

Monotonically Convergent Iterative Learning Control for Piecewise Affine Systems

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Abstract: Piecewise affine (PWA) systems enable modelling of systems that encompass hybrid dynamics and nonlinear effects. The aim of this paper is to develop an ILC framework for PWA systems. A new approach to analyse monotonic convergence is developed for PWA systems. This is achieved by exploiting the incremental ℓ_2 -gain leading to sufficient LMI conditions guaranteeing monotonic convergence. An example confirms the monotonic convergence property for ILC applied to a mass-spring-damper system with a one-sided spring.

Keywords: Learning Control

1. INTRODUCTION

Iterative learning control (ILC) can achieve high performance for systems that perform repetitive tasks (Bristow et al., 2006; Moore, 2012). The key idea of ILC is to iteratively determine an input signal that compensates for an unknown trial-invariant disturbance, e.g., a reference trajectory. By learning from the error signal observed during previous iterations a control input signal is computed that compensates the trial-invariant disturbance.

A key requirement in ILC is monotonic convergence of either the sequence of control inputs or the sequence of error signals. The monotonic convergence property ensures that large learning transients are avoided for any trial-invariant disturbance. When analysing an ILC algorithm, mainly linear time-invariant (LTI) models are considered. Analysis methods exist for ILC algorithms ranging from simple P -type ILC setups (Arimoto et al., 1984) to model inversion based ILC (van Zundert and Oomen, 2018).

Iterative learning control has been extended to classes of systems beyond LTI systems. Successful applications of this are, among many others: non-equidistant sampled systems that lead to a linear periodically time-varying (LPTV) model (van Zundert and Oomen, 2019); nonlinear systems, e.g., robotic manipulators (Horowitz et al., 1991); hybrid systems (Spiegel and Barton, 2019); linear parameter varying systems (LPV) (de Rozario et al., 2017).

Piecewise affine (PWA) systems can represent a broad range of behaviours, including dry friction (Shaw, 1986), systems with piecewise affine elements such as a one-sided spring (Heertjes et al., 1997), or linear systems that are in closed-loop with a hybrid controller, e.g., a reset controller

(Clegg, 1958). Moreover, PWA systems provide accurate approximations of nonlinear systems (Sontag, 1981).

Some existing nonlinear ILC analysis approaches may be applicable to the analysis of ILC for PWA systems, for an overview see, e.g., Xu (2011), but they do not exploit the specific properties of PWA systems, and feature some notable limitations. For example, incremental-output-dissipative systems are guaranteed to converge when exploiting a P -type ILC with a sufficiently small gain and assuming a zero-error trajectory is reachable (Arimoto and Naniwa, 2000; Quintanilla and Wen, 2008). This includes some PWA systems, but the reachability assumption is not always practical and the allowable controller class is limited. Monotonic convergence of a broader class of controllers can be checked via a differential ℓ_2 -gain analysis of the mapping of the input sequence from one trial to the next (Kong and Manchester, 2017). However, the analysis of this differential ℓ_2 -gain is dependent on the trial-invariant disturbance. Hence, guaranteeing monotonic convergence for any trial-invariant disturbance is not possible. Moreover, this method leads to solving an infinite-dimensional optimization problem, which is made tractable through gridding the state space, making the results local to the grid points rather than global.

Although PWA systems are an important model class for ILC algorithms, at present analysing monotonic convergence of general ILC algorithms applied to PWA systems is not addressed. The aim of this paper is to develop a global monotonic convergence analysis method for a general ILC setup applied to PWA systems subject to an unknown trial-invariant disturbance.

The main contribution of this paper is a monotonic convergence analysis method for a general ILC setup applied to a PWA system subject to an unknown trial-invariant disturbance. This analysis method is based on the incremental ℓ_2 -gain (van der Schaft, 2016), which is closely related to

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the incremental output-dissipativity property considered by Arimoto and Naniwa (2000) and the differential \mathcal{L}_2 -gain property considered by Kong and Manchester (2017). By exploiting LMI based analysis techniques for PWA systems this method leads to global guarantees on monotonic convergence.

This paper is organized as follows. In Section 2, the problem setup is outlined. In Section 3, an analysis method for a general ILC setup applied to nonlinear systems is introduced, including elaborating on its connection to existing methods. In Section 4 the analysis method is exploited to derive a computationally tractable method to analyse ILC applied to a PWA system. In Section 5, a numerical simulation study confirms monotonic convergence. Finally, in Section 6, conclusions are drawn.

Proofs will be published elsewhere.

1.1 Notation and Definitions

The ℓ_2 -norm of a discrete-time signal $x(k) \in \mathbb{R}^n, k \in \mathbb{Z}_{\geq 0}$ is given by $\|x\|_2 := \sqrt{\sum_{k=0}^{\infty} (x(k))^T x(k)}$. The set of signals with finite ℓ_2 -norm are denoted by ℓ_2 . The truncation of a signal x on the interval $[0, T]$ is defined as

$$x_{[0,T]}(k) = \begin{cases} x(k), & \text{if } 0 \leq k < T \\ 0, & \text{if } k \geq T \end{cases} \quad (1)$$

Definition 1.1. (Monotonic convergence towards a fixed point.) A sequence $\{Y_i\}_{i \in \mathbb{Z}_{\geq 0}}, Y_i \in X$ is said to converge monotonically in the p -norm, $p \in \{1, 2, \dots\}$, to a unique fixed point $Y_{\infty} \in X$, if there exists a $\kappa \in [0, 1)$ such that

$$\|Y_{j+1} - Y_{\infty}\|_p \leq \kappa \|Y_j - Y_{\infty}\|_p \quad (2)$$

is satisfied for all $Y_j \in X, j \in \mathbb{Z}_{\geq 0}$.

2. PROBLEM FORMULATION

2.1 Piecewise Affine Systems

Discrete-time piecewise affine (PWA) systems J of the form

$$\begin{aligned} x^J(k+1) &= A_i^J x^J(k) + a_i^J + B_i^J u^J(k) & \text{if } x^J(k) \in \Omega_i \\ y^J(k) &= C_i^J x^J(k) + D_i^J u^J(k) & \text{if } x^J(k) \in \Omega_i \end{aligned} \quad (3)$$

are considered, with $x^J \in \mathbb{R}^{n_x}$ the state, $u^J \in \mathbb{R}^{n_u}$ the control input, and $y^J \in \mathbb{R}^{n_y}$ the output. The matrices $A_i^J \in \mathbb{R}^{n_x \times n_x}, a_i^J \in \mathbb{R}^{n_x}, B_i^J \in \mathbb{R}^{n_x \times n_u}, C_i^J \in \mathbb{R}^{n_y \times n_x}, D_i^J \in \mathbb{R}^{n_y \times n_u}, i \in \{1, \dots, p\}$. Notice that if $a_i = 0, i \in \{1, \dots, p\}$ the system (3) reduces to a piecewise linear system. Each of the sets $\Omega_i, i \in \{1, \dots, p\}$ correspond to a mode of the PWA system such that if $x(k) \in \Omega_i$ at time $k \in \mathbb{N}$ then the system's dynamics at time k are defined by the i -th affine system given by $(A_i, B_i, C_i, D_i, a_i)$. The sets Ω_i are polyhedra that are defined by

$$\Omega_i = \{x \in \mathbb{R}^n \mid E_i x + \varepsilon_i \geq 0\} \quad (4)$$

with $E_i \in \mathbb{R}^{d \times n_x}, \varepsilon_i \in \mathbb{R}^d$. The intersection of the interiors of each two polyhedra is empty, i.e., $\text{int}(\Omega_i) \cap \text{int}(\Omega_j) = \emptyset$ for all $i \neq j, i, j \in \{1, \dots, p\}$ and the polyhedra span the complete state-space, i.e., $\cup_i \Omega_i = \mathbb{R}^{n_x}$. In case of discontinuities on the boundaries of the polyhedra, wellposedness of the solutions is ensured by changing some of the inequalities in (4) to strict inequalities, i.e., $(E_i)^j x - (\varepsilon_i)^j < 0$ such that the intersection of two polyhedra is empty, i.e., $\Omega_i \cap \Omega_j = \emptyset$ (Borrelli et al., 2017).

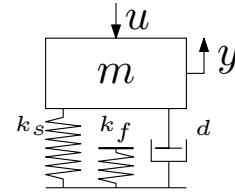


Fig. 1. Mass-spring-system with one-sided spring

2.2 Applications

Piecewise affine systems that can be written in the form (3) are relevant for a wide variety of applications, for instance:

- systems subject to dry friction, see, e.g., Shaw (1986),
- mechanical systems that include one-sided springs, see, e.g., Heertjes et al. (1997),
- linear systems that are in closed-loop with a hybrid controller such as a reset controller, see, e.g., Clegg (1958).

Next, it is shown how a mechanical system that includes a one-sided spring can be described by a PWA system.

Example 2.1. The mass-spring-damper system depicted in Fig. 1 is considered. This system can be modelled with the following equation of motion

$$m \frac{d^2}{dt^2} \bar{y}(t) + d \frac{d}{dt} \bar{y}(t) + k_s \bar{y}(t) + F(\bar{y}(t)) = \bar{u}(t) \quad (5)$$

where $\bar{y}(t)$ denotes the position, $\bar{u}(t)$ denotes the control input, and the parameters m, d and k_s denote the mass, damping coefficient, and spring constant of the linear spring, respectively. The one-sided spring is described by

$$F(\bar{y}) = \begin{cases} k_f \bar{y} & \text{if } \bar{y} \leq 0 \\ 0 & \text{if } \bar{y} > 0. \end{cases} \quad (6)$$

This system can be described by two linear mass-spring-damper systems that switch when $\bar{y} = 0$, when $\bar{y} \leq 0$ the stiffness of the spring is $k_s + k_f$ and when $\bar{y} > 0$ the stiffness of the spring is k_s . In state-space form this is can be described by the following system with state $\bar{x} := [\bar{y} \ \dot{\bar{y}}]^T$,

$$\begin{aligned} \dot{\bar{x}} &= \begin{cases} A_1 \bar{x} + B \bar{u} & \text{if } E \bar{x} \geq 0 \\ A_2 \bar{x} + B \bar{u} & \text{if } E \bar{x} < 0 \end{cases} \\ \bar{y} &= C \bar{x} \end{aligned} \quad (7)$$

with

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ -\frac{d}{m} & -\frac{k_s + k_f}{m} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -\frac{d}{m} & -\frac{k_s}{m} \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \ 0], E = [-1 \ 0] \end{aligned} \quad (8)$$

The output of the system is sampled with a constant sampling interval h , i.e.,

$$y(k) = \bar{y}(hk), k \in \mathbb{N}, \quad (9)$$

and the digital control input, $u(k)$, is connected to the analog world by a zero-order-hold device, i.e.,

$$\bar{u}(t) = u(k), \text{ if } t \in [hk, (k+1)h). \quad (10)$$

For simplicity it is assumed that the transition between contact and separation of the one-sided spring only takes place at the sampling instances. This allows for a discretization of each of the individual modes leading to a discrete-time PWA system of the form (3) with 2 modes defined by the following matrices

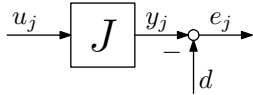


Fig. 2. ILC setup.

$$\begin{aligned} A_i^J &= e^{A_i h}, & B_i^J &= A_i^{-1}(e^{A_i h} - I)B, & C_i^J &= C, \\ D_i^J &= 0, & E_1 &= -E_2 = E, & \varepsilon_i &= 0 \end{aligned} \quad (11)$$

for $i \in \{1, 2\}$.

This shows that a mechanical system with a one-sided spring is described by a PWA system of the form (3).

2.3 What about ILC for PWA systems?

ILC applied to LTI systems ILC can lead to high performance gains, when systems are performing repeating tasks. The ILC setup considered in this paper depicted in Fig. 2. This setup is given by

$$y_j = Ju_j - d, \quad (12)$$

where J denotes a system that can represent either an open-loop or closed-loop system. The system performs trials with a finite length of $N_l \in \mathbb{N}$. The index $j \in \mathbb{Z}_{\geq 0}$ denotes the j -th trial. The control input and output at trial j are denoted by $u_j(k) \in \mathbb{R}^{n_u}$, $n_u \in \mathbb{N}$ and $y_j(k) \in \mathbb{R}^{n_y}$, $n_y \in \mathbb{N}$, respectively. Moreover, a trial-invariant disturbance d , e.g., a desired reference, is present, leading to the error

$$e_j(k) = y_j(k) - d(k). \quad (13)$$

The key idea of ILC is to minimize the error e_j by learning from past data. To achieve this the observed error e_j and control input u_j during trial j are used to determine the control input u_{j+1} for task $j+1$. One of the most common ILC update laws is of the following form,

$$u_{j+1} = Qu_j + Le_j, \quad (14)$$

where Q and L are LTI discrete-time filters.

When designing the ILC update law (14), typically the aim is to achieve convergence of the sequence of control inputs $\{u_j\}_{j \in \mathbb{Z}_{\geq 0}}$. To avoid large learning transients, monotonic convergence is desired (Bristow et al., 2006). A well-known result to analyse monotonic convergence in the 2-norm of the ILC setup (12), with J an LTI system, is checking if the condition,

$$\|Q - LJ\|_2 < 1 \quad (15)$$

is satisfied, see e.g., Moore et al. (1992).

ILC for PWA systems When considering a PWA system such as the mass-spring-damper system with one-sided-spring, the ILC update law can be designed in a similar fashion, evaluating condition (15) for both modes which will be referred to as J_1 and J_2 .

Next, an ILC update law (14) with Q and L filters is designed such that

$$\|Q - LJ_1\|_{\ell_2} = 0.9793 \text{ and } \|Q - LJ_2\|_{\ell_2} = 0.9877, \quad (16)$$

thereby satisfying (15) for each individual mode. Clearly, when applying ILC to each of the individual modes the ILC update law would lead to monotonic convergence of the sequence of control inputs. However, when applying these Q and L filters to the PWA system J convergence of the sequence of control inputs is not achieved. In Fig. 3 and Fig. 4, the norm of the control input and error at each

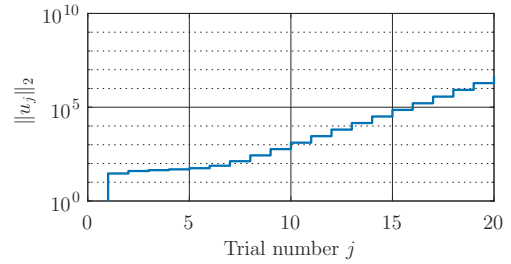


Fig. 3. Norm $\|u_j\|_2$ when applying ILC to a mass-spring-damper system with a one-sided spring. This norm is increasing indicating that convergence is not achieved.

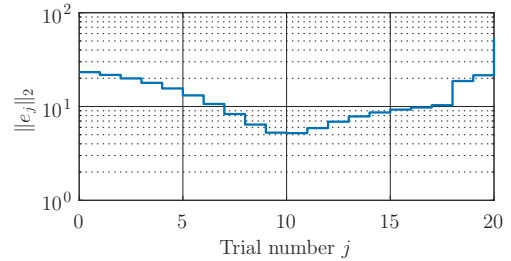


Fig. 4. Error norm $\|e_j\|_2$ when applying ILC to a mass-spring-damper system with a one-sided spring. This norm is not converging towards a fixed value, indicating convergence is not achieved.

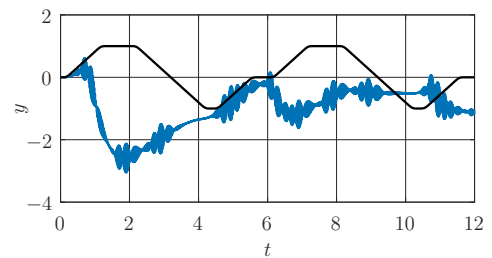


Fig. 5. Position output y at trial 20 (—). The reference is depicted in black (—). The position at trial 20 clearly is not converged towards the reference

trial are presented when the trial-invariant disturbance is given by a 4-th order reference. Clearly, convergence is not achieved. Moreover, evaluating the position of the mass at trial 20, depicted in Fig. 5, indicates that the position is not converging towards the desired reference.

From these results it is clear that the traditional monotonic convergence conditions for linear systems cannot be directly exploited for PWA systems. This is in part because of the state-dependent mode switching leading to iteration-varying switching times. This is one of challenges in deriving conditions for monotonic convergence of ILC applied to a PWA system.

This leads to the problem addressed in this paper: developing a framework for monotonic convergence analysis of the sequence of control inputs $\{u_j\}_{j \in \mathbb{Z}_{\geq 0}}$ of the ILC setup (12) where J represents a PWA system of the form (3).

3. ILC FOR PWA SYSTEMS

In this section, conditions for monotonic convergence of the ILC setup (12) are derived. The conditions derived in

this section employ the incremental ℓ_2 -gain and are applicable to general nonlinear discrete-time plants J . First, it is proven that when the incremental ℓ_2 -gain is smaller than 1, the sequence of control inputs is monotonically convergent. Next, it is shown that if the ILC update law is linear and time-invariant the incremental ℓ_2 -gain is independent of the unknown trial-invariant disturbance. This allows for an analysis that is independent of the unknown disturbance.

3.1 Assumptions

First, the following assumptions are imposed on the ILC setup.

Assumption 3.1. *The ILC update law only has access to the error and input signal of the previous trial.*

Assumption 3.2. *The trial-invariant disturbance d is unknown.*

Assumption 3.3. *The initial state $x(0)$ of the PWA system J is zero for each trial.*

3.2 ILC Setup for PWA systems

The ILC setup as depicted in Fig. 2 is considered. The ILC update law is given by (14) with causal linear filters Q and L that can be written in state space form as follows

$$Q : \begin{cases} x^Q(k+1) = A^Q x^Q(k) + B^Q u(k) \\ y^Q = C^Q x^Q(k) + D^Q u(k), \end{cases} \quad (17)$$

$$L : \begin{cases} x^L(k+1) = A^L x^L(k) + B^L e(k) \\ y^L = C^L x^L(k) + D^L e(k), \end{cases} \quad (18)$$

where $x^Q \in \mathbb{R}^{n^Q}$, $n^Q \in \mathbb{N}$, $x^L \in \mathbb{R}^{n^L}$, $n^L \in \mathbb{N}$ are the states of Q and L , respectively. The outputs of Q and L are denoted by $y^Q \in \mathbb{R}^m$, $y^L \in \mathbb{R}^m$, respectively.

3.3 The incremental ℓ_2 -gain

As observed in Section 2 the ℓ_2 -gain does not guarantee monotonic convergence for ILC applied to a PWA systems. Next, the incremental ℓ_2 -gain is exploited to derive guarantees for monotonic convergence of the ILC setup (12).

The incremental ℓ_2 -gain is defined as follows.

Definition 3.4. *(Incremental ℓ_2 -gain (van der Schaft, 2016, Definition 2.1.5))* The input-output map $G : \ell_2 \rightarrow \ell_2$ is said to have incremental ℓ_2 -gain if there exists a constant $\gamma \geq 0$ such that

$$\|(Gu^a)_{[0,T]} - (Gu^b)_{[0,T]}\|_2 \leq \gamma_0 \|u_{[0,T]}^a - u_{[0,T]}^b\|_2, \quad (19)$$

for all $T \in \mathbb{Z}_{\geq 0}$, $u^a, u^b \in \ell_2$. Furthermore, the incremental ℓ_2 -gain γ is defined as the infimum of all such γ_0 .

The incremental ℓ_2 -gain is a property that expresses the contraction, in terms of the ℓ_2 -norm, between all possible trajectories with respect to each other. Where the ℓ_2 -gain expresses contraction merely to a single trajectory.

To derive a condition that guarantees monotonic convergence of the sequence of control inputs the mapping U , depicted in Fig. 6 is exploited. This is the mapping from the control input at trial j , u_j , to the control input at trial $j+1$, u_{j+1} , i.e.,

$$u_{j+1} = U(u_j, d) = Qu_j + L(Ju_j - d) \quad (20)$$

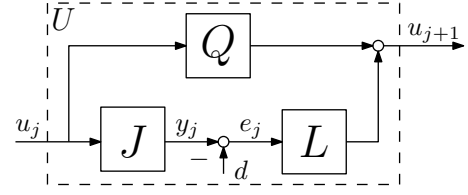


Fig. 6. Mapping from the control input at trial j to the control input at trial $j+1$

Exploiting the incremental ℓ_2 -gain of the mapping U the following result is obtained.

Theorem 3.5. *Consider the ILC setup (12) with J a PWA system (3), trial-invariant disturbance $d \in \ell_2$ and ILC update law given by (14). Then, the sequence of control inputs, $\{u_j\}_{j \in \mathbb{Z}_{\geq 0}}$, resulting from this ILC setup, converges monotonically in the 2-norm if the incremental ℓ_2 -gain of the corresponding mapping U , given by (20), is smaller than 1.*

Note that in general the incremental ℓ_2 -gain of U depends on d , but to guarantee convergence for all possible d a criterion independent of d is desired. Choosing L to be a linear time invariant mapping leads to such a result, given by the following.

Theorem 3.6. *Consider the mapping U with PWA system J and linear filters Q and L . Then, the incremental ℓ_2 -gain of U is independent of the disturbance d .*

The result of Theorem 3.6 and Theorem 3.5 allows to analyse the monotonic convergence property of the ILC applied to a PWA system independent of the trial-invariant disturbance d .

Remark 3.7. The incremental ℓ_2 -gain of LTI systems is equivalent to the ℓ_2 -norm (van der Schaft, 2016, Chapter 8). Hence, when considering LTI systems Theorem 3.5 reduces to the well-known condition (15).

4. COMPUTATIONALLY TRACTABLE ANALYSIS

In this section, a computationally tractable approach is derived to check the condition presented in Theorem 3.5. This approach will consist of checking a set of linear matrix inequalities (LMIs).

For ease of notation, the considered PWA system consist of two modes, i.e., $p = 2$ with the line $Ex^J = 0$ defining the hyperplane splitting the state space in Ω_1 and Ω_2 . Moreover linear dynamics are considered, i.e., $a_1, a_2 = 0$. However, note that the results in this section can easily be extended to the general PWA system (3).

4.1 Incremental ℓ_2 -gain for state-space systems

To analyse the incremental ℓ_2 -gain of the mapping U , first define its state-space form. Using the state $\underline{x} := [x^J \top \ (x^L) \top \ (x^Q) \top] \top \in \mathbb{R}^{\underline{n}}$, $\underline{n} = n^J + n^L + n^Q$, the system U is written in the state-space form

$$\begin{aligned} \underline{x}_j(k+1) &= \begin{cases} \underline{A}_1 \underline{x}_j(k) + \underline{B}_1 u_j(k) & \text{if } \underline{E} \underline{x} \geq 0 \\ \underline{A}_2 \underline{x}_j(k) + \underline{B}_2 u_j(k) & \text{if } \underline{E} \underline{x} < 0, \end{cases} \\ u_{j+1}(k) &= \begin{cases} \underline{C}_1 \underline{x}_j(k) + \underline{D}_1 u_j(k) & \text{if } \underline{E} \underline{x} \geq 0 \\ \underline{C}_2 \underline{x}_j(k) + \underline{D}_2 u_j(k) & \text{if } \underline{E} \underline{x} < 0 \end{cases}, \end{aligned} \quad (21)$$

with

$$\begin{aligned} \underline{A}_i &= \begin{bmatrix} A_i^J & O & O \\ B^L C^J & A^L & O \\ O & O & A^Q \end{bmatrix}, \underline{B}_i = \begin{bmatrix} B_i^J \\ B_i^L D^J \\ B^Q \end{bmatrix}, \\ \underline{C}_i &= [D^L C_i^J \ C^L \ C^Q], \underline{D}_i = D^L D_i^J + D^Q, \underline{E} = [E \ 0 \ 0]. \end{aligned} \quad (22)$$

For a state-space system of the form (21) the incremental ℓ_2 -gain is defined as

Definition 4.1. (*Incremental ℓ_2 -gain for state-space system (van der Schaft, 2016, Definition 8.2.8)*) Consider a state-space system of the form (21), with input space, \mathbb{R}^{n_u} , output space \mathbb{R}^{n_y} , and state space \mathbb{R}^n . The system (21) has incremental ℓ_2 -gain $\leq \gamma$ if there exists a function, called the incremental storage function,

$$S : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+ \quad (23)$$

such that

$$\begin{aligned} S(x_1(T), x_2(T)) - S(x_1(0), x_2(0)) \leq \\ \sum_{k=0}^T \gamma^2 \|u_1(k) - u_2(k)\|^2 - \|y_1(k) - y_2(k)\|^2 \end{aligned} \quad (24)$$

for all $T \in \mathbb{Z}_{\geq 0}$, and for all pairs of input signals $u_1, u_2 : [0, T] \rightarrow \mathbb{R}^{n_u}$ and all pairs of initial conditions $x_1(0), x_2(0) \in \mathbb{R}^n$, with resulting pairs of state and output trajectories $x_1, x_2 : [0, T] \rightarrow \mathbb{R}^n, y_1, y_2 : [0, T] \rightarrow \mathbb{R}^{n_y}$.

Without loss of generality that an incremental storage function $S(x_1, x_2)$ satisfies the symmetry property $S(x_1, x_2) = S(x_2, x_1)$, and $S(0, 0) = 0$.

This reduces the problem of checking the incremental ℓ_2 -gain of the mapping U to finding a storage function S such that (23) and (24) are satisfied for $\gamma \in [0, 1)$.

4.2 Linear Matrix Inequalities

In order to find an incremental storage function S such that (23) and (24) are satisfied for $\gamma \in [0, 1)$, define a piecewise quadratic storage function (Ferrari-Trecate et al., 2002) of the form

$$S(\underline{x}^a, \underline{x}^b) = \begin{cases} \begin{bmatrix} \underline{x}^a \\ \underline{x}^b \end{bmatrix}^\top P_{ij} \begin{bmatrix} \underline{x}^a \\ \underline{x}^b \end{bmatrix} & \text{if } \underline{E}_i \underline{x}^a \geq 0, \underline{E}_i \underline{x}^b \geq 0 \end{cases} \quad (25)$$

with $P_{ij} \in \mathbb{R}^{2n \times 2n}$, $i, j \in \{1, 2\}$, $\underline{E}_1 = \underline{E}$, and $\underline{E}_2 = -\underline{E}$.

Exploiting the storage function (25), the result of Theorem 3.5 can be converted into sufficient LMI based conditions. Three S-procedure relaxations (Yakubovich, 1997) are utilized to obtain the following result:

Theorem 4.2. *Consider the ILC setup (12) with J a PWA system (3), trial-invariant disturbance $d \in \ell_2$ and ILC update law given by (14). Moreover, consider the corresponding system U given by (21) with $\underline{A}_m, \underline{B}_m, \underline{C}_m, \underline{D}_m, m \in \{1, 2\}$. Then, the sequence of control inputs $\{u_j\}_{j \in \mathbb{Z}_{\geq 0}}$ of the ILC setup is monotonically convergent if the following LMI conditions hold for $\gamma \in [0, 1)$*

$$P_{ij} - E_{ij}^\top W_{ij} E_{ij} \succ 0, \quad (26a)$$

$$\begin{aligned} & \left[\begin{array}{c} A_{ij}^\top P_{kl} A_{ij} - P_{ij} + C_{ij}^\top T C_{ij} \\ * \\ E_{ij}^\top U_{ijkl} E_{ij} + A_{ij}^\top E_{kl}^\top V_{ijkl} E_{kl} A_{ij} \end{array} \quad \begin{array}{c} A_{ij}^\top P_{kl} B_{ij} + C_{ij}^\top T D_{ij} \\ B_{ij}^\top P_{kl} B_{ij} + D_{ij}^\top T D_{ij} - \gamma^2 T \\ A_{ij}^\top E_{kl}^\top V_{ijkl} E_{kl} B_{ij} \end{array} \right] + \\ & \left[\begin{array}{c} E_{ij}^\top U_{ijkl} E_{ij} + A_{ij}^\top E_{kl}^\top V_{ijkl} E_{kl} A_{ij} \\ * \\ B_{ij}^\top E_{kl}^\top V_{ijkl} E_{kl} B_{ij} \end{array} \right] \prec 0, \end{aligned} \quad (26b)$$

for all $i, j, k, l \in \{1, 2\}$, with $U_{ijkl}, W_{ij}, V_{ijkl} \in \mathbb{R}^{2n \times 2n}$ with only nonnegative entries, and where

$$\begin{aligned} A_{mn} &= \begin{bmatrix} \underline{A}_m & O \\ O & \underline{A}_n \end{bmatrix}, B_{mn} = \begin{bmatrix} \underline{B}_m & O \\ O & \underline{B}_n \end{bmatrix}, C_{mn} = \begin{bmatrix} \underline{C}_m & O \\ O & \underline{C}_n \end{bmatrix}, \\ D_{mn} &= \begin{bmatrix} \underline{D}_m & O \\ O & \underline{D}_n \end{bmatrix}, E_{mn} = \begin{bmatrix} \underline{E}_m & O \\ O & \underline{E}_n \end{bmatrix}, T = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

with $m, n \in 1, 2$.

Standard convex optimization tools can be utilized to find matrices P_{ij} , $i, j \in \{1, 2\}$ that satisfy the conditions of Theorem 4.2. Hence, this leads to a computationally tractable method to check monotonic convergence of the sequence of control inputs of an ILC algorithm applied to a PWA system.

5. APPLYING ILC TO A MASS-SPRING-DAMPER SYSTEM WITH A ONE-SIDED SPRING

In this section, ILC is applied to the mass-spring-damper system with a one-sided spring as described in Section 2.

The PWA system describing this system is governed by (11) and (8) with mass $m = 1$ [kg], damping coefficient $d = 1$ [N·m/s], spring constant of the linear spring $k_s = 1$ [N/m], and spring constant of the one-sided spring $k_f = 1$ [N/m], the system is sampled with a sample time $h = 0.01$ [s].

The ILC update law is given by (14), (17), (18). The filters Q and L are designed such that condition (15) is satisfied for the two LTI systems that define the two modes, leading to the following matrices

$$\begin{aligned} A^Q &= 0.5, B^Q = 1, C^Q = 0.5, D^Q = 0 \\ A^L &= \begin{bmatrix} -1.987 & 2.547 & 0 \\ -0.7721 & 0.99 & 0 \\ -0.3062 & 0 & 0.5 \end{bmatrix}, B^L = \begin{bmatrix} 140.3 \\ 36.28 \\ 14.39 \end{bmatrix} \\ C^L &= [-106.8 \ 0 \ 174.3], D^L = 5017 \end{aligned} \quad (27)$$

Note that by this design the incremental ℓ_2 -gain of the mapping U where J is an LTI system without the one-sided spring is smaller than 1. Moreover, the incremental ℓ_2 -gain of the mapping U where J is an LTI system where the one-sided spring is replaced by a linear spring is smaller than 1. Hence, in the situations where the system does not change modes this automatically guarantees monotonic convergence.

To check if the ILC update law yields global monotonic convergence, i.e, considering the state dependent mode changes, the results of Theorem 4.2 are exploited. It is confirmed that there exist matrices P_{ij} , $i, j \in \{1, 2\}$ and a constant $\gamma \in [0, 1)$ for which the LMIs (26a) and (26b) are satisfied. Hence, the sequence of control inputs $\{u_j\}_{j \in \mathbb{Z}_{\geq 0}}$ converges monotonically in the 2-norm for any trial-invariant disturbance d .

This is confirmed through simulations where a 4-th order reference signal is applied as a trial-invariant disturbance. In Fig. 7, the norm $\|u_j - u_\infty\|_2$ is depicted. From this figure, it can be observed that the sequence of input signals $\{u_j\}_{j \in \mathbb{Z}_{\geq 0}}$ is monotonically convergent toward u_∞ . In Fig. 8 the error norm $\|e_j\|_2$ indicates a high performance improvement. Moreover, in Fig. 9, the position at various trials is depicted together with the reference. From this figure, it is observed that the time instance of switching between the two modes varies each iteration.

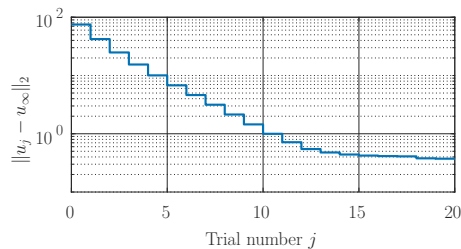


Fig. 7. Norm $\|u_j - u_\infty\|_2$ when applying ILC to a mass-spring-damper system with one-sided spring. This norm is decreasing, thereby confirming monotonic convergence of the sequence of control inputs $\{u_j\}_{j \in \mathbb{Z}_{\geq 0}}$.

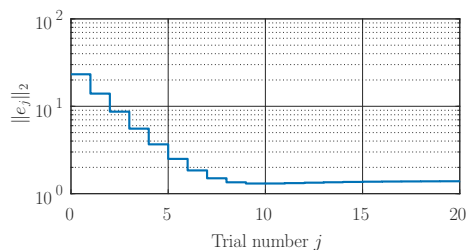


Fig. 8. Error norm $\|e_j\|_2$ when applying ILC to a mass-spring-damper system with one-sided spring.

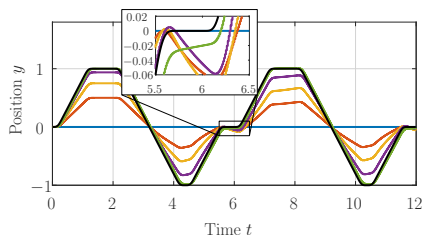


Fig. 9. Position output y at trials: 0 (—), 1 (—), 3 (—), 5 (—), and 20 (—). The reference is depicted in black (—). The zoom in this figure highlights the time instants of switching between the two modes that varies each iteration.

6. CONCLUSION

In this paper, a framework is developed to analyse the monotonic convergence property of ILC applied to PWA systems, opening up the possibility to apply ILC to a wide variety of applications. The analysing method exploits the incremental ℓ_2 -gain to determine if the mapping from the control input at trial j to the control input at trial $j + 1$ is a contraction mapping, thereby guaranteeing monotonic convergence in the 2-norm. This result is exploited to derive LMI conditions that guarantee monotonic convergence, leading to a computationally tractable analysis approach. A simulation example of a mass-spring-damper system with one-sided spring confirms the results.

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