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Laguerre-domain Modeling and Identification of Linear Discrete-time Delay Systems ⋆

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Abstract: A closed-form Laguerre-domain representation of discrete linear time-invariant systems with constant input time delay is derived. It is shown to be useful in a $\mathbf{l}_2 \to \mathbf{l}_2$ system identification setup (with $\mathbf{l}_2$ denoting square-summables signals) often arising in biomedical applications, where the experimental protocol does not allow for persistent excitation of the system dynamics. The utility of the proposed system representation is demonstrated on a problem of drug kinetics estimation from clinical data.

Keywords: Linear systems, discrete systems, time delay, system identification, Laguerre functions

1. INTRODUCTION

Pure time delays are ubiquitous in technological and biological systems. Delay operators in mathematical models describe e.g. transport phenomena or the aggregation of fast dynamics in model reduction. In continuous time, introducing a time delay into a model necessitates an infinite-dimensional state-space description that is usually written in terms of a difference-differential equation. In discrete time, a time delay is a finite-dimensional operator and, therefore, it can be captured using difference equations. Yet, the difference equations corresponding to time-delay systems possess a specific structure that can be exploited in solving control and estimation problems for the system in question (Fridman, 2004).

In linear time-invariant discrete systems, it is straightforward to convert a state-space model of a time-delay system to an equivalent delay-free model by state vector augmentation. For long and moderate-length time delays, this procedure results in a model of high-dimension, which properties usually translate to poor conditioning of control and estimation design methods. Models with an explicit parameterization of the delay are therefore preferable in engineering.

The Laguerre domain is the functional space spanned by the Laguerre functions. One distinguishes between continuous and discrete Laguerre functions depending on whether the argument is a continuous or discrete variable. This conveniently corresponds to the continuous- and discrete-time frameworks of dynamical systems. The use of Laguerre functions in systems theory is twofold. They are applied either to approximate the dynamics of linear (Heuberger et al., 2005) and nonlinear systems (Marmarelis, 1993) or to represent their input and output signals (Fischer and Medvedev, 1998). The former approach is suitable for systems whose solutions are asymptotically stable, while the latter demands in addition that the input decays to zero. Thus representing the input and output signals of a systems as Laguerre series, which is often termed as Laguerre domain representation, restricts the consideration to the input signals that are square-integrable ($\mathbb{L}_2$) or square-summable ($\mathbf{l}_2$), depending on the selected time framework. Naturally, such a signal is not persistently exciting and vanishes at infinity.

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In the time domain, either differentiation or time shift is used to describe the dynamics of the system. In the Laguerre domain, the Laguerre shift operator is utilized for the same purpose (Mäkilä and Partington, 1999). Being applied to a Laguerre function, the Laguerre shift produces the next (in order) Laguerre function. In continuous time, the Laguerre representation implementing a map $\mathbb{L}_2[0,\infty) \to \mathbf{l}_2[0,\infty)$ can be handled in the framework of discrete systems. In discrete time, a Laguerre-domain representation of a system is essentially a re-parameterization and can be obtained by re-writing the system (difference) equations (Nurges and Yaaksoo, 1982). Since the Laguerre functions possess a parameter, its value can be selected to improve the numerical properties of the algorithms performed on the mathematical model, e.g. controller or observer design (Dumont et al., 1990), system identification (Wahlberg, 1991), order reduction (Amghayrir et al., 2005), etc.

The identification of stable continuous linear time-delay systems in Laguerre domain from an impulse response is covered in Hidayat and Medvedev (2012). A classical approach to the estimation of Volterra models kernels is to approximate the kernels by (truncated) series of Laguerre functions (Marmarelis, 1993). The identification of continuous Volterra-Laguerre models with explicit time delay is studied in Bro and Medvedev (2019). To the best of our knowledge, the Laguerre domain modeling of discrete linear time-invariant delay systems has not yet been treated and the present paper fills this gap.

The main contribution of this work is a closed-form expression for the Laguerre spectrum of the output signal of a linear discrete-time system with an explicit time delay. Further, proofs of previously published properties of Laguerre-domain representations of discrete-time systems are provided.

The rest of the paper is organized as follows: in Section 2, some necessary background on discrete Laguerre functions is provided. Then, the concept of Laguerre domain representation of a discrete time-invariant system is revisited in Section 3. Further in Section 4, analytical expressions for the output Laguerre spectrum of the linear time-invariant discrete system with a delay in the input (or output) signal are derived. The utility of the obtained parameterization is illustrated in Section 5 by identifying the pharmacokinetics of levodopa from plasma concentration data from a single dose response. Finally, some conclusions are given in Section 6.

2. THE DISCRETE LAGUERRE FUNCTIONS

The discrete Laguerre functions in $\mathcal{Z}$-domain are given by

$$L_j(z) = \frac{\sqrt{1-p}}{z-\sqrt{p}} T^j(z), \quad T(z) = \frac{1-\sqrt{p}z}{z-\sqrt{p}}, \tag{1}$$

for all $j \in \mathbb{N}$, where the constant $0 < p < 1$ is the discrete Laguerre parameter. Let $\mathbb{H}_d^2$ be the Hardy space of analytic functions on the complement of the unit disc that are square-integrable on the unit circle. The functions L_j, $j \in \mathbb{N}$ constitute an orthonormal complete basis in $\mathbb{H}_d^2$ w. r. t. the inner product

$$\langle W, V \rangle = \frac{1}{2\pi i} \oint_D W(z)\overline{V(z)} \frac{\mathrm{d}z}{z}, \tag{2}$$

where $\overline{V(z)} = V(z^{-1})$ and D is taken the unit circle .

From (1), the complete set of Laguerre functions can be obtained in terms of the discrete-time Laguerre shift operator T by the recursion formula:

$$L_{j+1}(z) = T(z)L_j(z), \; j = 0, 1, \ldots$$

Furthermore, the j-th Laguerre coefficient of $W \in \mathbb{H}_d^2$ is evaluated as the projection of W onto L_j

$$w_j = \langle W, L_j \rangle, \tag{3}$$

and the set $\{w_j\}_{j\in\mathbb{N}}$ is referred to as the *Laguerre spectrum* of W. This parallels the notion of the Fourier spectrum that is obtained by projecting a function onto a set of harmonic functions, while the Laguerre spectrum is calculated by projecting it on a set of weighted (real) exponentials.

The time domain representations of the Laguerre functions $\ell_j = \mathcal{Z}^{-1}\{L_j(z)\}$, $j \in \mathbb{N}$ yield an orthonormal basis in $\mathbf{l}_2[0,\infty)$, the space of square-summable sequences defined for non-negative arguments, where $\mathcal{Z}$ denotes the $\mathcal{Z}$-transform.

3. PROBLEM FORMULATION

Consider the state-space description of a system with a time delay in the input signal

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t-\tau), \\ y(t) &= Cx(t), \end{aligned} \tag{4}$$

where $x : \mathbb{R} \to \mathbb{R}^n$, A, B, C are real matrices of suitable dimensions, and $\tau \in \mathbb{N}$. The problem at hand is to calculate the Laguerre spectrum of the output $\{y_j\}_{j\in\mathbb{N}}$ in terms of A, B, C, and τ given the Laguerre spectrum of the input $\{u_j\}_{j\in\mathbb{N}}$.

3.1 Linear discrete time-invariant system in Laguerre domain

Consider (4) to be delay-free, i.e. $\tau = 0$. In this special case, the state-space equations of (4) in the Laguerre domain were originally derived in Nurges and Yaaksoo (1982) via an ingenuous algebraic approach. To devise a common framework for the treatment of continuous- and discrete-time systems and highlight the role of the Laguerre shift operator, a direct calculation of the expressions for the Laguerre state-space equations has been performed in Fischer and Medvedev (1998). Unfortunately, the latter work is only published in an abbreviated version and without proofs. This deficiency is rectified below by providing a direct proof of the Laguerre-domain description in the form of a discrete convolution.

Proposition 1. Consider the discrete linear time-invariant system in (4), with $y, u \in \mathbf{l}_2$ and $\tau = 0$. For the input Laguerre spectrum $\{u_j\}_{j\in\mathbb{N}}$, the coefficients of the output Laguerre spectrum $\{y_j\}_{j\in\mathbb{N}}$ are given by

$$\begin{aligned} y_j = (1-p)C(I-\sqrt{p}A)^{-2} \sum_{k=0}^{j-1} \overline{T(A)}^{j-k-1} Bu_k + \\ + \sqrt{p}C(I-\sqrt{p}A)^{-1}Bu_j. \end{aligned} \tag{5}$$

Proof. See Appendix A.

Two important properties of the input-output Laguerre domain representation of a discrete LTI system are revealed in (5). First, there is always an algebraic coupling between the input and output signal, i.e. y_j depends on u_j. Second, the convolution representation in (5) possesses "casuality" in the sense that y_j does not depend on $u_k, k > j$. This is despite the fact that a Laguerre coefficient is calculated over the whole support of the function in question, which is $t \in [0, \infty)$ in the present case.

3.2 Equivalent augmented system

An implicit solution to (4) is to augment the state vector with the delayed values of the input signal and thus reduce it to a delay-free description. Then the solution is readily provided by Proposition 1.

Indeed, introduce the matrix $S_\tau \in \mathbb{R}^{\tau\times\tau}$

$$S_\tau = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix},$$

and let $\mathbb{O}$ be a zero matrix of the specified dimensions. Then the augmented system (7)

$$x_\mathrm{a}(t+1) = A_\mathrm{a}x_\mathrm{a}(t) + B_\mathrm{a}u(t), \tag{6}$$

$$y(t) = C_\mathrm{a}x_\mathrm{a}(t), \tag{7}$$

with $x_\mathrm{a} : \mathbb{R} \to \mathbb{R}^{n+\tau}$ and

$$A_\mathrm{a} = \begin{bmatrix} A & [B \; \mathbb{O}_{n,\tau-1}] \\ \mathbb{O}_{\tau,n} & S_\tau \end{bmatrix}, \; B_\mathrm{a} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}, \; C_\mathrm{a}{}^\mathsf{T} = \begin{bmatrix} C^\mathsf{T} \\ \mathbb{O}_{\tau,1} \end{bmatrix},$$

possesses the same dynamics as (4). Yet this approach suffers from high dimensionality of the resulting system description and does not explicitly include the delay τ as a parameter.

4. TIME DELAY SYSTEM

Consider the special case of (4) representing a pure time delay

$$y(t) = u(t-\tau). \tag{8}$$

The Laguerre spectrum of $y(t)$ for a given spectrum of u has been treated in Fischer and Medvedev (1999). However, the formulae in this paper contain typos and no proof of the result is provided.

Naturally, (8) can be written as a delay-free system with S_τ as the system matrix. At first glance, it should be straightforward to obtain the output Laguerre spectrum of (8) from Proposition 1. Yet, since S_τ is a Jordan block, the computation of $\overline{T(S_\tau)}$ requires multiple differentiation of $\overline{T(s)}$ and is cumbersome. Furthermore, τ appears as the dimension and not a parameter of the augmented system. Thus, a direct evaluation of the output Laguerre spectrum by means of (3) is performed below.

Proposition 2. Let the input and output signals of (8) be

$$U(z) = \sum_{k=0}^{\infty} u_k L_k(z), \quad Y(z) = \sum_{k=0}^{\infty} y_k L_k(z).$$

Then the Laguerre spectrum of the output is related to that of the input by

$$y_j = (1-p)(-1)^{j-\tau} \sum_{k=0}^{j-1} L_{j-k}^{(\tau)}(\sqrt{p}) u_k + \sqrt{p}^{\tau} u_j, \tag{9}$$

where

$$L_m^{(\tau)}(\sqrt{p}) = (-\sqrt{p})^{m-\tau} \sum_{n=0}^{\tau-1} \binom{m+n}{n} \binom{m-1}{\tau-n-1} (-p)^n,$$

and it is agreed that $\binom{n}{k} = 0$ for $k > n$ by definition.

Proof. See Appendix B.

The sum in (9) is a convolution of the polynomials $L_m^{(\tau)}(\sqrt{p})$ with the Laguerre spectrum of the input signal.

Now the problem defined in Section 3 is solved by combining the results of Proposition 1 and Proposition 2.

Proposition 3. Consider the discrete linear time-invariant system in (4), with $y, u \in \mathbf{l}_2$. For the input Laguerre spectrum $\{u_j\}_{j\in\mathbb{N}}$, the coefficients of the output Laguerre spectrum $\{y_j\}_{j\in\mathbb{N}}$ are given by

$$y_j = (1-p)C(I-\sqrt{p}A)^{-2} \sum_{k=0}^{j-1} \overline{T(A)}^{j-k-1} B v_k +$$
$$+ \sqrt{p} C (I - \sqrt{p}A)^{-1} B v_j,$$

where

$$v_j = (1-p)(-1)^{j-\tau} \sum_{k=0}^{j-1} L_{j-k}^{(\tau)}(\sqrt{p}) u_k + \sqrt{p}^{\tau} u_j,$$

5. NUMERICAL EXAMPLE

The examples in this section are solely intended to illustrate a potential use of the obtained theoretical results and should not be considered as a system identification approach *per se*, as the properties of the estimates are not investigated.

5.1 Simulated data

The noise-free unit impulse response of (4) with

$$A = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad B = \begin{bmatrix} 0.7 \\ 1.3 \end{bmatrix}, \quad C = [1.1 \ \ 0.6]$$

under zero initial conditions was simulated. The Laguerre spectrum of the input is then $u_k = \sqrt{1-p}(-\sqrt{p})^k$, and the Laguerre spectrum for the output was determined by numerically evaluating (3) in time domain.

Gridding was used to find a suitable value of the Laguerre parameter p. For each grid point, a second-order Laguerre domain model was then estimated in two different ways. First, an ARX model was fitted by least squares to the input and estimated output Laguerre spectra of order $L = 15$, and a state-space realization was computed from the estimated transfer function. Second, a state-space model was computed directly, using the *n4sid* subspace identification algorithm. Using the inverse transformation in Fischer and Medvedev (1998), a time-domain state-space representation is calculated from a Laguerre-domain state-space description. The Laguerre parameter p was then chosen to yield the least (time-domain) output MSE. Fig. 1 shows the MSE as a function of $\sqrt{p}$, and Fig. 2 presents the model outputs together with the measured output. For the ARX model, the estimated Laguerre parameter was $\sqrt{p} = 0.44$ with an output MSE of 0.0023. In the subspace case, the estimated Laguerre parameter was $\sqrt{p} = 0.77$, with an output MSE of 0.0163. The ARX model thus provides a better output data fit and the corresponding loss function possesses no local minima, in contrast to the loss function for the subspace method. This is expected as subspace identification demands a higher order of system excitation.

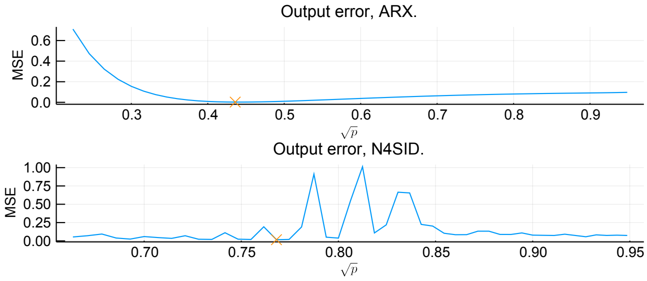

Fig. 1. MSE as a function of the Laguerre parameter p when using ARX (top) and subspace identification (bottom), simulated data.

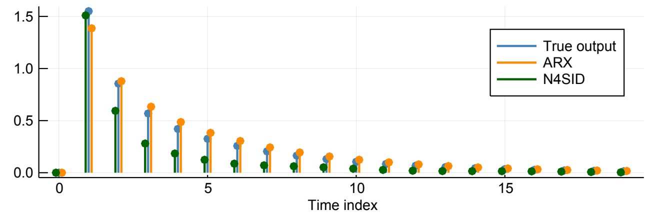

Fig. 2. Model outputs vs true simulated output.

5.2 Experimental data

The experimental data represent the blood concentration of the anti-Parkinsonian drug Levodopa (LD) after a single dose administered to a patient. The experimental protocol is described in detail in Senek et al. (2017). The blood sampling is irregular, with the shortest time interval in between two sample points being 15 min. The series was therefore re-sampled to a constant sampling interval of 10 min using linear interpolation between the measured values.

The Laguerre parameter p and the delay τ were estimated by gridding, i.e. by identifying the system as described above for each grid point. Fig. 3 shows the MSE as a function of p for the optimal delay estimate, 20 min, and Fig. 4 compares the estimated model outputs with the measured LD blood concentration. The estimated Laguerre parameters were $\sqrt{p} = 0.32$ in the ARX case and $\sqrt{p} = 0.21$ in the subspace case. The corresponding output MSE values were 0.022 for the ARX model and 0.015 using the subspace method. Interestingly,

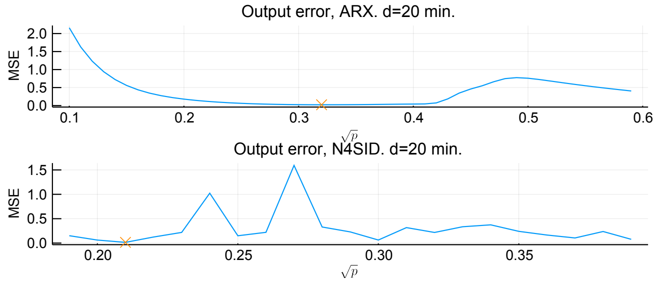

Fig. 3. MSE as a function of the Laguerre parameter p when using ARX (top) and subspace identification (bottom).

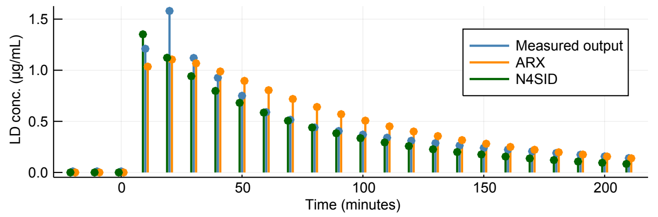

Fig. 4. Model outputs vs true LD measurements.

subspace identification performs better than ARX in this case, a phenomenon that is worth investigating further. Notice also a nonlinear system behavior for higher LD concentrations in the beginning of the data set.

6. CONCLUSION

The problem of modeling linear discrete time-delay systems with the input and output signals in Laguerre domain was considered in this paper. The study was primarily motivated by biomedical applications, where the input signal vanishes asymptotically or has finite support. A closed-form expression for the Laguerre spectrum of the system output given the Laguerre spectrum of the input and a state-space model of the system was derived. In the Laguerre domain, due to the delay, the input-output map involves a certain class of polynomials, whose nature was not readily recognized. An application of the developed modeling approach produced promising results on pharmacokinetic data representing the administration of a bolus dose of levodopa to a Parkinsonian patient.

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Appendix A.

Proof of Proposition 1

Under the initial conditions $x(0) = x_0$, the solution to the state equation is given by

$$x(t) = A^t x_0 + \sum_{k=0}^{t-1} A^{t-k-1} Bu(k).$$

By taking the $\mathcal{Z}$-transform, one obtains

$$X(z) = z(zI_n - A)^{-1} x_0 + (zI_n - A)^{-1} BU(z),$$

where the capital letters denote the corresponding signal in the transform domain. The output is then

$$Y(z) = zC(zI_n - A)^{-1} x_0 + C(zI_n - A)^{-1} BU(z).$$

Projecting the first term onto the Laguerre functions, the j-th coefficient is

$$\begin{aligned}&\langle zC(zI_n - A)^{-1} x_0, L_j(z)\rangle \\ &= \frac{1}{2\pi i} C \oint_D (zI_n - A)^{-1} \overline{L_j(z)}\, dz\, x_0 = C\overline{L_j(A)} x_0.\end{aligned}$$

The last equality is due to the definition of the matrix function through Cauchy's integral formula since all the eigenvalues of A are within D. Since

$$L_j(z) = \frac{\sqrt{1-p}}{1-\sqrt{p}z} \left(\frac{z-\sqrt{p}}{1-\sqrt{p}z} \right)^j,$$

it applies, for the case $u \equiv 0$, that

$$\begin{aligned}y_0 &= C\overline{L_0(A)} x_0, \\ y_j &= C\overline{L_0(A)} \overline{T^j(A)} x_0, \quad j = 0, 1, \ldots\end{aligned}$$

with

$$\begin{aligned}\overline{L_0(A)} &= \sqrt{1-p}\,(I - \sqrt{p}A)^{-1}, \\ \overline{T(A)} &= (A - \sqrt{p}I)(I - \sqrt{p}A)^{-1}.\end{aligned}$$

Under the assumptions made, the function $u(\cdot)$ is uniquely represented by its Laguerre spectrum, i.e.

$$U(z) = \sum_{k=0}^{\infty} u_k L_k(z),$$

where $u_k = \langle U(z), L_k(z) \rangle$, for all $k \in \mathbb{N}$. Therefore

$$Y(z) = \sum_{k=0}^{\infty} u_k C(zI - A)^{-1} B L_k(z).$$

For the forced response in the output signal, the Laguerre coefficients of the output are

$$y_j = \langle Y(z), L_j(z) \rangle = \sum_{k=0}^{\infty} \langle u_k C(zI-A)^{-1} B L_k(z), L_j(z) \rangle = = \frac{C}{2\pi i} \sum_{k=0}^{\infty} u_k \oint_D (zI-A)^{-1} L_k(z) \overline{L_j(z)} \frac{\mathrm{d}z}{z} B.$$

Notice that for the Laguerre shift operator, it applies

$$\overline{T(z)} = \frac{|z|^2 - \sqrt{p}\mathrm{Re}(z) + p}{p|z|^2 - \sqrt{p}\mathrm{Re}(z) + 1} T^{-1}(z).$$

Hence on the unit circle D, it follows $\overline{T(z)} = T^{-1}(z)$. Then, by direct evaluation

$$\frac{1}{z} L_k(z) \overline{L_j(z)} = \frac{1-p}{(z-\sqrt{p})(1-\sqrt{p}z)} T^{k-j}(z).$$

Consider now the products

$$\langle C(zI-A)^{-1} B L_k(z), L_j(z) \rangle = \frac{1-p}{2\pi i} C \oint_D (zI-A)^{-1} \frac{T^{k-j}(z)}{(z-\sqrt{p})(1-\sqrt{p}z)} \mathrm{d}z B \tag{A.1}$$

Direct term: For the case $k = j$, (A.1) turns into

$$\langle C(zI-A)^{-1} B L_k(z), L_k(z) \rangle = \frac{1-p}{2\pi i} C \oint_D (zI-A)^{-1} \frac{\mathrm{d}z}{(z-\sqrt{p})(1-\sqrt{p}z)} B.$$

The product is evaluated to

$$\langle C(zI-A)^{-1} B L_k(z), L_k(z) \rangle = C\sqrt{p}(I - \sqrt{p}A)^{-1} B.$$

Therefore, a direct term is present in the Laguerre domain description of the system even though there is no such term in the time-domain description, i.e. y_k always depends on u_k.

Causality: Due to the previously mentioned similarity between the Laguerre shift and the conventional discrete backward shift, casuality is preserved in the Laguerre domain in the sense that y_k depends only on $u_m, m = 0, \ldots, k$ and not on $u_m, m > k$. To confirm this, consider (A.1) for the case $j > k$, or, equivalently $j - k = r > 0$

$$\langle C(zI-A)^{-1} B L_k(z), L_j(z) \rangle = \frac{1-p}{2\pi i} C \oint_D (zI-A)^{-1} \frac{(z-\sqrt{p})^{r-1}}{(1-\sqrt{p}z)^{r+1}} \mathrm{d}z B = 0.$$

The latter equality holds since all the singularities of the integrand are outside of D.

Convolution: Finally, evaluate the Laguerre spectrum of the output $y(t)$ under zero initial conditions by making use of the results above. One has

$$\begin{aligned} y_j &= \sum_{k=0}^{\infty} \langle u_k C(zI-A)^{-1} B L_k(z), L_j(z) \rangle = \\ &= \frac{C(1-p)}{2\pi i} \sum_{k=0}^{\infty} u_k \oint_D (zI-A)^{-1} L_k(z) \overline{L_j(z)} \frac{\mathrm{d}z}{z} B \\ &= \frac{C(1-p)}{2\pi i} \sum_{k=0}^{j} u_k \oint_D (zI-A)^{-1} L_k(z) \overline{L_j(z)} \frac{\mathrm{d}z}{z} B, \end{aligned}$$

since the higher-order terms in the sum are zero due to the property of casuality. Taking into account the expression for the direct term yields

$$y_j = \frac{C(1-p)}{2\pi i} \sum_{k=0}^{j-1} u_k \oint_D (zI-A)^{-1} L_k(z) \overline{L_j(z)} \frac{\mathrm{d}z}{z} B + C(\sqrt{p}I - A)^{-1} B u_j.$$

Then, the j-th Laguerre coefficient of the output is given by

$$y_j = (1-p) C (I - \sqrt{p}A)^{-2} \sum_{k=0}^{j-1} \overline{T(A)}^{j-k-1} u_k B + + \sqrt{p} C (I - \sqrt{p}A)^{-1} B u_j.$$

Similarly to time domain, the output of the LTI system in Laguerre domain is a convolution of the Laguerre domain Markov parameters and the Laguerre coefficients of the input.

Appendix B.

Proof of Proposition 2

By taking $\mathcal{Z}$-transform of (8) it holds that

$$Y(z) = U(z) z^{-\tau}.$$

Calculating the Laguerre coefficients of the output gives

$$\begin{aligned} y_j &= \langle Y(z), L_j(z) \rangle = \frac{1}{2\pi i} \oint_D Y(z) \overline{L_j(z)} \frac{\mathrm{d}z}{z} \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} u_k \oint_D L_k(z) \overline{L_j(z)} z^{-\tau-1} \mathrm{d}z. \end{aligned}$$

Now, the contour integrals have to be evaluated. The integrand on the unit circle D is

$$L_k(z) \overline{L_j(z)} z^{-\tau-1} = \frac{(1-p) T^{k-j}(z)}{(z-\sqrt{p})(1-\sqrt{p}z) z^{\tau}}$$

Direct term. Consider the direct term, i.e. $k = j$

$$\frac{(1-p)}{2\pi i} \oint_D \frac{\mathrm{d}z}{(z-\sqrt{p})(1-\sqrt{p}z) z^{\tau}} = (1-p)\left(\mathrm{Res}(\cdot, \sqrt{p}) + \mathrm{Res}(\cdot, 0)\right),$$

since $z = \frac{1}{\sqrt{p}}$ is outside the unit circle. Now

$$\mathrm{Res}(\cdot, \sqrt{p}) = \frac{1}{(1-p)\sqrt{p}^{\tau}},$$

$$\mathrm{Res}(\cdot, 0) = \frac{1}{(\tau-1)!} \lim_{z \to 0} \frac{\mathrm{d}^{\tau-1}}{\mathrm{d}z^{\tau-1}} \frac{1}{(z-\sqrt{p})(1-\sqrt{p}z)}.$$

Since

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} \frac{1}{z-\sqrt{p}} = (-1)^n n! (z-\sqrt{p})^{-n-1},$$

$$\frac{d^n}{dz^n} \frac{1}{1-\sqrt{p}z} = \sqrt{p}^n n! (1-\sqrt{p}z)^{-n-1},$$

$$\frac{1}{(z-\sqrt{p})(1-\sqrt{p}z)} = \frac{1}{1-p} \left(\frac{1}{z-\sqrt{p}} + \frac{\sqrt{p}}{1-\sqrt{p}z} \right),$$

it follows that

$$\frac{d^{\tau-1}}{dz^{\tau-1}}\frac{1}{(z-\sqrt{p})(1-\sqrt{p}z)}=$$
$$=\frac{(\tau-1)!}{1-p}\left(\frac{(-1)^{\tau-1}}{\left(z-\sqrt{p}\right)^{\tau}}+\frac{\sqrt{p}^{\tau}}{\left(1-\sqrt{p}z\right)^{\tau}}\right).$$

Thus,

$$\mathrm{Res}(\cdot,0)=\frac{1}{1-p}\left(\sqrt{p}^{\tau}-\sqrt{p}^{-\tau}\right).$$

To summarize, for $k=j$:

$$\frac{(1-p)}{2\pi i}\oint_D\frac{\mathrm{d}z}{(z-\sqrt{p})(1-\sqrt{p}z)z^{\tau}}=$$
$$=(1-p)\left(\frac{1}{(1-p)\sqrt{p}^{\tau}}+\frac{1}{1-p}\left(\sqrt{p}^{\tau}-\sqrt{p}^{-\tau}\right)\right)=\sqrt{p}^{\tau}.$$

Convolution. The general case calls for evaluating the following expression

$$\begin{aligned}y_j&=\frac{1}{2\pi i}\sum_{k=0}^{\infty}u_k\oint_D L_k(z)\overline{L_j(z)}z^{-\tau-1}\,\mathrm{d}z\\&=\frac{1-p}{2\pi i}\sum_{k=0}^{\infty}u_k\oint_D\frac{T^{k-j}(z)}{(z-\sqrt{p})(1-\sqrt{p}z)z^{\tau}}\,\mathrm{d}z\\&=\frac{1-p}{2\pi i}\sum_{k=0}^{\infty}u_k\oint_D\frac{(z-\sqrt{p})^{j-k-1}}{(1-\sqrt{p}z)^{j-k+1}z^{\tau}}\,\mathrm{d}z \qquad \text{(B.1)}\end{aligned}$$

For $j>k$, there are multiple singularities at $z=0$ while $z=\frac{1}{\sqrt{p}}$ is outside the unit circle. Thus, we only calculate

$$\mathrm{Res}(\cdot,0)=\frac{1}{(\tau-1)!}\lim_{z\to 0}\frac{\mathrm{d}^{\tau-1}}{\mathrm{d}z^{\tau-1}}\frac{(z-\sqrt{p})^{j-k-1}}{(1-\sqrt{p}z)^{j-k+1}}.$$

It holds that

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}(z-\sqrt{p})^{j-k-1}=\frac{(j-k-1)!}{(j-k-1-n)!}(z-\sqrt{p})^{j-k-1-n},$$
$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}(1-\sqrt{p}z)^{-(j-k+1)}=\sqrt{p}^n\frac{(j-k+n)!}{(j-k)!}(1-\sqrt{p}z)^{-(j-k+1+n)}.$$

The derivative of the rational function is then

$$\frac{\mathrm{d}^{\tau-1}}{\mathrm{d}z^{\tau-1}}\frac{(z-\sqrt{p})^{j-k-1}}{(1-\sqrt{p}z)^{j-k+1}}=$$
$$=\sum_{n=0}^{\tau-1}\binom{\tau-1}{n}\frac{(j-k-1)!(j-k+n)!}{(j-k-\tau+n)!(j-k)!}\sqrt{p}^n\frac{(z-\sqrt{p})^{j-k-\tau+n}}{(1-\sqrt{p}z)^{j-k+1+n}}.$$

Then, the residue is

$$\frac{1}{(\tau-1)!}\lim_{z\to 0}\frac{\mathrm{d}^{\tau-1}}{\mathrm{d}z^{\tau-1}}\frac{(z-\sqrt{p})^{j-k-1}}{(1-\sqrt{p}z)^{j-k+1}}$$
$$=\sum_{n=0}^{\tau-1}\frac{(j-k-1)!(j-k+n)!}{n!(\tau-1-n)!(j-k-\tau+n)!(j-k)!}\sqrt{p}^n(-\sqrt{p})^{j-k-\tau+n}$$
$$=\sum_{n=0}^{\tau-1}\binom{j-k+n}{n}\binom{j-k-1}{\tau-1-n}(-1)^{j-k-\tau+n}\sqrt{p}^{j-k-\tau+2n}.$$

For $j<k$, (B.1) has multiple singularities in $z=0$ and a pole of order $k-j+1$ at $z=\sqrt{p}$. Hence

$$\mathrm{Res}(\cdot,\sqrt{p})=\frac{1}{(k-j)!}\lim_{z\to\sqrt{p}}\frac{\mathrm{d}^{k-j}}{\mathrm{d}z^{k-j}}\frac{(1-\sqrt{p}z)^{k-j-1}}{z^{\tau}}.$$

Since

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}(1-\sqrt{p}z)^{k-j-1}=$$
$$(-1)^n\sqrt{p}^n\frac{(k-j-1)!}{(k-j-1-n)!}(1-\sqrt{p}z)^{k-j-1-n}$$

and

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{1}{z^{\tau}}=(-1)^n\frac{(\tau-n+1)!}{(\tau-1)!}(z)^{-\tau-n}$$

then

$$\mathrm{Res}(\cdot,\sqrt{p})=$$
$$\frac{1}{(k-j)!}\lim_{z\to\sqrt{p}}\sum_{n=0}^{k-j}\binom{k-j}{n}(-1)^n\sqrt{p}^n\frac{(k-j-1)!}{(k-j-1-n)!}$$
$$(-1)^{k-j-n}\frac{(\tau-k+j+n+1)!}{(\tau-1)!}z^{-\tau-k+j+n}(1-\sqrt{p}z)^{k-j-1-n}$$
$$=\frac{1}{(k-j)!}\sum_{n=0}^{k-j}\binom{k-j}{n}p^n\frac{(k-j-1)!}{(k-j-1-n)!}$$
$$(-1)^{k-j}\frac{(\tau-k+j+n+1)!}{(\tau-1)!}\sqrt{p}^{-\tau-k+j}(1-p)^{k-j-1-n}$$
$$=\sum_{n=0}^{k-j}\binom{k-j-1}{n}\binom{\tau-k+j+n+1}{\tau-1}p^n$$
$$(-1)^{k-j}\sqrt{p}^{-\tau-k+j}(1-p)^{k-j-1-n}$$

Also, the following residue has to be calculated

$$\mathrm{Res}(\cdot,0)=\frac{1}{(\tau-1)!}\lim_{z\to 0}\frac{\mathrm{d}^{\tau-1}}{\mathrm{d}z^{\tau-1}}\frac{(1-\sqrt{p}z)^{k-j-1}}{(z-\sqrt{p})^{k-j+1}}.$$

It holds that

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}(z-\sqrt{p})^{-(k-j+1)}=$$
$$=\frac{(-1)^n(k-j+n)!}{(k-j)!}(z-\sqrt{p})^{-(k-j+1+n)}.$$

The derivative of the rational function is then

$$\frac{\mathrm{d}^{\tau-1}}{\mathrm{d}z^{\tau-1}}\frac{(1-\sqrt{p}z)^{k-j-1}}{(z-\sqrt{p})^{k-j+1}}=$$
$$=\sum_{n=0}^{\tau-1}\binom{\tau-1}{n}\frac{(k-j-1)!(k-j+n)!}{(k-j-\tau+n)!(k-j)!}(-1)^{\tau-n}\sqrt{p}^{\tau-n}$$
$$(-1)^n(z-\sqrt{p})^{-(k-j+1+n)}(1-\sqrt{p}z)^{k-j-\tau+n}$$
$$=\sum_{n=0}^{\tau-1}\binom{\tau-1}{n}\frac{(k-j-1)!(k-j+n)!}{(k-j-\tau+n)!(k-j)!}(-1)^{\tau}\sqrt{p}^{\tau-n}$$
$$(z-\sqrt{p})^{-(k-j+1+n)}(1-\sqrt{p}z)^{k-j-\tau+n}.$$

Then, the residue is

$$\frac{1}{(\tau-1)!}\lim_{z\to 0}\frac{d^{\tau-1}}{dz^{\tau-1}}\frac{(z-\sqrt{p})^{j-k-1}}{(1-\sqrt{p}z)^{j-k+1}}$$
$$=\sum_{n=0}^{\tau-1}\binom{\tau-1}{n}\frac{(k-j-1)!(k-j+n)!}{(k-j-\tau+n)!(k-j)!}$$
$$\times(-1)^{\tau-(k-j+1+n)}\sqrt{p}^{\tau-n-(k-j+1+n)}$$
$$=\sum_{n=0}^{\tau-1}\binom{k-j-n}{n}\binom{k-j-1}{\tau-n-1}(-1)^{\tau-(k-j+1+n)}$$
$$\times\sqrt{p}^{\tau-(k-j+1)}p^{-n}.$$

Finally, this yields the Laguerre coefficient

$$y_j=(1-p)\sum_{k=0}^{j-1}\left(\sum_{n=0}^{\tau-1}\binom{j-k+n}{n}\binom{j-k-1}{\tau-n-1}\right.$$
$$\left.\times(-1)^{j-k-\tau+n}\sqrt{p}^{j-k-\tau+2n}\right)u_k+\sqrt{p}^{\tau}u_j.$$