Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Approximated Constrained Optimal Control subject to Variable Parameters

Christian Kallies [*,⋆] **Mohamed Ibrahim** [*] **Rolf Findeisen** [*,⋆]

[*] *Laboratory for Systems Theory and Automatic Control, Otto von Guericke University, Magdeburg, Germany (e-mail: christian.kallies, mohamed.ibrahim, rolf.findeisen@ovgu.de).*

Abstract: Implementing optimal controllers on embedded systems can be challenging as it requires the solution of an optimization problem in real-time. Furthermore, the a priory verification of stability, e.g. not relying on the possibly numerical solution of an optimization problem is often not possible. We propose a non-linear control synthesis based on an approximated explicit solution of a constrained optimal control problem, which can be efficiently implemented and verified. The control law is derived based on a series expansion of an infinite horizon optimal control problem via Al'brekht's Method. In comparison to existing approaches we consider parametric uncertainties. The proposed method provides under certain conditions an approximated solution of the Hamilton–Jacobi–Bellman (HJB) equation. The feedback control law uses a finite number of terms of the series expansion, and therefore the evaluation does not require intensive online computation. Furthermore, the optimal control strategy does not only achieve an approximated infinite horizon performance but is also parameterized in terms of the varying parameters which are assumed to be known. We provide a proof of convergence and existence of the optimal control law. Simulation results with a non-linear quadcopter example show the effectiveness of the proposed strategy.

Keywords: Approximated Infinite Horizon Optimal Control, Al'brekht's Method, Non-linear Infinite Horizon Adaptive Control, Parametric Uncertainties.

1. EXPLICIT OFFLINE OPTIMAL CONTROL

Optimal control has gained increasing interest, over the recent years, especially for constrained systems. Optimal control aims to design a control law that minimizes a performance criteria. Analytic solutions to constrained optimal control problems are often difficult to find. Solving the resulting optimization problem which can be computational challenging on an embedded platform. Consequently, computationally efficient methods for solving the optimal control problem by calculating an explicit solution offline have been investigated, e.g. (Na and Herrmann, 2014). There are several works in the literature investigating approximation approaches to obtain suboptimal solutions for non-linear optimal control problems, e.g. Luo et al. (2018); Vamvoudakis and Lewis (2010). Al'brekht (1961) developed a formal iterative procedure to solve an optimal control problem via a power series expansion. Garrard et al. (1992) developed an approach which is applicable to a wider class of non-linear systems by expanding both the optimal cost and the non-linear dynamics as a power series in terms of the states. However the overall complexity increases dramatically with the system order. Due to the complexity of solving the Hamilton–Jacobi–Bellman equations for non-linear systems, inverse optimal design approaches have been investigated, e.g. in (Margaliot and Langholz, 2001). Xin and Balakrishnan (2005) presented the so called $\theta - D$ approximation method for solving non-linear optimal control problems characterized by a quadratic cost function and an affine model. A further approach, named model predictive static programming (Padhi and Kothari, 2009) has turned out to be computationally efficient for finite-horizon optimal control problems with terminal constraints. Dalamagkidis et al. (2010) implemented an explicit model predictive controller (MPC) to calculate the inputs via integrating the optimal control derivatives, which are approximated using recurrent neural networks. Neural networks are also used to approximate the solution of the HJB equations (Liu et al., 2018) for non-linear discrete time systems (Kiumarsi and Lewis, 2014). Liu et al. (2012) reduced the complexity of the explicit MPC problem by a hierarchical framework, wherein the explicit solution is derived using Taylor expansions considering the vehicle model. Modares and Lewis (2014) presented a new formulation of optimal control problems using reinforcement learning for non-linear partially-unknown constrained-input systems. Papachristos et al. (2014) provided explicit solution approximations of piecewise affine models which leads to a reduction of the computational time to solve a finite-horizon optimal control problem. To solve the optimal tracking problem for non-linear systems, Batmani et al. (2017) presented a suitable technique based on a state-dependent Riccati equation method (Cimen, 2012).More recently an approach that provides a proximal averaged Newton-type methods (Sathya et al., 2018; Stella et al., 2017) to solve optimal control problems has been presented

[⋆] RF and CK are also affiliated to the International Max Planck Research School (IMPRS) for Advanced Methods in Process and Systems Engineering, Magdeburg, Germany.

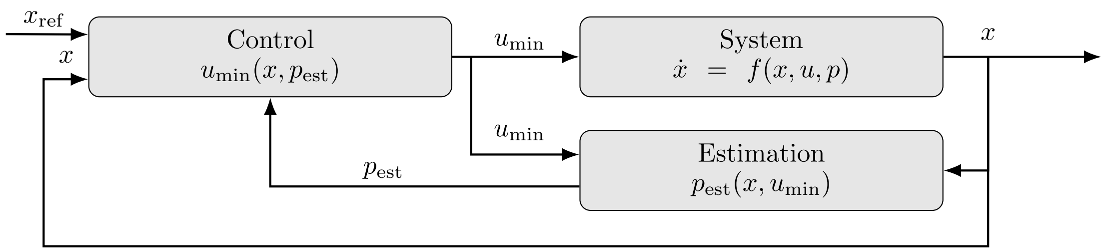

Fig. 1. Structure diagram showing the interaction between the optimal control policy with the dynamical system and the parameter estimator. The explicit solution of an optimal control problem $u_{\min}(x, p_{\mathrm{est}})$ is calculated once offline. The control parameters p_{est} are updated online via parameter estimation.

which requires only computation of Jacobian vector products.

Most of the existing methods are limited to linear systems or do not consider constraints and assume that the system dynamics does not change. We propose an explicit optimal control approach for non-linear systems based on the power series expansion method (Al'brekht, 1961; Lucia et al., 2015; Krener, 2018a), which can exploit a non-linear continuous-time model of the autonomous vehicle.

The approach allows to consider variable - uncertain - but known parameters. The main idea of the proposed approach is to express the approximated optimal control law as a power series expression of the current states x and the parameters p_{est}, see Fig. 1. Therefore, the optimal control strategy not only minimizes the performance index, but also considers the variable - uncertain - but known parameters p_{est}, which leads to alleviate the disturbance effect.

The remainder of the paper is organized as follows. An overview of Al'brekht's Method is presented in Section 2. Section 3 defines, based on Al'brekht's Method, an approach for constrained optimal control problems which are subject to variables. Section 4 presents simulation results of a non-linear quadcopter example with uncertainties which illustrates the effectiveness and capabilities of the proposed method. Conclusions and directions for future work are provided in Section 5.

2. AL'BREKHT'S METHOD

The original method from E.G. Al'brekht (1961) developed a formal iterative procedure to obtain an optimal control law for a non-linear continuous time system expressed as a power series in terms of the states. The considered infinite horizon optimal control problem takes the form

$$V\big(x(0)\big) = \min_{u(\cdot)} \int_0^\infty \ell\big(x(t), u(t)\big)\,\mathrm{d}t \tag{1a}$$

$$\text{s.t. } \dot{x} = f(x, u), \tag{1b}$$

$$x(0) = x_0 \in \mathbb{R}^{n_x}, \tag{1c}$$

where $f\colon \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}^{n_x}$ represents the dynamics of a system and $\ell\colon \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}$ is the cost function. Both are assumed to be analytic functions in both variables and can be represented as

$$f(x, u) = \sum_{i=1}^{\infty} f^{[i]}(x, u)$$
$$\text{and } \ell(x, u) = \sum_{i=2}^{\infty} \ell^{[i]}(x, u).$$

Here $f^{[i]}(x, u)$ and $\ell^{[i]}(x, u)$ represent all terms of f respectively ℓ which are homogeneous with degree i. It is crucial that $f(0, 0)$ equals zero and ℓ does not contain a linear part. Furthermore the control law $u_{\min}(\cdot)$ and the value function $V(\cdot)$ are assumed to be analytical in a domain which contains the origin and can be written as

$$u_{\min}(x) = \sum_{i=1}^{\infty} u_{\min}^{[i]}(x) \text{ and } V(x) = \sum_{i=2}^{\infty} V^{[i]}(x). \tag{2}$$

$u_{\min}(0)$ and $V(0)$ have to vanish since clearly no control has to be applied at $x = 0$. $V^{[1]}(x)$ is zero as $\ell^{[1]}(x, u)$ is zero. As shown in (Al'brekht, 1961; Aguilar and Krener, 2014) given these conditions one can solve the HJB equations

$$0 = \nabla_x V(x) \cdot f\big(x, u_{\min}(x)\big) + \ell\big(x, u_{\min}(x)\big) \tag{3a}$$

$$0 = \nabla_x V(x) \cdot \nabla_u f\big(x, u_{\min}(x)\big) + \nabla_u \ell\big(x, u_{\min}(x)\big) \tag{3b}$$

under the following assumptions:
(I) The linear system $\dot{x} = f^{[1]}(x, u)$ is stabilizable
(II) $\big(f^{[1]}(x, 0), \ell^{[2]}(x, 0)\big)$ is detectable
(III) $\ell^{[2]}(x, u)$ is convex and $\ell^{[2]}(0, u)$ is strictly convex

The basic idea to do so is to first collect all terms of degree one in (3b) to obtain a formula for $u_{\min}^{[1]}(x)$. This formula is then used in the equation which is obtained by taking all terms homogeneous with degree two from (3a). The resulting Riccati equation can be solved provided that conditions (I)-(III) (Aguilar and Krener, 2014) are holding and provides us with $V^{[2]}(x)$ which leads to $u_{\min}^{[1]}(x)$. This first iteration step leads to the linear quadratic regulator. In later iteration steps, terms homogeneous with degree i in (3b) and degree $i+1$ in (3a) are collected. The resulting equations are linear in the unknowns $u_{\min}^{[i]}(x)$ and $V^{[i+1]}(x)$ and are solvable since the linear part of the control law is stabilizing the linear part of the system. The convergence of both power series in (2) has been established by E.G. Al'brekht for systems which are linear in the input and quadratic cost functions. While this method can calculate an approximation of the exact solution of (1) it does not include uncertain parameters as well as constraints.

3. APPROXIMATED EXPLICIT SOLUTIONS SUBJECT TO VARIABLE PARAMETERS

In this section we outline how to derive approximated solutions using Al'brekht's Method for constraint optimal control problems subject to variable parameters. To this end we consider the following optimal control problem:

$$V\big(x(0), p\big) = \min_{u(\cdot)} \int_0^\infty \ell\big(x(t), u(t)\big)\, \mathrm{d}t \tag{4a}$$

$$\text{s.t. } \dot{x} = f(x, u, p), \tag{4b}$$

$$x(0) = x_0 \in \mathbb{R}^{n_x}, \tag{4c}$$

where $p \in \mathbb{R}^{n_p}$ is a variable uncertain parameter. The functions $f\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ and $\ell\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ are assumed to be analytic in all variables, e.g.

$$f(x, u, p) = \sum_{i=1}^{\infty} f^{[i]}(x, u, p)$$

$$\text{respectively } \ell(x, u) = \sum_{i=2}^{\infty} \ell^{[i]}(x, u).$$

Note that the dimension of the control input is arbitrary. Again it is indispensable that ℓ does not contain a linear part and that

$$\forall p \in \mathbb{R}^{n_p} : f(0, 0, p) = 0. \tag{5}$$

The control law $u_{\min}(\cdot, \cdot)$ and the value function $V(\cdot, \cdot)$ are assumed to be analytic in a domain which contains the origin and can be written as

$$u_{\min}(x, p) = \sum_{i=1}^{\infty} u_{\min}^{[i]}(x, p) \tag{6a}$$

$$\text{and } V(x, p) = \sum_{i=2}^{\infty} V^{[i]}(x, p). \tag{6b}$$

As in the previous section, the property (5) implies $u_{\min}(0, p) = 0$ and $V(0, p) = 0$. Furthermore, $\ell^{[1]}(x, u) = 0$ and (5) show $V^{[1]}(x, p) = 0$. Similar to the original method, one can exploit the power series expressions to the solve the HJB equations which now become

$$\begin{aligned} 0 &= \nabla_x V(x, p) \cdot f\big(x, u_{\min}(x, p), p\big) \\ &\quad + \ell\big(x, u_{\min}(x, p)\big), \end{aligned} \tag{7a}$$

$$\begin{aligned} 0 &= \nabla_x V(x, p) \cdot \nabla_u f\big(x, u_{\min}(x, p), p\big). \\ &\quad + \nabla_u \ell\big(x, u_{\min}(x, p)\big) \end{aligned} \tag{7b}$$

To guarantee that the optimal control problem is solvable, the following conditions must hold:

(I') The linear system $\dot{x} = f^{[1]}(x, u, p)$ is stabilizable
(II') $\big(f^{[1]}(x, 0, 0), \ell^{[2]}(x, 0)\big)$ is detectable
(III') $\ell^{[2]}(x, u)$ is convex and $\ell^{[2]}(0, u)$ is strictly convex

As for Al'brekht's Method we collect terms which are homogeneous with degree i from (7b) and terms homogeneous with degree $i + 1$ from (7a). Here x and p are considered as variables. So the parameters p can be seen as states with vanishing derivatives. Since the cost function does not depend on the parameters, (5) implies the same for $f^{[1]}$, the solution of the lowest degree $\big(u_{\min}^{[1]},\ V^{[2]}\big)$ is identical to the one without any parameters. Additional it can be shown that with our setup $p \mapsto \nabla_x V(0, p)$ is vanishing and since this function is also analytic the same holds for each degree of its power series expansion. Higher degrees will depend on the parameters but are again the same as in the non-parametric case if p is set to zero. Furthermore the unknowns $u_{\min}^{[i]}(x, p)$ and $V^{[i+1]}(x, p)$ only appear linear in the equations and can be calculated since the linear part of the control law is stabilizing the linear part of the system. The power series can be stopped at a desired degree to obtain an approximation of $u_{\min}$. One should know that the approximation error does not necessarily decrease with increasing degrees.

3.1 Convergence and Existence

Even though the power series in (6) can be calculated degree wise it is not clear if they exist/converge. However, one can show that the following holds.

Theorem 1. (Existence of the power series). Consider the optimal control problem (4) for cost functions which are quadratic and for system dynamics which are linear in the input. If the conditions (I')-(III') hold then the power series (6) exists locally around the origin and solves (7).

Remark 1. The proof follows along the lines of (Al'brekht, 1961). The parameters can be treated as states with vanishing derivatives.

Remark 2. Since the solution of the lowest degree $\big(V^{[2]}(x, p),\, u_{\min}^{[1]}(x, p)\big)$ equals a linear quadratic regulator, local stability is ensured if the uncertain parameters p are sufficiently small.

The proof furthermore allows to derive the following inequality to lower bound the region of convergence.

$$\sum_{i=2}^{\infty} C_i^f \cdot ||(x, p)||^i \le C_1 \cdot ||(x, p)||, \tag{8}$$

Here the constants C_1 and C_2 are defined as

$$C_1 \coloneqq C_2 - \sqrt{C_2^2 - \alpha^2} \text{ and } C_2 := \alpha + 2C_u^2 \cdot C_{uu},$$

while the C_i^f for every degree i are given by the following inequality.

$$\left|\left|f^{[i]}(x, 0, p)\right|\right| \le C_i^f \cdot ||(x, p)||^i$$

The constants C_u and C_{uu} are determined from the system dynamics and the cost function, respectively, as

$$C_u = ||\nabla_u f(0, 0, 0)||_2,$$
$$C_{uu} = \left|\left|\nabla_{uu}^2 \ell(0, 0)^{-1}\right|\right|_2,$$

whereas α is such that for the solution of the initial value problem

$$\dot{x} = f^{[1]}\big(x, u_{\min}^{[1]}(x, p), p\big), \quad x_0 = x(0)$$

the following holds:

$$||x(t)|| \le ||x_0|| \cdot \mathrm{e}^{-\alpha t}$$

The convergence of the power series (6) is given for all pairs (x, p) such that (8) is holding.

3.2 Handling inequality Constraints

We propose to handle inequality constraints via logarithmic barrier functions which will be added to the cost function. Barrier functions are widely used in optimal control schemes, e.g. (Feller and Ebenbauer, 2015; Wu and

Christofides, 2019). In the following we consider the $m \in \mathbb{N}$ inequality constraints

$$g(x,u) = g_0 + \sum_{i=1}^{\infty} g^{[i]}(x,u) \leq 0,$$

where $g\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^m$ is assumed to be analytic in all variables. Furthermore g_0 is assumed to be component-wise negative such that the origin is a feasible point. We define

$$\tilde{g}(x,u) := \sum_{i=1}^{\infty} g^{[i]}(x,u) \leq -g_0$$

and use the Taylor series of the logarithm component-wise to see

$$-\log\left(1 - \frac{\tilde{g}(x,u)}{-g_0}\right) = \sum_{i=1}^{\infty} \frac{1}{i} \cdot \left(\frac{\tilde{g}(x,u)}{-g_0}\right)^i.$$

After removing the linear part of this series and multiplying everything with a vector of penalty factors $c \in \mathbb{R}^m_{>0}$ we adjust the cost function as follows.

$$\tilde{\ell}(x,u) = \ell(x,u) + c^{\mathrm{T}} \cdot \sum_{i=2}^{\infty} \frac{1}{i} \cdot \left(\frac{\tilde{g}(x,u)}{-g_0}\right)^i$$

Remark 3. If the constraints g are convex, then $\tilde{\ell}^{[2]}$ is also convex. Furthermore the conditions (II') and (III') stay valid.

Therefore the calculation of the power series (6) does not change. Solvability is guaranteed for every choice of c.

Remark 4. The stability of the closed loop system can be checked afterwards by using the polynomial approximation of V as a Lyapunov function candidate which has been done in (Lucia et al., 2015).

The proposed strategy to include inequality constraints does not guarantee constraint satisfaction since it is usually not possible to calculate and use the whole series of $u_{\min}$. In fact if the control law is only calculated up to a certain degree d then from the cost function $\tilde{\ell}^{[2]}, \ldots \tilde{\ell}^{[d+1]}$ are used which implies that the cost at $g(x,u) = 0$ is finite. We note that similar approaches omit constraints (6) (Lucia et al., 2015; Krener, 2018a,b) or vector norms of x and u with even degrees are added to the cost function. While the later is similar to the presented approach, it is less general. Other approaches (Xin and Balakrishnan, 2005) are usually restricted to convex state constraints, while the proposed one can also consider non-convex constraints in the problem formulation.

4. QUADCOPTER EXAMPLE

Quadcopter are widely used in many applications as they are highly maneuverable and capable of diverse tasks such as hovering, vertical takeoff and landing. The use of numerically based optimal controller design is often challenged due to the limited computational power present on many autonomous vehicles such as quadcopter.

We consider a 10D scaled quadrotor model (Hu et al., 2018; Köhler et al., 2019):

$$\begin{aligned}
\dot{p}_x &= v_x + \omega_x & \dot{v}_x &= \mathrm{g} \cdot \tan(\theta) \\
\dot{p}_y &= v_y + \omega_y & \dot{v}_y &= \mathrm{g} \cdot \tan(\phi) \\
\dot{p}_z &= v_z & \dot{v}_z &= -\mathrm{g} + k_t \cdot u_z \\
\dot{\theta} &= -d_1 \cdot \theta + v_\theta & \dot{v}_\theta &= -d_0 \cdot \theta + n_0 \cdot u_\theta \\
\dot{\phi} &= -d_1 \cdot \phi + v_\phi & \dot{v}_\phi &= -d_0 \cdot \phi + n_0 \cdot u_\phi
\end{aligned}$$

the states x are defined as:

$$x = [p_x \quad p_y \quad p_z \quad v_x \quad v_y \quad v_z \quad \theta \quad \phi \quad v_\theta \quad v_\phi]^\top.$$

Herein, p_x, p_y, p_z define the position coordinates, v_x, v_y, v_z are translational velocities. $\phi, \theta, v_\phi, v_\theta$ are roll and pitch angles and rates respectively. While the control vector $u = [u_z \quad u_\phi \quad u_\theta]^\top$ includes the adjustable vertical thrust, roll and pitch angles. $\mathrm{g} = 9{,}81$, $k_t = 0{,}91$, $n_0 = 10$, $d_0 = 10$, $d_1 = 8$ represent the known quadcopter parameters. Compared to the model used in (Hu et al., 2018; Köhler et al., 2019), we consider state-dependent uncertainties (ω_x, ω_y) which is typically the case for aerial vehicles (Ibrahim et al., 2020).

$$\begin{aligned}
\omega_x &= p_{xx} \cdot v_x + p_{yx} \cdot v_y + p_{zx} \cdot v_z, \\
\omega_y &= p_{xy} \cdot v_x + p_{yy} \cdot v_y + p_{zy} \cdot v_z
\end{aligned}$$

Here p_{ij} represent the effect on the j axes due to the velocity component i. The parameters can vary and we assume that they can be obtained via a suitable estimation approach. In order to make the approach computational efficient for embedded implementation, we propose to use multivariate learning regression for online learning of the uncertainty structure based on the state measurements and control inputs at the previous time steps. To do so, we propose a linear model to learn the state-dependent uncertainty via Gradient descend utilizing the observed data gathered during the mission. For more details, the reader is referred to (Ibrahim et al., 2020). Using the disturbance estimation the performance and robustness of the optimal control is been improved. In the problem formulation (1), we use a quadratic cost function

$$\ell(x,u) = \frac{1}{2} x^{\mathrm{T}} Q x + \frac{1}{2} u^{\mathrm{T}} R u$$

with $Q = \mathrm{diag}\{1, 1, 1, 0, \ldots, 0\}$ and $R = I_3$.

The outlined approach is used to derive a parametric approximated explicit solution of (4). The resulting control law $u_{\min}(x,p)$ only requires the evaluation of the power series expansion of the current states x and the uncertain parameters p. This allows for real-time implementation even in case of limited computational power. For safety and reliability verification the resulting control law can furthermore be validated under different circumstances (e.g. obstacle avoidance and wind disturbance).

4.1 SIMULATION RESULTS

In this section, the effectiveness of the proposed strategy to mitigate the effect of the external disturbances is validated via simulation results for a quadcopter stabilization problem (Hu et al., 2018; Köhler et al., 2019). The simulations consider different random scenarios of the external disturbances. In the simulations, the quadcopter starts with different initial conditions, while the control objective is to guide the quadcopter to the origin $x_{ref} = 0 \in \mathbb{R}^{10}$. For reducing the disturbance effect, the controller is parametrized in terms of the uncertain variable parameters:

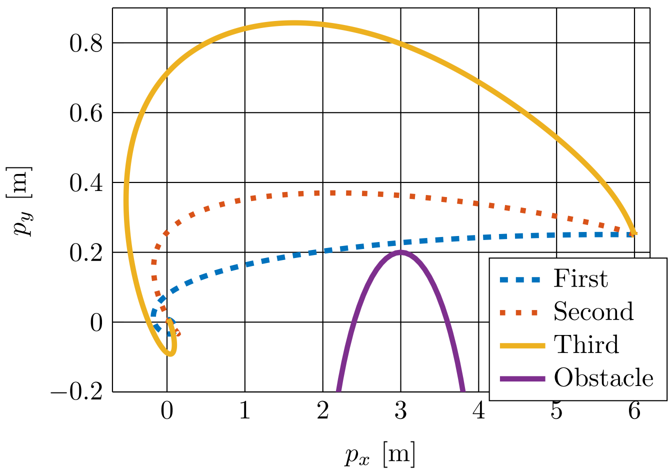

Fig. 2. Comparison of first, second, and third order approximation for initial conditions $x_0 = (6, 0.25, 10)$

$$u_{\min}(x,p) = \sum_{i=1}^{d} u_{\min}^{[i]}(x,p).$$

We first analyze the influence of increasing the approximation order d of the optimal control law. For all scenarios (see Fig. 2 and Fig. 3), the third order approximation of the control law avoids the obstacle despite the disturbances. This is achieved by a sufficiently high penalty value in the objective function for obstacle avoidance. Furthermore, one can see that considering the disturbance estimation improves the control performance and increase the safety.

Fig. 4 depicts a comparison of the third order approximation of the explicit control law and a certainty-equivalent non-linear MPC (Hu et al., 2018), which is implemented via the ACADO Toolkit (Houska et al., 2011). Not surprising also the nominal MPC allows to reject disturbances.

Table 1 outlines the required solution time. Using the explicit solution reduces the computational time significantly (10 times) which allows to implement the optimal controller in real-time even on computationally limited embedded systems. The numerical simulations were implemented on an Intel® Core^{TM} i7-6700 CPU @ 3.40GHz. The dynamics integration was done via the ACADO Toolkit (Houska et al., 2011) using multiple shooting with a 4th order Runge-Kutta integrator (RK4).

As outlined in section 3 the convergence area of the power series (6) for the quadcopter example has been inner approximated. The constants then result in $C_u = 10$, $C_{uu} = 1$, $\alpha = 0.6745$ and the convergence is given for all $r = ||(x,p)||$ such that

$$9.81 \cdot \tan(r) - 9.81 \cdot r + \frac{1}{4} \cdot r^2 \leq 0.0011 \cdot r$$

holds, e.g. $r \leq 0.0041723$.

Table 1. Computational Times

Controller	First	Second	Third	MPC
CPU Time [μs]	6.28	11.13	17.3	120.52

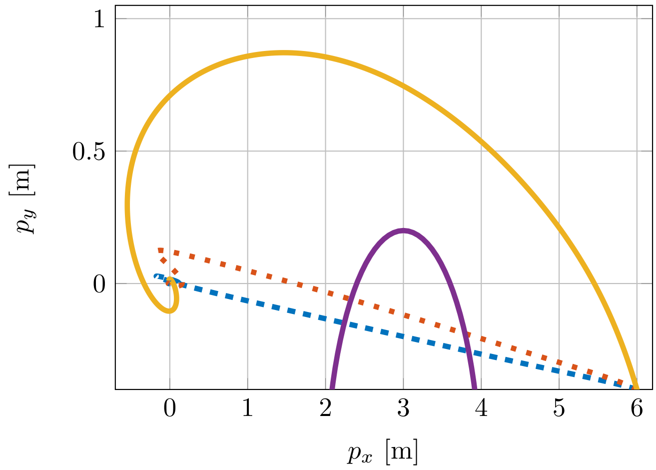

Fig. 3. Comparison of first, second, and third order approximation for initial conditions $x_0 = (6, -0.4, 10)$

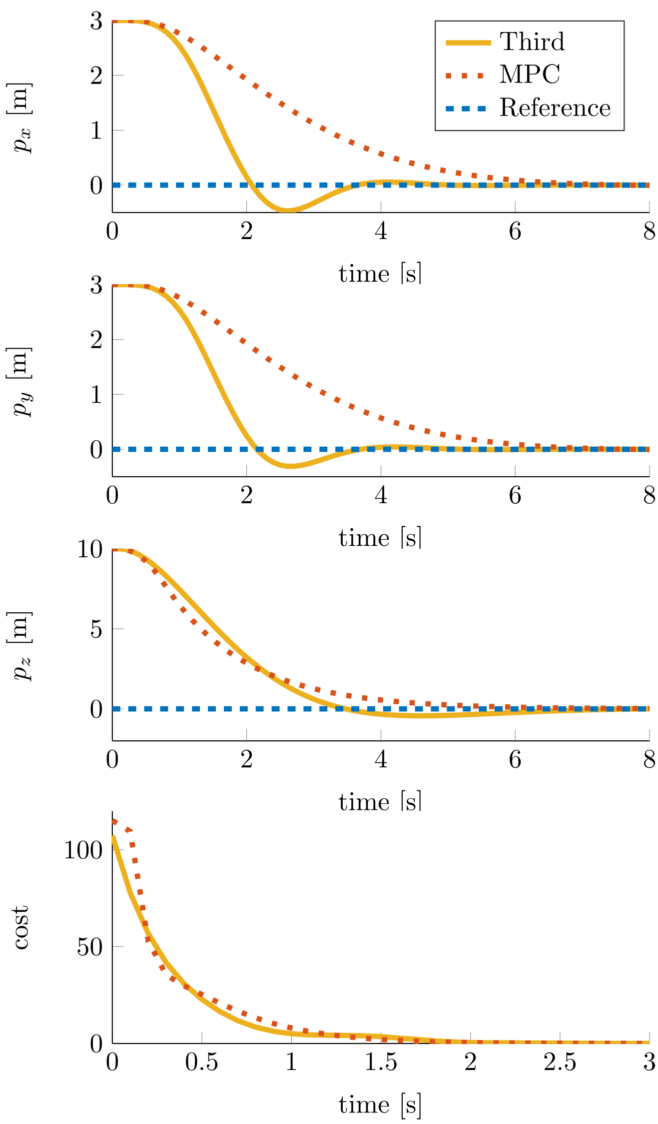

Fig. 4. Third order approximation vs MPC.

5. CONCLUSION AND FUTURE WORK

Motivated by the need of efficient computational implementation of optimal control, this work proposed an approximated optimal control approach based on Al'brekht's Method. The strategy obtains an approximated explicit solution of the optimal control problem allowing for variable parameters to capture uncertainties. The efficiency and performance of the proposed approximated optimal control is evaluated via a quadcopter simulation study. The simulation studies underline the capability of the approach with respect to efficient implementation, constraint handling and disturbance rejection. Future work will focus on validating the region of attraction as well as the achieved performance. Furthermore, we plan to validate the approach in real experiments and will consider implementation on hardware platforms such as FPGAs.

REFERENCES

Aguilar, C.O. and Krener, A.J. (2014). Numerical solutions to the bellman equation of optimal control. *J. of Optimization Theory and Applications*, 160, 527–552.

Al'brekht, E. (1961). On the optimal stabilization of nonlinear systems. *Applied Mathematics and Mechanics*, 25(5), 836–844.

Batmani, Y., Davoodi, M., and Meskin, N. (2017). Nonlinear suboptimal tracking controller design using state-dependent riccati equation technique. *IEEE Transactions on Control Systems Technology*, 25(5), 1833–1839.

Cimen, T. (2012). Survey of state-dependent riccati equation in nonlinear optimal feedback control synthesis. *J. of Guidance, Control, and Dynamics*, 35(4), 1025–1047.

Dalamagkidis, K., Valavanis, K.P., and Piegl, L.A. (2010). Nonlinear model predictive control with neural network optimization for autonomous autorotation of small unmanned helicopters. *IEEE Transactions on Control Systems Technology*, 19(4), 818–831.

Feller, C. and Ebenbauer, C. (2015). Input-to-state stability properties of relaxed barrier function based mpc. *IFAC-PapersOnLine*, 48(23), 302–307.

Garrard, W.L., Enns, D.F., and Antony Snell, S. (1992). Nonlinear feedback control of highly manoeuvrable aircraft. *International journal of control*, 56(4), 799–812.

Houska, B., Ferreau, H.J., and Diehl, M. (2011). ACADO toolkit - An open-source framework for automatic control and dynamic optimization. *Optimal Control Applications and Methods*, 32(3), 298–312.

Hu, H., Feng, X., Quirynen, R., Villanueva, M.E., and Houska, B. (2018). Real-time tube mpc applied to a 10-state quadrotor model. In *2018 Annual American Control Conference (ACC)*, 3135–3140. IEEE.

Ibrahim, M., Kallies, C., and Findeisen, R. (2020). Learning-supported approximated optimal control for autonomous vehicles in the presence of state dependent uncertainties. In *Proceedings of the 19th European Control Conference*. Saint Petersburg, Russia.

Kiumarsi, B. and Lewis, F.L. (2014). Actor–critic-based optimal tracking for partially unknown nonlinear discrete-time systems. *IEEE Transactions on Neural Networks and Learning Systems*, 26(1), 140–151.

Köhler, J., Soloperto, R., Müller, M.A., and Allgöwer, F. (2019). A computationally efficient robust model predictive control framework for uncertain nonlinear systems. *submitted to IEEE Transactions on Automatic Control*.

Krener, A. (2018a). Adaptive horizon model predictive control. *IFAC-PapersOnLine*, 51(13), 31–36.

Krener, A. (2018b). Adaptive horizon model predictive regulation. *IFAC-PapersOnLine*, 51(20), 54–59.

Liu, C., Chen, W.H., and Andrews, J. (2012). Explicit non-linear model predictive control for autonomous helicopters. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, 226(9), 1171–1182.

Liu, Y., Zhang, H., Yu, R., and Qu, Q. (2018). Data-driven optimal tracking control for discrete-time systems with delays using adaptive dynamic programming. *Journal of the Franklin Institute*, 355(13), 5649–5666.

Lucia, S., Rumschinski, P., Krener, A., and Findeisen, R. (2015). Improved design of nonlinear model predictive controllers. *IFAC-PapersOnLine*, 48(23), 254–259.

Luo, B., Yang, Y., and Liu, D. (2018). Adaptive q-learning for data-based optimal output regulation with experience replay. *IEEE transactions on cybernetics*, 48(12), 3337–3348.

Margaliot, M. and Langholz, G. (2001). Some nonlinear optimal control problems with closed-form solutions. *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, 11(14), 1365–1374.

Modares, H. and Lewis, F.L. (2014). Optimal tracking control of nonlinear partially-unknown constrained-input systems using integral reinforcement learning. *Automatica*, 50(7), 1780–1792.

Na, J. and Herrmann, G. (2014). Online adaptive approximate optimal tracking control with simplified dual approximation structure for continuous-time unknown nonlinear systems. *IEEE/CAA Journal of Automatica Sinica*, 1(4), 412–422.

Padhi, R. and Kothari, M. (2009). Model predictive static programming: a computationally efficient technique for suboptimal control design. *Int. J. of innovative computing, information and control*, 5(2), 399–411.

Papachristos, C., Alexis, K., and Tzes, A. (2014). Technical activities execution with a tiltrotor uas employing explicit model predictive control. *IFAC Proceedings Volumes*, 47(3), 11036–11042.

Sathya, A., Sopasakis, P., Van Parys, R., Themelis, A., Pipeleers, G., and Patrinos, P. (2018). Embedded nonlinear model predictive control for obstacle avoidance using panoc. In *2018 European Control Conference (ECC)*, 1523–1528. IEEE.

Stella, L., Themelis, A., Sopasakis, P., and Patrinos, P. (2017). A simple and efficient algorithm for nonlinear model predictive control. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, 1939–1944.

Vamvoudakis, K.G. and Lewis, F.L. (2010). Online actor–critic algorithm to solve the continuous-time infinite horizon optimal control problem. *Automatica*, 46(5), 878–888.

Wu, Z. and Christofides, P.D. (2019). Optimizing process economics and operational safety via economic MPC using barrier functions and recurrent neural network models. *Chem. Eng. Res. and Des.*, 152, 455–465.

Xin, M. and Balakrishnan, S. (2005). A new method for suboptimal control of a class of non-linear systems. *Opt. Con. App. and Meth.*, 26(2), 55–83.