, A) is observable, we need to show that $\mathcal{O}(A, C) := \begin{pmatrix} C \\ \vdots \\ CA^{m-1} \end{pmatrix}$ is full column rank. Since $T(x)$ is *linearly independent*, there exist N points $\{\tilde{x}^1, \cdots, \tilde{x}^N\}$ such that $\text{span}\{T(\tilde{x}^1), \cdots, T(\tilde{x}^N)\} = \mathbb{R}^m$. Thus, for any $z \in \mathbb{R}^m$, there exist $\{\alpha_1, \cdots, \alpha_N\}$ such that $z = \sum_{i=1}^N \alpha_i T(\tilde{x}^i)$, which implies that $Pz = \sum_{i=1}^N \alpha_i PT(\tilde{x}^i) = \sum_{i=1}^N \alpha_i \mathcal{F}_M(x^i)$, where the second equality follows from (11). Since $AT(x) = T(f(x))$ and $CT(x) = x$, $\begin{pmatrix} Cz \\ \vdots \\ CA^M z \end{pmatrix} = \begin{pmatrix} C \\ \vdots \\ CA^M \end{pmatrix} \sum_{i=1}^N \alpha_i T(\tilde{x}^i) = Pz, \forall z \in \mathbb{R}^m$, which implies that $\begin{pmatrix} C \\ \vdots \\ CA^M \end{pmatrix} = P$. Hence, $\text{rank}(\mathcal{O}(A, C)) = m$ and (C, A) is observable. Finally, we show that A is Schur stable under **A2** & **A3**. As $T(x)$ is *linearly independent* and $T(X)$ is compact and contains the origin in the interior, we can choose N points $\{\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^N\}$ inside X such that $\text{span}\{T(\tilde{x}^1), T(\tilde{x}^2), \cdots, T(\tilde{x}^N)\} = \mathbb{R}^m$. Hence, for any $z \in \mathbb{R}^m$, there exist $\{\alpha_1, \alpha_2, \cdots, \alpha_N\}$ such that $z = \sum_{i=1}^N \alpha_i T(\tilde{x}^i)$, which leads to $A^k z = \sum_{i=1}^N \alpha_i A^k T(\tilde{x}^i) = \sum_{i=1}^N \alpha_i T(f^k(\tilde{x}^i)), \forall k \in \mathbb{Z}^+$. The asymptotic stability of the system (1) implies that $A^k z \to 0$ as $k \to \infty$ for any $z \in \mathbb{R}^m$. Therefore, A is asymptotically stable and thus Schur stable. □

3.2 Set invariance under immersion

For systems that are immersible into a linear system (see Definition 3), we can also establish the immersion on invariant sets of the nonlinear system and its linear equivalent, as shown in the following proposition.

Proposition 3. Suppose there exist a continuous map $T : \mathbb{R}^n \to \mathbb{R}^m$ and matrices $A \in \mathbb{R}^{m\times m}, C \in \mathbb{R}^{n\times m}$ such that $AT(x) = T(f(x))$ and $CT(x) = x$ for all $x \in \mathbb{R}^n$. Given the constraint set X, let $Z \subseteq X$ be an invariant set for the system (1) and $\Xi \subseteq \{\xi \in \mathbb{R}^m : C\xi \in X\}$ be an invariant set for $\Pi(A, C)$. Then, (i) $T^{-1}(\Xi) := \{x \in \mathbb{R}^n : T(x) \in \Xi\} \subseteq X$ is invariant for the system (1); (ii) $T(Z) \subseteq \{\xi \in \mathbb{R}^m : C\xi \in X\}$ is invariant for $\Pi(A, C)$.

Proof: (i) As $CT(x) = x$ and $\Xi \subseteq \{\xi \in \mathbb{R}^m : C\xi \in X\}$, we can see that $T^{-1}(\Xi) = \{x \in \mathbb{R}^n : T(x) \in \Xi\} \subseteq \{x \in \mathbb{R}^n : CT(x) \in X\} = X$. Hence, we only need to show the invariance of $T^{-1}(\Xi)$. For any $x \in T^{-1}(\Xi)$, we want to show that $f(x) \in T^{-1}(\Xi)$, i.e., $T(x) \in \Xi$ implies $T(f(x)) \in \Xi$. From the invariance of Ξ, we know that $AT(x) \in \Xi$. This, together with the fact that $AT(x) = T(f(x))$, proves the invariance of $T^{-1}(\Xi)$. (ii) Similarly, we will first show that $T(Z) \subseteq \{\xi \in \mathbb{R}^m : C\xi \in X\}$. For any $\xi' \in T(Z)$, there exists an $x \in Z \subseteq X$ such that $\xi' = T(x)$. From the fact that $CT(x) = x$, we can see that $C\xi' = CT(x) = x \in X$, which implies $T(Z) \subseteq \{\xi \in \mathbb{R}^m : C\xi \in X\}$. Then, we will show the invariance of $T(Z)$. As Z is invariant for system (1), we know that $f(x) \in Z$. Hence, $A\xi = AT(x) = T(f(x)) \in T(Z)$. This completes the proof. □

The results in Proposition 3 allow us to use the lifted linear system to compute the maximal invariant set O_∞ of the system (1). Given any pair (A, C) with $A \in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{n\times m}$, let us define

$$O_\infty^L(A, C) := \{x \in \mathbb{R}^m : CA^k x \in X, \forall k \in \mathbb{Z}^+\} \tag{12}$$

From Theorem 4.1 in (Gilbert and Tan, 1991), $O_\infty^L(A, C)$ exists and can be finitely determined when (C, A) is observable and A is Schur stable. Based on the discussions above, the following theorem can be obtained.

Theorem 1. Suppose **A1-A3** hold, let O_∞ be defined as in (5) for the system (1) with the constraint set X. Assume that the system (1) is immersible into a linear system $\Pi(A_\xi, C_\xi)$ in (6). Then, there exist a continuous *linearly independent* map $T : \mathbb{R}^n \to \mathbb{R}^m$ and an observable pair (C, A) with A being Schur stable such that $O_\infty = T^{-1}(O_\infty^L(A, C))$, where $O_\infty^L(A, C)$ is defined in (12).

Proof: This result can be derived from Propositions 2 & 3 and hence the proof is omitted due to page limitation. □

3.3 Approximate immersion and robustness

Since linear equivalents exist only for very particular classes of systems, we will discuss approximate state immersion and robustness issues. In general cases, we want to find a transformation map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(f^{t+1}(x)) - AT(f^t(x))$ is within some given tolerance for all $x \in X$ and $t \in \mathbb{Z}^+$. With **A1-A3**, the system (1) can be arbitrarily close to the projection of a linear system by increasing the dimension of the lifted space, which is formally stated in the following lemma.

Lemma 1. Suppose **A1-A3** hold, for any given $\delta > 0$, there exist a continuous *linearly independent* map $T : \mathbb{R}^n \to \mathbb{R}^m$, an observable pair (C, A) with A being Schur stable and a matrix $B \in \mathbb{R}^{m\times n}$ such that

$$CT(x) = x, T(f^{t+1}(x)) - AT(f^t(x)) \in BW_\delta \tag{13}$$

for all $x \in X$ and $t \in \mathbb{Z}^+$, where

$$W_\delta := \{w \in \mathbb{R}^n : \|w\|_\infty \le \delta\}. \tag{14}$$

Proof: From **A3**, there always exist M and matrices $\{\gamma_\ell \in \mathbb{R}^{n\times n}\}_{\ell=0}^M$ such that

$$\|f^{M+1+k}(x) - \sum_{\ell=0}^{M} \gamma_\ell f^{\ell+k}(x)\|_\infty \le \delta \tag{15}$$

for all $x \in X$ and $k \in \mathbb{Z}^+$. Hence, $\mathcal{F}_M(f^{k+1}(x)) - \Gamma(\boldsymbol{\gamma}_M)\mathcal{F}_M(f^k(x)) \in \begin{pmatrix} \mathbf{0}_{Mn\times n} \\ I_n \end{pmatrix} W_\delta$ for all $x \in X$ and $k \in \mathbb{Z}^+$. Let $T(x)$ be the m *linearly independent* functions that form a basis for the spanning set of $\{x_1, x_2, \cdots, f_n^M(x)\}$. We can find a full column rank matrix $P \in \mathbb{R}^{(M+1)n\times m}$ such that $T(f^{k+1}(x)) - P^+\Gamma(\boldsymbol{\gamma}_M)PT(f^k(x)) \in P^+ \begin{pmatrix} \mathbf{0}_{Mn\times n} \\ I_n \end{pmatrix} W_\delta, \forall x \in \mathbb{R}^n, k \in \mathbb{Z}^+$. Letting $A = P^+\Gamma(\boldsymbol{\gamma}_M)P, C = [I_n \ \mathbf{0}_{n\times(M+1)n}]P$ and $B = P^+ \begin{pmatrix} \mathbf{0}_{Mn\times n} \\ I_n \end{pmatrix}$, we can get (13). Then, we will show that we can always find an observable (C, A) with A being Schur stable. Since $T(x)$ is *linearly independent*, there exist N points $\{\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^N\}$ inside X such that $\text{span}\{T(\tilde{x}^1), T(\tilde{x}^2), \cdots, T(\tilde{x}^N)\} = \mathbb{R}^m$. Hence, for any $z \in \mathbb{R}^m$, there exist $\{\alpha_1, \alpha_2, \cdots, \alpha_N\}$ such that $z = \sum_{i=1}^N \alpha_i T(\tilde{x}^i)$. By increasing M, $AT(x) - T(f(x))$ can be made arbitrary small for any $x \in X$. Following similar arguments in Proposition 2, we can see that $\begin{pmatrix} C \\ \vdots \\ CA^M \end{pmatrix} z - Pz$ can be made arbitrarily small for any z. Hence, we can always find M such that $\begin{pmatrix} C \\ \vdots \\ CA^M \end{pmatrix}$ is full column rank, which implies (C, A) is observable. Similarly, we can also find a Schur stable A by increasing the dimension. □

To account for the mismatch between the system (1) and the linear system $\Pi(A, C)$ that satisfies (13) in Lemma 1, we compute a tightened subset of $O_\infty^L(A, C)$, instead of $O_\infty^L(A, C)$. Given (A, B, C) and $\delta > 0$, let us define

$$O_\infty^{L,\delta}(A, B, C) := \{x \in \mathbb{R}^m : \ CA^k x \in X \ominus \sum_{\ell=0}^{k-1} CA^\ell BW_\delta, \ \forall k \in \mathbb{Z}^+\} \tag{16}$$

where W_δ is given in (14). From (Kolmanovsky and Gilbert, 1998), the set $O_\infty^{L,\delta}(A, B, C)$ is nonempty when $\sum_{\ell=0}^\infty CA^\ell BW_\delta \subseteq X$ and it is the *maximal robust invariant set*, that is, the maximal invariant set for the disturbed system $x^+ = Ax + Bw$ where the disturbance w is constrained in W_δ.

Theorem 2. Suppose **A1-A3** hold, let O_∞ be defined as in (5) for the system (1). For any given $\delta > 0$, let W_δ be defined in (14). Consider a continuous *linearly independent* map $T : \mathbb{R}^n \to \mathbb{R}^m$, an observable pair (C, A) with A being Schur stable and a matrix $B \in \mathbb{R}^{m\times n}$ that satisfy (13), the following results hold: (i) $T^{-1}(O_\infty^{L,\delta}(A, B, C)) \subseteq O_\infty$; (ii) $T^{-1}(O_\infty^{L,\delta}(A, B, C))$ is invariant for the system (1), where $O_\infty^{L,\delta}(A, B, C)$ is defined as in (16).

Proof: First, we show that $T^{-1}(O_\infty^{L,\delta}(A, B, C)) \subseteq O_\infty$. For any $x \in T^{-1}(O_\infty^{L,\delta}(A, B, C)) \subseteq O_\infty$, we know that $CA^kT(x) \in X \ominus \sum_{\ell=0}^{k-1} CA^\ell BW_\delta$ for all $k \in \mathbb{Z}^+$. From the fact that $T(f^{k+1}(x)) - AT(f^k(x)) \in BW_\delta$ for all $k \in \mathbb{Z}^+$, we know that $T(f^k(x)) \in A^kT(x) + \sum_{\ell=0}^{k-1} A^\ell BW_\delta$ for all $k \in \mathbb{Z}^+$. Hence, for all $k \in \mathbb{Z}^+$, $f^k(x) = CT(f^k(x)) \in CA^kT(x) + \sum_{\ell=0}^{k-1} CA^\ell BW_\delta \subseteq X \ominus \sum_{\ell=0}^{k-1} CA^\ell BW_\delta + \sum_{\ell=0}^{k-1} CA^\ell BW_\delta \subseteq X$, where the last inclusion follows from the properties of the Minkowski difference, see, e.g., Theorem 2.1 in (Kolmanovsky and Gilbert, 1998). This implies that $x \in O_\infty$. Therefore, $T^{-1}(O_\infty^{L,\delta}(A, B, C)) \subseteq O_\infty$. To prove the invariance of $T^{-1}(O_\infty^{L,\delta}(A, B, C))$, we need to show that $f(x) \in T^{-1}(O_\infty^{L,\delta}(A, B, C))$, which means that $CA^kT(f(x)) \in X \ominus \sum_{\ell=0}^{k-1} CA^\ell BW_\delta$ for all $k \in \mathbb{Z}^+$. Since $T(f(x)) \in AT(x) + BW_\delta$, for any $k \in \mathbb{Z}^+$, $CA^kT(f(x)) \in CA^{k+1}T(x) + CA^kBW_\delta \subseteq X \ominus \sum_{\ell=0}^{k} CA^\ell BW_\delta + CA^kBW_\delta \subseteq X \ominus \sum_{\ell=0}^{k-1} CA^\ell BW_\delta$, where the last inclusion again follows from Theorem 2.1 in (Kolmanovsky and Gilbert, 1998). □

4. COMPUTATIONAL ISSUES

From the discussions in the previous section, to obtain a linear approximation and the associated transformation, for the given $\delta > 0$, we need to find $M \in \mathbb{Z}^+$ and $\boldsymbol{\gamma}_M := \{\gamma_\ell \in \mathbb{R}^{n\times n}\}_{\ell=0}^M$ such that

$$\|g_k^M(x, \boldsymbol{\gamma}_M)\|_\infty \le \delta, \forall x \in X, k \in \mathbb{Z}^+ \tag{17}$$

where

$$g_k^M(x, \boldsymbol{\gamma}_M) = f^{k+M+1}(x) - \sum_{\ell=0}^{M} \gamma_\ell f^{k+\ell}(x). \tag{18}$$

We will solve this problem numerically by griding or random sampling. Suppose we take N initial states inside X, denoted by $\omega_N := \{x^1, x^2, \cdots, x^N\}$, and generate the trajectory $\{x, f(x), \cdots, f^{t_f}(x)\}$ with a sufficiently long horizon $t_f \in \mathbb{Z}^+$ for each $x \in \omega_N$, the following least squares regression problem is formulated for any $M \in \mathbb{Z}^+$,

$$\min_{\boldsymbol{\gamma}_M} \sum_{x\in\omega_N} \sum_{k=0}^{t_f-M-1} \|g_k^M(x, \boldsymbol{\gamma}_M)\|_2^2 + \rho\|\boldsymbol{\gamma}_M\|_F^2 \tag{19}$$

where $g_k^M(x, \boldsymbol{\gamma}_M)$ is given in (18) and $\rho > 0$. Let the solution of Problem (19) be denoted by $\hat{\boldsymbol{\gamma}}_M$. With this solution, we can compute

$$\hat{\delta}_M = \max_{x\in\omega_N, 0\le k\le t_f-M-1} \|g_k^M(x, \hat{\boldsymbol{\gamma}}_M)\|_\infty. \tag{20}$$

From $\hat{\boldsymbol{\gamma}}_M$, a linear system $\Pi(\Gamma(\hat{\boldsymbol{\gamma}}_M), [I_n \ \mathbf{0}_{n\times(M+1)n}])$ can be obtained with the transformation $\mathcal{F}_M(x)$. By checking and removing the redundancy, we get a *linearly independent* transformation map $T : \mathbb{R}^n \leftarrow \mathbb{R}^m$ and a full column rank matrix $P \in \mathbb{R}^{(M+1)n\times m}$ such that $\mathcal{F}_M(x) = PT(x), \forall x$, which implies that

$$T(x) = P^+\mathcal{F}_M(x) \tag{21}$$

where P^+ denotes the pseudo inverse. Then, we can get a linear system $\Pi(A_M, C_M)$ with $A_M = P^+\Gamma(\boldsymbol{\gamma}_M)P$ and $C_M = [I_n \ \mathbf{0}_{n\times(M+1)n}]P$, and a matrix $B_M = P^+ \begin{pmatrix} \mathbf{0}_{Mn\times n} \\ I_n \end{pmatrix}$. Note that $\mathcal{F}_M(x)$ is already *linearly independent* in many real applications. We will increase M and repeat the procedure above until $\hat{\delta}_M$ is smaller than some given $\delta > 0$. Let the solution from the computations above be denoted by $(\hat{M}, \hat{\delta}_{\hat{M}}, A_{\hat{M}}, B_{\hat{M}}, C_{\hat{M}})$. As N increases, it can be made arbitrary close to the actual solution.

For a given $\delta > 0$, we can obtain $\Pi(A_{\hat{M}}, C_{\hat{M}})$ and $B_{\hat{M}}$, as shown above. Then, we will compute $O_\infty^{L,\hat{\delta}_{\hat{M}}}(A_{\hat{M}}, B_{\hat{M}}, C_{\hat{M}})$ using the standard algorithm (Gilbert and Tan, 1991). For notational simplicity, let

$$\Omega_\delta := O_\infty^{L,\hat{\delta}_{\hat{M}}}(A_{\hat{M}}, B_{\hat{M}}, C_{\hat{M}}). \tag{22}$$

If Ω_δ is empty, we will have to reduce δ and repeat the computations above again.

5. SIMULATION RESULTS

Example 1. We consider the following nonlinear system: $x^+ = f(x) := [2x_1^2 + x_2, \;\; -2\left(2x_1^2 + x_2\right)^2 - 0.8x_1]^T$, where $x = [x_1, \;\; x_2]^T$. This system is globally stable at the origin. With the transformation $T(x) = \mathcal{F}_1(x)$ it is globally immersible to $\Pi(A, C)$ with $A = [0\;0\;1\;0; 0\;0\;0\;1; -0.8\;0\;0\;0; 0\;0.64\;\; -1.44\;0], C = [I_2 \;\; \mathbf{0}_{2\times 2}]$ Let us consider the state constraint set $X := \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$. We can easily compute $O_\infty^L(A, C)$, expressed by $\{\xi \in \mathbb{R}^4 : G\xi \leq \mathbf{1}_4, \|\xi\|_\infty \leq 1\}$ with $G = [1.152\;0\;0\;0.64; -1.152\;0\;0\; -0.64; 0\;0.64\; -1.44\;0; 0\; -0.64 1.44\;0]$. Then, the maximal invariant set O_∞ of the original system can be expressed as $T^{-1}(O_\infty^L(A, C)) = \{x \in \mathbb{R}^2 : GT(x) \leq \mathbf{1}_4, \|T(x)\|_\infty \leq 1\}$, which is shown in Figure 1.

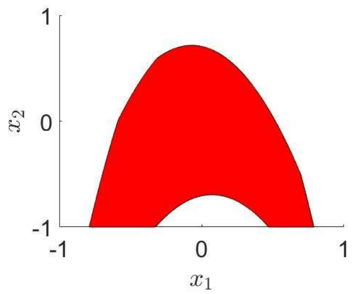


Fig. 1. The maximal invariant set O_∞ of Example 1.

Example 2. Consider a controlled Lorenz system: $\dot{x}_1 = 10(x_2 - x_1), \dot{x}_2 = 28x_1 - x_2 - x_1x_3 + u, \dot{x}_3 = x_1x_2 - \frac{8}{3}x_3$, where $x = [x_1, \; x_2, \; x_3]^T$. As shown in (Wan and Bernstein, 1995), this system can be globally stabilized at the origin with the linear control law $u = -38x_1$. The closed-loop system is given by $\dot{x}_1 = 10(x_2 - x_1), \dot{x}_2 = -10x_1 - x_2 - x_1x_3, \dot{x}_3 = x_1x_2 - \frac{8}{3}x_3$. We consider the maximal invariant set in $X := \{x \in \mathbb{R}^3 : \|x\|_\infty \leq 30\}$. The closed-loop system is then discretized by the Runge–Kutta–Fehlberg method with discretization period 0.03. It can be verified that the discretized system does not satisfy the condition in Proposition 1 and thus is not exactly immersible into any linear system. We obtain ω_N by gridding over X with $N = 6.4 \times 10^4$ and set the trajectory horizon to be $t_f = 100$ and the regularization weight to be $\rho = 10^{-4}$. For different values of M, we compute $\hat{\delta}_M$ and $O_\infty^{L,\hat{\delta}_M}(A_M, B_M, C_M)$. It turns out the set $O_\infty^{L,\hat{\delta}_M}(A_M, B_M, C_M)$ is non-empty only when $M \geq 18$. The curve of $\hat{\delta}_M$ is shown in Figure 2. As we can see from this figure, for small values of M, $\hat{\delta}_M$ is large, which leads to an empty robust invariant set $O_\infty^{L,\hat{\delta}_M}(A_M, B_M, C_M)$ as defined in (16). To illustrate the effect of M on the accuracy of the lifted linear model, trajectories are also shown in Figure 2 for several choices of M starting from the initial state $[-25.8 \;\; -5.6 \; 29.7]^T$.

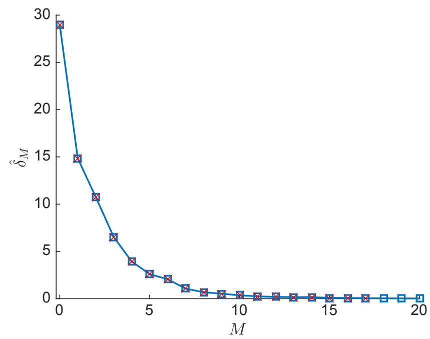


Fig. 2. Mismatch errors for different values of M: those with empty $O_\infty^{L,\hat{\delta}_M}(A_M, B_M, C_M)$ are labeled by ×.

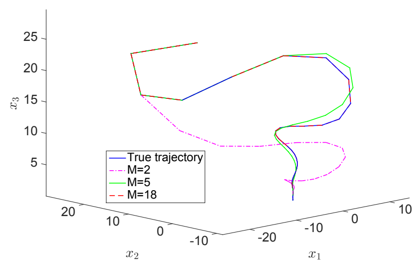


Fig. 3. Trajectories for different values of M with the initial state $[-25.8 \;\; -5.6 \; 29.7]^T$.

From Figure 2, to get a non-empty set, δ has to be smaller than $\hat{\delta}_{17} = 5.34 \times 10^{-2}$. Without the preprocessing in Figure 2, the value of δ can be obtained by bisection. In the simulation, we set $\delta = 5 \times 10^{-2}$ and obtain $\Omega_\delta = O_\infty^{L,\hat{\delta}_{18}}(A_{18}, B_{18}, C_{18})$, which is a 57-dimensional polytope expressed as $\{\xi \in \mathbb{R}^{57} : H\xi \leq \mathbf{1}_{202}\}$, where $H \in \mathbb{R}^{202\times 57}$. Then, the invariant set for the original system $T^{-1}(\Omega_\delta)$ is shown in Figure 4. To verify the invariance of the set $T^{-1}(\Omega_\delta)$, we generate a random data set $\mathcal{D}$ inside X with 10^6 points. For $k \in \mathbb{Z}^+$, let $|\mathcal{D} \bigcap O_k|$ denote the number of points inside O_k. Similarly, $|\mathcal{D} \bigcap T^{-1}(\Omega_\delta)|$ is the number of points inside $T^{-1}(\Omega_\delta)$. From the Monte Carlo simulation in Figure 5, we can see that O_3 can be approximately considered as O_∞ and that $T^{-1}(\Omega_\delta)$ and O_3 have almost the same size. Finally, we validate the invariance of $T^{-1}(\Omega_\delta)$ with the data set $\mathcal{D}$ directly. For any $x \in \mathcal{D}_{in} := \mathcal{D} \bigcap T^{-1}(\Omega_\delta)$, we check the feasibility of its successor x^+ with respect to $T^{-1}(\Omega_\delta)$. Let the feasibility of any point $x \in \mathbb{R}^n$ be measured by $\max\{HT(x) - \mathbf{1}_{202}\}$. For the convenience of visualization, the points in $\mathcal{D}_{in}$ are sorted in descending order according to the feasibility measure $\max\{HT(x) - \mathbf{1}_{202}\}$. Let $\mathcal{D}_{in}^+$ denote the successors of the states in $\mathcal{D}_{in}$. The feasibility measures for $\mathcal{D}_{in}$ and $\mathcal{D}_{in}^+$ are given in Figure 5b, from which we can conclude the invariance of $T^{-1}(\Omega_\delta)$. Note that $T^{-1}(\Omega_\delta)$ is an invariant set of the discretized system but may not be invariant for the original continuous system.

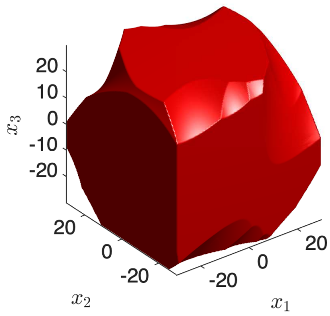
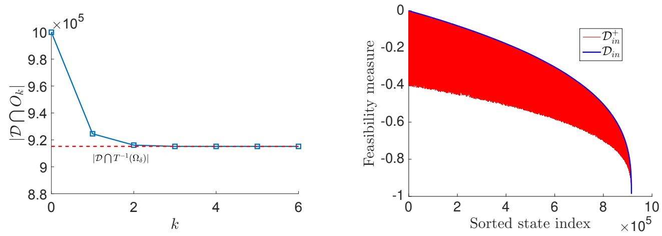


Fig. 4. The set $T^{-1}(\Omega_\delta)$.



(a) (b)

Fig. 5. Monte Carlo verification for the invariance of $T^{-1}(\Omega_\delta)$.

6. CONCLUSIONS

We propose an immersion-based method for computing the maximal invariant set of discrete-time nonlinear systems. It characterizes the maximal invariant set using a lifted linear model of the nonlinear system. For certain nonlinear systems, the set computed from the linear model is the exact maximal invariant set. For general cases, the lifted linear system is not exactly equivalent to the nonlinear system and the computed set is can be only considered as an inner approximation of the actual maximal invariant set. In spite of that, this inner approximation is in fact an invariant set that can be made arbitrary close to the actual maximal invariant set by increasing the dimension of the lifted system. Finally, the proposed method is demonstrated on two nonlinear examples.

REFERENCES

Alamo, T., Cepeda, A., Fiacchini, M., and Camacho, E.F. (2009). Convex invariant sets for discrete-time lur'e systems. *Automatica*, 45(4), 1066–1071.

Blanchini, F. (1999). Set invariance in control. *Automatica*, 35(11), 1747–1767.

Blanchini, F. and Miani, S. (2008). *Set-Theoretic Methods in Control*. Birkhauser.

Bravo, J.M., Limón, D., Alamo, T., and Camacho, E.F. (2005). On the computation of invariant sets for constrained nonlinear systems: An interval arithmetic approach. *Automatica*, 41(9), 1583–1589.

Fiacchini, M., Alamo, T., and Camacho, E.F. (2010). On the computation of convex robust control invariant sets for nonlinear systems. *Automatica*, 46(8), 1334–1338.

Gilbert, E. and Tan, K.T. (1991). Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36, 1008–1020.

Henrion, D. and Korda, M. (2014). Convex computation of the region of attraction of polynomial control systems. *IEEE Transactions on Automatic Control*, 59(2), 297–312.

Jungers, R.M. and Tabuada, P. (2019). Non-local linearization of nonlinear differential equations via polyflows. In *Proceedings of the American Control Conference*.

Khalil, H.K. (2002). *Nonlinear systems*. Upper Saddle River, NJ:Prentice-Halll.

Kolmanovsky, I. and Gilbert, E.G. (1998). Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4, 317–367.

Korda, M., Henrion, D., and Jones, C.N. (2014). Convex computation of the maximum controlled invariant set for polynomial control systems. *SIAM Journal on Control and Optimization*, 52(5), 2944–2969.

Lee, H. and Marcus, S.I. (1988). Immersion and immersion by nonsingular feedback of a discrete-time nonlinear system into a linear system. *IEEE transactions on Automatic Control*, 33(5), 479–483.

Mayne, D.Q., Rawlings, J.B., Rao, C.V., and Scokaert, P.O.M. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, 36(6), 789–814.

Menini, L. and Tornambè, A. (2009). Linearization through state immersion of nonlinear systems admitting lie symmetries. *Automatica*, 45(8), 1873–1878.

Monaco, S. and Normand-Cyrot, D. (1983). The immersion under feedback of a multidimensional discrete-time non-linear system into a linear system. *International Journal of Control*, 38(1), 245–261.

Ong, C.J. and Gilbert, E.G. (2006). The minimal disturbance invariant set: Outer approximations via its partial sums. *Automatica*, 42(9), 1563–1568.

Rakovic, S.V., Kerrigan, E.C., Kouramas, K.I., and Mayne, D.Q. (2005). Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on Automatic Control*, 50(3), 406–410.

Sassi, M.A.B. and Girard, A. (2012). Computation of polytopic invariants for polynomial dynamical systems using linear programming. *Automatica*, 48(12), 3114–3121.

Trodden, P. (2016). A one-step approach to computing a polytopic robust positively invariant set. *IEEE Transactions on Automatic Control*, 61(12), 4100–4105.

Wan, C.J. and Bernstein, D.S. (1995). Nonlinear feedback control with global stabilization. *Dynamics and Control*, 5(4), 321–346.

Wang, Z. and Jungers, R.M. (2019). Data-driven computation of invariant sets of discrete time-invariant black-box systems. *arXiv preprint arXiv:1907.12075*.

Wang, Z. and Jungers, R.M. (2020). A data-driven immersion technique for linearization of discrete-time nonlinear systems. *IFAC-PapersOnLine*.

Wang, Z., Jungers, R.M., and Ong, C.J. (2019). Computation of the maximal invariant set of linear systems with quasi-smooth nonlinear constraints. In *Proceedings of the European Control Conference*, 3803–3808.