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An Impossibility Result Concerning Bounded Average-Moment Control of Linear Stochastic Systems ⋆

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Abstract: It is known that strictly unstable linear systems that are subject to nonvanishing additive stochastic noise with unbounded supports cannot be stabilized by using deterministically bounded control inputs. In this paper, we explore similar impossibility results for scenarios where the expected value of the squared control input norm is subject to constraints and the support of the noise distribution is not necessarily unbounded. Specifically, we consider the stabilization problem with control policies that have bounded time-averaged second moments. We obtain values of such average second moment bounds, below which stabilization is not possible and the second moment of the state diverges regardless of the choice of the control policy and the initial state distribution. The results are illustrated with a numerical example.

Keywords: Stochastic systems, constrained control, instability analysis, networked control

1. INTRODUCTION

In the last few decades, the stabilization problem under control input constraints has been investigated for deterministic systems extensively (see Saberi et al. (2012) for an overview). It has been established that linear deterministic systems with strictly unstable system matrices cannot be globally stabilized under bounded control inputs (Sontag, 1984; Sussmann et al., 1994).

More recently, with the increasing interest in stochastic optimal and model predictive control approaches, constrained control of stochastic systems has become an important topic (Mesbah, 2016). As in the deterministic case, stochastic systems also yield impossibility results. It has been shown by Chatterjee et al. (2012) that bounded control policies cannot stabilize unstable linear stochastic systems with nonvanishing and unbounded additive stochastic noise. Specifically, Chatterjee et al. (2012) considered the control problem of a discrete-time linear stochastic system and proved that when the distribution of the additive noise has unbounded support and the system matrix has an eigenvalue outside the unit circle of the complex plane, the second moment of the system state diverges under deterministically constrained control inputs.

To overcome the difficulties in the stabilization of strictly unstable linear stochastic systems, considering probabilistic constraints instead of hard deterministic constraints can be a viable option. In particular, chance constraints (Farina et al., 2015; Lorenzen et al., 2016) and constraints based on the expected values of the states and control inputs can be utilized. In this paper, we are interested in deriving conditions under which stabilization is again not possible even with probabilistic control input constraints.

Specifically, we consider linear discrete-time stochastic systems with additive noise and strictly unstable system matrices. For the stabilization problem, we investigate a class of control policies that have bounded time-averaged second moments. This class is fairly large and contains many existing control policies that are constrained deterministically or probabilistically. Our goal is to identify the scenarios in which a stochastic system cannot be stabilized with controllers from this class. To this end, we obtain conditions on the system dynamics and additive noise properties, under which the second moment of the state *diverges* regardless of the particular choice of the controller. Based on these conditions, we also provide a method for using the eigenstructures of system matrices in *instability* assessment. Our result allows us to obtain impossibility results for networked control systems subject to packet losses. We show that designing stabilizing constrained controllers for networked control systems under noise is not possible if the average probability of successful transmissions of control commands over the network is known to be too small.

In our instability analysis, we use nonnegative-definite Hermitian matrices to characterize the variance of the noise projected on an unstable mode of the uncontrolled system. This approach allows us to check instability of stochastic systems with both unbounded and bounded noise distributions. We note that Nair and Evans (2004) and Chatterjee et al. (2012) previously discussed instability problems under unbounded noise distributions and deterministically constrained controllers. The approaches in those works are different from ours in that they are based on the observation that unbounded distributions guarantee nonzero probabilities for the events where the noise norm exceeds certain values.

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The *instability problem* due to *stochastic noise* and *probabilistic control constraints* was previously not considered in the literature. Existing works mostly deal with deterministically-constrained controllers and explore their use in the stabilization problem. For instance, in the case of bounded noise with sufficiently small norms, control laws with deterministic norm constraints were used for achieving practical stability by Blanchini (1990), Braslavsky and Middleton (1993), Lin and Saberi (1995), Kolmanovsky and Gilbert (1995), and Zhou et al. (2015). Such deterministic norm constraints were also extended to deterministic rate and increment constraints by Bateman and Lin (2002) as well as Mesquine et al. (2004). Moreover, in the case of stochastic noise, deterministic control input norm constraints have been incorporated in model predictive control (Chatterjee et al., 2011; Hokayem et al., 2012; Korda et al., 2014), nonlinear control (Min et al., 2018), adaptive control (Yao and Zhang, 2019), and networked control systems (Mishra et al., 2018). Stabilization results presented in those works for stochastic noise with unbounded supports exclude systems with strictly unstable system matrices, since such systems cannot be stabilized by deterministically-constrained control inputs as pointed out by Chatterjee et al. (2012). On the the hand, there are also a few works that explore chance constraints in optimization problems faced in control (e.g., Farina et al. (2015) and Lorenzen et al. (2016)). However, instability analysis is not the main concern in those works. We note that instability of networked control systems was previously discussed in Cetinkaya et al. (2017), but the systems analyzed in that work are noise-free and instability is caused by the actions of a malicious attacker.

We organize the rest of the paper as follows. In Section 2, we describe the bounded average-moment control of discrete-time linear stochastic systems. We then present our impossibility results on stabilization in Section 3. We illustrate our results with a numerical example in Section 4 and conclude the paper in Section 5.

In this paper, $\mathbb{N}$ and $\mathbb{N}_0$ respectively denote the sets of positive and nonnegative integers, $\|\cdot\|$ denotes the Euclidean norm, $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ respectively denote the minimum and maximum eigenvalues of a Hermitian matrix $H \in \mathbb{C}^{n\times n}$. Moreover, $H^{\frac{1}{2}}$ represents the unique nonnegative-definite Hermitian square root of a nonnegative-definite Hermitian matrix $H \in \mathbb{C}^{n\times n}$, satisfying $H^{\frac{1}{2}}H^{\frac{1}{2}} = H$ and $(H^{\frac{1}{2}})^* = H^{\frac{1}{2}}$. The notations $\mathbb{P}[\cdot]$ and $\mathbb{E}[\cdot]$ respectively denote the probability and expectation on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with sample space Ω and σ-algebra $\mathcal{F}$. We use $\mathcal{B}(\mathbb{R}^n)$ to denote the Borel σ-algebra associated with $\mathbb{R}^n$. Furthermore, $\overline{c}$ denotes the complex conjugate of a complex number $c \in \mathbb{C}$, and C^* denotes the complex conjugate transpose of a complex matrix $C \in \mathbb{C}^{n\times m}$, i.e., $C^*_{i,j} = \overline{C_{j,i}}$, $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$.

2. BOUNDED AVERAGE-MOMENT CONTROL

Consider the linear stochastic dynamical system given by

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t), \quad t \in \mathbb{N}_0, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $w(t) \in \mathbb{R}^n$ is the noise. We assume that $w(0), w(1), \ldots$ are independent and identically distributed. Their distribution is represented with probability measure $\phi\colon \mathcal{B}(\mathbb{R}^n) \to [0,1]$ satisfying $\mathbb{P}[w(t) \in W] = \phi(W)$, $W \in \mathcal{B}(\mathbb{R}^n)$, $t \in \mathbb{N}_0$. We further assume that the noise has zero mean, that is, $\mathbb{E}[w(t)] = \int_{\mathbb{R}^n} w\phi(\mathrm{d}w) = 0$, and moreover, the initial state x_0 and the noise process $\{w(t) \in \mathbb{R}^n\}_{t\in\mathbb{N}_0}$ are mutually independent.

To characterize the class of control policies that we investigate, we consider a filtration $\{\mathcal{F}_t \subset \mathcal{F}\}_{t\in\mathbb{N}_0}$ such that 1) x_0 is $\mathcal{F}_0$-measurable, 2) for each $t \in \mathbb{N}$, the random variables $x(0),\ldots,x(t)$, and $w(0),\ldots,w(t-1)$ are $\mathcal{F}_t$-measurable, and 3) $w(t)$ is independent of $\mathcal{F}_t$.

In this paper, we focus on $\mathcal{F}_t$-adapted control policies, that is, for each $t \in \mathbb{N}_0$, $u(t)$ is $\mathcal{F}_t$-measurable. Notice that the static state feedback policy $u(t) = \tau(x(t))$ and feedback policies of the form $u(t) = \pi(t, x(0),\ldots,x(t))$ are $\mathcal{F}_t$-measurable. We note that the σ-algebra $\mathcal{F}_t$ is allowed to include events associated with random variables other than $x(0),\ldots,x(t)$, $w(0),\ldots,w(t-1)$, and $u(t)$. For instance, in the case of a networked control problem over a communication channel that is subject to random packet losses, an $\mathcal{F}_t$-measurable binary-variable $l(t) \in \{0,1\}$ may be used for denoting the status of the channel. In such a case, the control input $u(t)$ may depend on $l(t)$.

Since $w(t)$ is an additive noise and does not converge to zero, the system state and its moments cannot converge to 0 regardless of the control input. Therefore, instead of asymptotic stabilization, we explore a weaker notion of stabilization. In particular, we consider the *bounded second-moment stabilization* notion, where the control goal is to achieve $\sup_{t\in\mathbb{N}_0} \mathbb{E}[\|x(t)\|^2] < \infty$.

It is shown by Chatterjee et al. (2012) that if the system matrix A is strictly unstable (having one or more eigenvalues outside the unit circle of the complex plane) and the projection of the noise $w(t)$ onto an eigenspace associated with an unstable eigenvalue has unbounded support, then the system (1) cannot be stabilized by control laws satisfying deterministic constraints of the form $\|u(t)\| \le \tilde{u}$, $t \in \mathbb{N}_0$, with $\tilde{u} \in (0,\infty)$. In this paper, we investigate similar impossibility results for the case where the *expected value* of the norm of the control input is bounded. In particular, we consider control policies that are bounded in *their time-averaged second moments* as characterized in the following definition.

Definition 1. A control policy has bounded average second moment if

$$\frac{1}{t}\sum_{i=0}^{t-1} \mathbb{E}\left[\|u(i)\|^2\right] \le \hat{u}, \quad t \in \mathbb{N}, \tag{2}$$

where $\hat{u} \ge 0$.

Notice that bounded control laws have bounded second moments and thus have bounded average second moments. This is represented by a chain of implications given as

$$\begin{aligned} &\mathbb{P}[\|u(t)\|^2 \le \hat{u},\ t \in \mathbb{N}_0] = 1 \\ &\quad\implies \mathbb{E}[\|u(t)\|^2] \le \hat{u},\ t \in \mathbb{N}_0 \\ &\quad\implies \frac{1}{t}\sum_{i=0}^{t-1} \mathbb{E}\left[\|u(i)\|^2\right] \le \hat{u},\ t \in \mathbb{N}. \end{aligned} \tag{3}$$

Similarly, we have

$$\mathbb{P}[\|u(t)\|^2 \leq \hat{u},\ t \in \mathbb{N}_0] = 1$$
$$\implies \mathbb{P}\left[\frac{1}{t}\sum_{i=0}^{t-1}\|u(i)\|^2 \leq \hat{u},\ t \in \mathbb{N}\right] = 1$$
$$\implies \frac{1}{t}\sum_{i=0}^{t-1}\mathbb{E}\left[\|u(i)\|^2\right] \leq \hat{u},\ t \in \mathbb{N}, \quad (4)$$

which indicates that control laws that are bounded (on average) also have bounded average second moments. Note also that Definition 1 allows $\mathbb{E}[\|u(t)\|^2]$ to exceed $\hat{u}$ at certain times, as long as (2) holds for all $t \in \mathbb{N}$. Moreover, risk-constrained control policies can also be considered within the characterization of Definition 1. For instance, if the Conditional-Value-at-Risk associated with $\|u(t)\|^2$ is bounded by $\hat{u}$, then (2) holds. This is because $\mathbb{E}[\|u(t)\|^2]$ is upper-bounded by the Conditional-Value-at-Risk associated with $\|u(t)\|^2$ (Uryasev and Rockafellar, 2001).

In this paper, we are interested in (2) as a way of characterizing a large class of control policies that encapsulates some of the existing (deterministically or probabilistically) constrained control laws.

3. AN IMPOSSIBILITY RESULT ON STABILIZATION

Our goal in this section is to identify the cases where the linear stochastic system (1) cannot be stabilized with control policies that satisfy (2). The following result provides a characterization of those cases. Specifically, it provides conditions on the system dynamics and $\hat{u}$, under which the second moment of the state *diverges*.

Theorem 2. Consider the linear stochastic system (1). Assume there exist a nonnegative-definite Hermitian matrix $R \neq 0$ and scalars $\alpha_{\mathrm{L}}, \alpha_{\mathrm{U}} > 1$ such that

$$\alpha_{\mathrm{L}} R \leq A^T R A \leq \alpha_{\mathrm{U}} R, \quad (5)$$
$$\int_{\mathbb{R}^n} w^{\mathrm{T}} R w \phi(\mathrm{d}w) > 0. \quad (6)$$

If the control policy is $\mathcal{F}_t$-adapted and satisfies (2) with

$$\hat{u} < \begin{cases} \sigma^2\left(\alpha_{\mathrm{L}} - 1\right) / \left(\alpha_{\mathrm{U}}\beta_{\mathrm{U}} - \alpha_{\mathrm{L}}\beta_{\mathrm{L}} + \beta_{\mathrm{L}}\right), & \text{if } \beta_{\mathrm{U}} \neq 0, \\ \infty, & \text{otherwise,} \end{cases} \quad (7)$$

where $\beta_{\mathrm{L}} \triangleq \lambda_{\min}(B^{\mathrm{T}} R B)$, $\beta_{\mathrm{U}} \triangleq \lambda_{\max}(B^{\mathrm{T}} R B)$, and $\sigma \triangleq \sqrt{\int_{\mathbb{R}^n} w^{\mathrm{T}} R w \phi(\mathrm{d}w)}$, then the second moment of the state diverges, that is,

$$\lim_{t\to\infty} \mathbb{E}[\|x(t)\|^2] = \infty, \quad (8)$$

for any initial state distribution.

Proof. First, we define the nonnegative-definite function $V : \mathbb{R}^n \to [0, \infty)$ by $V(x) \triangleq x^{\mathrm{T}} R x$. Our initial goal is to show that $\lim_{t\to\infty} \mathbb{E}[V(x(t))] = \infty$.

Note that $w(t)$ and $u(t)$ are independent, and similarly $w(t)$ and $x(t)$ are independent. Therefore, by $\mathbb{E}[w(t)] = 0$, we have $\mathbb{E}[x^{\mathrm{T}}(t)A^{\mathrm{T}} R w(t)] = \mathbb{E}[x^{\mathrm{T}}(t)A^{\mathrm{T}} R]\mathbb{E}[w(t)] = 0$ and $\mathbb{E}[u^{\mathrm{T}}(t)B^{\mathrm{T}} R w(t)] = \mathbb{E}[u^{\mathrm{T}}(t)B^{\mathrm{T}} R]\mathbb{E}[w(t)] = 0$. Thus, it follows from (1) that

$$\begin{aligned}\mathbb{E}[V(x(t+1))] &= \mathbb{E}[x^{\mathrm{T}}(t+1)Rx(t+1)] \\ &= \mathbb{E}[x^{\mathrm{T}}(t)A^{\mathrm{T}}RAx(t)] + \mathbb{E}[x^{\mathrm{T}}(t)A^{\mathrm{T}}RBu(t)] \\ &\quad + \mathbb{E}[u^{\mathrm{T}}(t)B^{\mathrm{T}}RAx(t)] + \mathbb{E}[u^{\mathrm{T}}(t)B^{\mathrm{T}}RBu(t)] \\ &\quad + \mathbb{E}[w^{\mathrm{T}}(t)Rw(t)].\end{aligned} \quad (9)$$

By (5), we get $\mathbb{E}[x^{\mathrm{T}}(t)A^{\mathrm{T}}RAx(t)] \geq \alpha_{\mathrm{L}}\mathbb{E}[x^{\mathrm{T}}(t)Rx(t)] = \alpha_{\mathrm{L}}\mathbb{E}[V(x(t))]$. By using this inequality and $\mathbb{E}[w^{\mathrm{T}}(t)Rw(t)] = \int_{\mathbb{R}^n} w^{\mathrm{T}}Rw\phi(\mathrm{d}w) = \sigma^2$ in (9), we obtain

$$\begin{aligned}\mathbb{E}[V(x(t+1))] &\geq \alpha_{\mathrm{L}}\mathbb{E}[V(x(t))] + \mathbb{E}[x^{\mathrm{T}}(t)A^{\mathrm{T}}RBu(t)] \\ &\quad + \mathbb{E}[u^{\mathrm{T}}(t)B^{\mathrm{T}}RAx(t)] \\ &\quad + \mathbb{E}[u^{\mathrm{T}}(t)B^{\mathrm{T}}RBu(t)] + \sigma^2.\end{aligned} \quad (10)$$

Notice that $\beta_{\mathrm{U}} = \lambda_{\max}(B^{\mathrm{T}}RB) \geq 0$. We show divergence of $\mathbb{E}[V(x(t))]$ separately for the case where $\beta_{\mathrm{U}} = 0$ and the case where $\beta_{\mathrm{U}} > 0$.

First, consider the case where $\beta_{\mathrm{U}} = 0$. Notice that $\beta_{\mathrm{U}} = 0$ implies $R^{\frac{1}{2}}B = 0$, and thus $RB = 0$. Therefore, if $\beta_{\mathrm{U}} = 0$, then it follows from (10) that $\mathbb{E}[V(x(t+1))] \geq \alpha_{\mathrm{L}}\mathbb{E}[V(x(t))] + \sigma^2$. Hence,

$$\begin{aligned}\mathbb{E}[V(x(t))] &\geq \alpha_{\mathrm{L}}^t\mathbb{E}[V(x(0))] + \sigma^2\sum_{i=0}^{t-1}\alpha_{\mathrm{L}}^{t-1-i} \\ &= \alpha_{\mathrm{L}}^t\mathbb{E}[V(x(0))] + \sigma^2\left(\alpha_{\mathrm{L}}^t - 1\right)/\left(\alpha_{\mathrm{L}} - 1\right),\end{aligned}$$

implying $\lim_{t\to\infty}\mathbb{E}[V(x(t))] = \infty$, since $\sigma > 0$ and $\alpha_{\mathrm{L}} > 1$.

Next, we consider the case where $\beta_{\mathrm{U}} > 0$. It follows from (10) with $u^{\mathrm{T}}(t)B^{\mathrm{T}}RBu(t) \geq \lambda_{\min}(B^{\mathrm{T}}RB)\|u(t)\|^2 = \beta_{\mathrm{L}}\|u(t)\|^2$ that

$$\begin{aligned}\mathbb{E}[V(x(t+1))] &\geq \alpha_{\mathrm{L}}\mathbb{E}[V(x(t))] + \mathbb{E}[x^{\mathrm{T}}(t)A^{\mathrm{T}}RBu(t)] \\ &\quad + \mathbb{E}[u^{\mathrm{T}}(t)B^{\mathrm{T}}RAx(t)] \\ &\quad + \beta_{\mathrm{L}}\mathbb{E}[\|u(t)\|^2] + \sigma^2, \quad t \in \mathbb{N}_0.\end{aligned} \quad (11)$$

Now, let

$$\mathcal{C} \triangleq \left\{c > 1 : \hat{u} < \sigma^2\left(\alpha_{\mathrm{L}} - 1\right)/\left(c\alpha_{\mathrm{U}}\beta_{\mathrm{U}} - \alpha_{\mathrm{L}}\beta_{\mathrm{L}} + \beta_{\mathrm{L}}\right)\right\}. \quad (12)$$

Since $\hat{u}$ satisfies (7), we have $\mathcal{C} \neq \emptyset$. Next, let $\hat{c} \in \mathcal{C}$ and define $\gamma(\hat{c}) \triangleq \hat{c}\alpha_{\mathrm{U}}/\left(\alpha_{\mathrm{L}} - 1\right)$. Consider the square root $R^{\frac{1}{2}}$ of matrix R. Since $R^{\frac{1}{2}}$ is a nonnegative-definite Hermitian matrix and $\gamma(\hat{c}) > 0$, we have

$$\begin{aligned}0 &\leq \left(\frac{1}{\sqrt{\gamma(\hat{c})}}R^{\frac{1}{2}}Ax(t) + \sqrt{\gamma(\hat{c})}R^{\frac{1}{2}}Bu(t)\right)^* \\ &\quad \cdot\left(\frac{1}{\sqrt{\gamma(\hat{c})}}R^{\frac{1}{2}}Ax(t) + \sqrt{\gamma(\hat{c})}R^{\frac{1}{2}}Bu(t)\right) \\ &= \frac{1}{\gamma(\hat{c})}x^{\mathrm{T}}(t)A^{\mathrm{T}}RAx(t) + x^{\mathrm{T}}(t)A^{\mathrm{T}}RBu(t) \\ &\quad + u^{\mathrm{T}}(t)B^{\mathrm{T}}RAx(t) + \gamma(\hat{c})u^{\mathrm{T}}(t)B^{\mathrm{T}}RBu(t),\end{aligned}$$

which implies

$$\begin{aligned}&x^{\mathrm{T}}(t)A^{\mathrm{T}}RBu(t) + u^{\mathrm{T}}(t)B^{\mathrm{T}}RAx(t) \\ &\quad \geq -\frac{1}{\gamma(\hat{c})}x^{\mathrm{T}}(t)A^{\mathrm{T}}RAx(t) - \gamma(\hat{c})u^{\mathrm{T}}(t)B^{\mathrm{T}}RBu(t) \\ &\quad \geq -\frac{1}{\gamma(\hat{c})}\alpha_{\mathrm{U}}V(x(t)) - \gamma(\hat{c})\beta_{\mathrm{U}}\|u(t)\|^2.\end{aligned} \quad (13)$$

Now by (11) and (13),

$$\begin{aligned}\mathbb{E}[V(x(t+1))] &\geq \mu(\hat{c})\mathbb{E}[V(x(t))] \\ &\quad + \eta(\hat{c})\mathbb{E}[\|u(t)\|^2] + \sigma^2, \quad t \in \mathbb{N}_0,\end{aligned} \quad (14)$$

where $\mu(\hat{c}) \triangleq \left(\alpha_{\mathrm{L}} - \frac{1}{\gamma(\hat{c})}\alpha_{\mathrm{U}}\right)$ and $\eta(\hat{c}) \triangleq (\beta_{\mathrm{L}} - \gamma(\hat{c})\beta_{\mathrm{U}})$. It then follows from (14) that

$$\begin{aligned}\mathbb{E}\left[V(x(t))\right] \geq \mu^t(\hat{c})\mathbb{E}\left[V(x(0))\right] \\ + \eta(\hat{c})\sum_{i=0}^{t-1}\mu^{t-1-i}(\hat{c})\mathbb{E}[\|u(i)\|^2] \\ + \sigma^2\sum_{i=0}^{t-1}\mu^{t-1-i}(\hat{c}), \quad t \in \mathbb{N}. \end{aligned} \tag{15}$$

Since $1 < \alpha_{\mathrm{L}} \leq \alpha_{\mathrm{U}}$ and $\hat{c} > 1$, we have $\gamma(\hat{c}) > 1$, and thus $\mu(\hat{c}) > 1$ and $\eta(\hat{c}) < 0$. It follows from $\mu(\hat{c}) > 1$ that $\mu^{t-1-i}(\hat{c})$ is a nonincreasing function of i. Consequently, by using Lemma A.1 (see Appendix) with $f(i) \triangleq \mu^{t-1-i}(\hat{c})$ and $g(i) \triangleq \mathbb{E}[\|u(i)\|^2]$, $i \in \mathbb{N}_0$, we obtain

$$\sum_{i=0}^{t-1}\mu^{t-1-i}(\hat{c})\mathbb{E}[\|u(i)\|^2] \leq \hat{u}\sum_{i=0}^{t-1}\mu^{t-1-i}(\hat{c}). \tag{16}$$

Furthermore, since $\eta(\hat{c}) < 0$, it follows from (16) that

$$\eta(\hat{c})\sum_{i=0}^{t-1}\mu^{t-1-i}(\hat{c})\mathbb{E}[\|u(i)\|^2] \geq \hat{u}\eta(\hat{c})\sum_{i=0}^{t-1}\mu^{t-1-i}(\hat{c}). \tag{17}$$

Then by (15) and (17), we arrive at

$$\begin{aligned}\mathbb{E}\left[V(x(t))\right] &\geq \mu^t(\hat{c})\mathbb{E}\left[V(x(0))\right] \\ &\quad + \hat{u}\eta(\hat{c})\sum_{i=0}^{t-1}\mu^{t-1-i}(\hat{c}) + \sigma^2\sum_{i=0}^{t-1}\mu^{t-1-i}(\hat{c}) \\ &= \mu^t(\hat{c})\mathbb{E}\left[V(x(0))\right] + \left(\hat{u}\eta(\hat{c}) + \sigma^2\right)\frac{\mu^t(\hat{c}) - 1}{\mu(\hat{c}) - 1}. \end{aligned} \tag{18}$$

Since $\mathbb{E}[V(x(0))] \geq 0$, it follows from (18) that

$$\mathbb{E}\left[V(x(t))\right] \geq \left(\hat{u}\eta(\hat{c}) + \sigma^2\right)\frac{\mu^t(\hat{c}) - 1}{\mu(\hat{c}) - 1}. \tag{19}$$

Notice that since $\hat{c} \in \mathcal{C}$, we have

$$\hat{u} < \sigma^2\left(\alpha_{\mathrm{L}} - 1\right) / \left(\hat{c}\alpha_{\mathrm{U}}\beta_{\mathrm{U}} - \alpha_{\mathrm{L}}\beta_{\mathrm{L}} + \beta_{\mathrm{L}}\right),$$

and hence, $\hat{u}\eta(\hat{c}) + \sigma^2 = \hat{u}(\beta_{\mathrm{L}} - \frac{\alpha_{\mathrm{U}}}{\alpha_{\mathrm{L}}-1}\hat{c}\beta_{\mathrm{U}}) + \sigma^2 > 0$. Since $\hat{u}\eta(\hat{c}) + \sigma^2 > 0$ and $\lim_{t\to\infty}\mu^t(\hat{c}) = \infty$, it follows from (19) that $\lim_{t\to\infty}\mathbb{E}\left[V(x(t))\right] = \infty$.

Finally, since $R \neq 0$, we have $\lambda_{\max}(R) > 0$. Thus, noting that $V(x(t)) \leq \lambda_{\max}(R)\|x(t)\|^2$, we get $\|x(t)\|^2 \geq (1/\lambda_{\max}(R))\, V(x(t))$, $t \in \mathbb{N}_0$, which implies (8). ☐

Theorem 2 provides sufficient conditions under which the closed-loop system is unstable and the second moment of the state diverges. First, condition (5) in Theorem 2 is used for quantifying the instability of the uncontrolled ($u(t) \equiv 0$) system. Notice that if A is a strictly unstable matrix (i.e., some of its eigenvalues are strictly outside the unit circle of the complex plane), then there always exists a nonnegative-definite Hermitian matrix that satisfies (5). This is further discussed in Section 3.1, where we provide a method to find R satisfying (5) by exploiting the eigenstructure of the matrix A. Notice that when a nonnegative-definite matrix R satisfies (5), then the term $\int_{\mathbb{R}^n} w^{\mathrm{T}}Rw\phi(\mathrm{d}w)$ in condition (6) represents the variance of noise $w(t)$ projected on an unstable mode of the open-loop system. In particular, $\mathbb{E}[w^{\mathrm{T}}(t)Rw(t)] = \int_{\mathbb{R}^n} w^{\mathrm{T}}Rw\phi(\mathrm{d}w)$ for each $t \in \mathbb{N}_0$. Theorem 2 indicates that when conditions (5) and (6) hold, then it is *not possible* to stabilize the system by using control laws with too small average second moments. In particular, if the bound $\hat{u}$ of the time-averaged second moment of the control input is small such that (7) holds, then the second moment of the state diverges.

We conduct the divergence analysis in Theorem 2 by evaluating the expected trajectory of a Lyapunov-like function $V(x(t)) = x^{\mathrm{T}}(t)Rx(t)$. Typically for showing *stability* of stochastic systems, positive-definite symmetric matrices are utilized. The analysis for the *divergence* differ from stability analysis in two ways. First, R is a nonnegative-definite matrix and thus the sublevel sets $\{x \in \mathbb{R}^n\colon V(x) \leq v\}$ need not be bounded. This allows us to deal with A matrices that possess both stable and unstable eigenvalues. Secondly, R is a Hermitian matrix with possibly complex entries. This aspect is utilized for systems where A may possess complex eigenvalues (see Section 3.1). Notice that even though entries of R may be complex, $V(x(t))$ takes real and nonnegative values, because R is a Hermitian nonnegative-definite matrix.

3.1 Instability Conditions Based on the Eigenstructure of the System Matrix

We now utilize the eigenstructure of the matrix A to establish conditions for instability. Let $n_{\mathrm{G}} \in \{1, \ldots, n\}$ denote the sum of the geometric multiplicities of the eigenvalues of A. Furthermore, let $\lambda_i \in \mathbb{C}$, $i \in \{1, \ldots, n_{\mathrm{G}}\}$, be the distinct eigenvalues of A and $v_i \in \mathbb{C}^n$ be vectors that satisfy

$$A^{\mathrm{T}}v_i = \lambda_i v_i, \quad i \in \{1, \ldots, n_{\mathrm{G}}\}. \tag{20}$$

Notice that if λ_i is a complex eigenvalue (i.e., $\lambda_i \notin \mathbb{R}$), then the complex conjugate $\overline{\lambda_i}$ is also an eigenvalue of A. In particular, by (20), we have

$$v_i^* A = \overline{\lambda_i}v_i^*, \quad i \in \{1, \ldots, n_{\mathrm{G}}\}, \tag{21}$$

where v_i^* is the complex conjugate transpose of vector v_i. In the result below, we use the eigenvalues λ_i, $i \in \{1, \ldots, n_{\mathrm{G}}\}$, and the left-eigenvectors $v_i \in \mathbb{C}^n$, $i \in \{1, \ldots, n_{\mathrm{G}}\}$ to characterize instability conditions.

Corollary 3. Consider the linear stochastic system (1) with system matrix A and the pairs (λ_i, v_i) in (20). Let

$$\mathcal{I} \triangleq \{i \in \{1, \ldots, n_{\mathrm{G}}\}\colon |\lambda_i| > 1 \text{ and } \sigma_i > 0\},$$

where $\sigma_i \triangleq \sqrt{\int_{\mathbb{R}^n} w^{\mathrm{T}}v_i v_i^* w\phi(\mathrm{d}w)}$. Suppose $\mathcal{I} \neq \emptyset$. If the control policy is $\mathcal{F}_t$-adapted and satisfies (2) with

$$\hat{u} < \max_{i\in\mathcal{I}}\varphi_i, \tag{22}$$

where

$$\varphi_i \triangleq \begin{cases} \dfrac{\sigma_i^2\left(|\lambda_i|^2 - 1\right)}{|\lambda_i|^2(\beta_{\mathrm{U},i} - \beta_{\mathrm{L},i}) + \beta_{\mathrm{L},i}}, & \text{if } \beta_{\mathrm{U},i} \neq 0, \\ \infty, & \text{otherwise,} \end{cases}$$

$\beta_{\mathrm{L},i} \triangleq \lambda_{\min}(B^{\mathrm{T}}v_i v_i^* B)$, $\beta_{\mathrm{U},i} \triangleq \lambda_{\max}(B^{\mathrm{T}}v_i v_i^* B)$, $i \in \mathcal{I}$, then the second moment of the state diverges (i.e., (8) holds) for any initial state distribution.

Proof. By (20) and (21), we have

$$A^{\mathrm{T}}v_i v_i^* A = \lambda_i\overline{\lambda_i}v_i v_i^* = |\lambda_i|^2 v_i v_i^*, \quad i \in \mathcal{I}. \tag{23}$$

Here, notice that for each $i \in \mathcal{I}$, $v_i v_i^* \in \mathbb{C}^{n\times n}$ is a nonnegative-definite Hermitian matrix. Thus, it follows

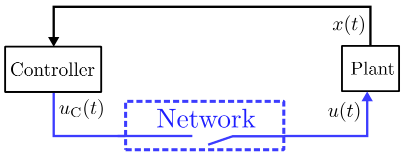

Fig. 1. Control over a network subject to packet losses.

from (23) that (5) holds with $R = v_i v_i^*$, $\alpha_{\mathrm{L}} = \alpha_{\mathrm{U}} = |\lambda_i|^2$. Furthermore, by definition, $\sigma_i > 0$ for $i \in \mathcal{I}$. Therefore, for each $i \in \mathcal{I}$, it follows from Theorem 2 by setting $\beta_{\mathrm{L}} = \beta_{\mathrm{L},i}$, $\beta_{\mathrm{U}} = \beta_{\mathrm{U},i}$, and $\sigma = \sigma_i$ that under control policies satisfying (2) with $\hat{u} < \varphi_i$ the second moment of the state diverges. Finally, (22) implies that there exists $\tilde{i} \in \mathcal{I}$ such that $\hat{u} < \varphi_{\tilde{i}}$, implying divergence. $\square$

Corollary 3 provides an approach for checking instability of the system (1) by using the eigenvalues and the left-eigenvectors of the matrix A.

3.2 Instability of Networked Control Systems

The results presented above can be used in obtaining impossibility results for networked control systems. Specifically, consider the networked control problem depicted in Fig. 1, where the plant dynamics are given by (1).

In this problem, control input information that is transmitted from the controller to the plant is subject to packet losses. In the case of a packet loss, the input at the plant-side is set to 0. In this setting, the plant input is given by $u(t) = (1 - l(t))u_{\mathrm{C}}(t)$, where $u_{\mathrm{C}}(t)$ represents the control input transmitted from the controller, and $l(t) \in \{0, 1\}$ is an indicator of a packet loss at time t (i.e., $l(t) = 1$ represents a packet loss and $l(t) = 0$ represents a successful transmission). Suppose that the transmitted control input $u_{\mathrm{C}}(t)$ is subject to the constraint $\mathbb{P}[\|u_{\mathrm{C}}(t)\|^2 \le \hat{u}_{\mathrm{C}}] = 1$, $t \in \mathbb{N}_0$, (with $\hat{u}_{\mathrm{C}} \in (0, \infty)$) enforced by the controller. Furthermore, assume for each $t \in \mathbb{N}_0$ that $u_{\mathrm{C}}(t)$ and $l(t)$ are mutually independent. With $p_{\mathrm{S}}(t) \triangleq \mathbb{P}[l(t) = 0]$ denoting the probability of a successful transmission at time $t \in \mathbb{N}_0$, we have $\mathbb{E}[\|u(t)\|^2] = \mathbb{E}[(1-l(t))\|u_{\mathrm{C}}(t)\|^2] = \mathbb{E}[(1-l(t))]\mathbb{E}[\|u_{\mathrm{C}}(t)\|^2] \le \mathbb{P}[1 - l(t) = 1]\hat{u}_{\mathrm{C}} = p_{\mathrm{S}}(t)\hat{u}_{\mathrm{C}}$. As a result, we obtain $\frac{1}{t}\sum_{i=0}^{t-1}\mathbb{E}[\|u(i)\|^2] \le \hat{u}_{\mathrm{C}}\frac{1}{t}\sum_{i=0}^{t-1} p_{\mathrm{S}}(i)$ for $t \in \mathbb{N}$. Hence, if $\hat{u}_{\mathrm{C}}\frac{1}{t}\sum_{i=0}^{t-1} p_{\mathrm{S}}(i) \le \hat{u}$, $t \in \mathbb{N}$, then (2) holds.

Theorem 2 indicates that the second moment of the networked control system's state diverges, if the average successful transmission probability $\frac{1}{t}\sum_{i=0}^{t-1} p_{\mathrm{S}}(i)$ is consistently small such that $\hat{u}_{\mathrm{C}}\frac{1}{t}\sum_{i=0}^{t-1} p_{\mathrm{S}}(i) \le \hat{u}$ holds for every $t \in \mathbb{N}$ with $\hat{u}$ satisfying the inequality (7). In other words, knowing the probability of successful transmissions is small, it is not possible to design a control law $u(t)$ that guarantees stabilization.

4. NUMERICAL EXAMPLE

We consider the linear discrete-time model of the cart-pendulum system from Kolmanovsky and Gilbert (1995). This model is obtained via linearization in a sampled-data control setup, and it is given by (1) with

$$A = \begin{bmatrix} 1 & 0.1 & -0.0506 & -0.0017 \\ 0 & 1 & -1.0240 & -0.0506 \\ 0 & 0 & 1.0723 & 0.1024 \\ 0 & 0 & 1.4628 & 1.0723 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0101 \\ 0.2024 \\ -0.0072 \\ -0.1463 \end{bmatrix}.$$

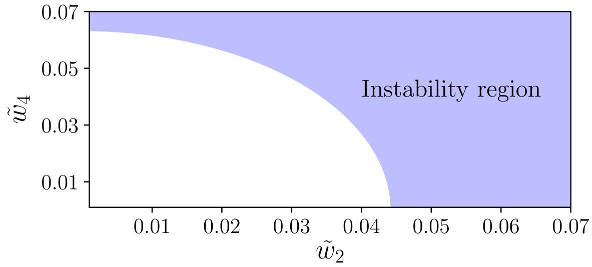

Fig. 2. Region of instability with respect to noise distribution parameters $\tilde{w}_2$ and $\tilde{w}_4$.

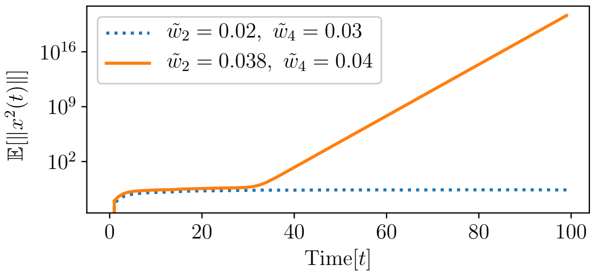

Fig. 3. Trajectories of the second moment for different noise distribution parameters.

It is shown by Kolmanovsky and Gilbert (1995) that the control law

$$u(t) = \mathrm{sign}(Kx(t)) \min\{0.5, |Kx(t)|\} \tag{24}$$

with $K = [0.5451,\ 1.8357,\ 27.2815,\ 8.6552]$ guarantees that the state of the system stays bounded if the initial state x_0 and the disturbance vectors $w(t)$, $t \in \mathbb{N}_0$, have sufficiently small norms.

Different from the earlier work by Kolmanovsky and Gilbert (1995), we explore the scenarios where $\{w(t)\}_{t\in\mathbb{N}_0}$ is a stochastic process and look for conditions under which the second moment of the system state diverges.

Suppose that the noise vector entries $w_2(t) \in \mathbb{R}$ and $w_4(t) \in \mathbb{R}$ are independent random variables that are uniformly distributed in the respective ranges $[-\tilde{w}_2, \tilde{w}_2]$ and $[-\tilde{w}_4, \tilde{w}_4]$, where $\tilde{w}_2, \tilde{w}_4 > 0$. Furthermore, suppose that $w_1(t) = 0$, $w_3(t) = 0$, $t \in \mathbb{N}_0$. Our results obtained for control policies with bounded-average second moments are applicable here, since (24) guarantees $|u(t)| \le 0.5$, which implies (2) with $\hat{u} = 0.25$.

We check instability of the system with respect to parameters $\tilde{w}_2$ and $\tilde{w}_4$. Specifically, (22) holds for the noise distribution parameters in the shaded region of Fig. 2. Thus, by Corollary 3, the second moment of the state diverges for those parameter values. In Fig. 3, we show the second moment $\mathbb{E}[\|x(t)\|^2]$ of the state approximated over 1000 sample trajectories with initial condition $x_0 = 0$. For parameter values $\tilde{w}_2 = 0.038, \tilde{w}_4 = 0.04$ (from the instability region), the second moment is divergent. On the other hand, for smaller values $\tilde{w}_2 = 0.02, \tilde{w}_4 = 0.03$ (from outside the region), the second moment of the state stays bounded. Notice that even in the divergent case, $\mathbb{E}[\|x(t)\|^2]$ stays small until around time 30. This happens because the initial state is 0; therefore, the control law (24) does not face saturation and is able to keep the state norm at relatively small values for a while. For the setting with

$\tilde{w}_2 = 0.038, \tilde{w}_4 = 0.04$ there are sample noise realizations where the control law saturates ($u(t) \in \{-0.5, 0.5\}$) as time progresses, and the state starts to increase rapidly. With the smaller values $\tilde{w}_2 = 0.02, \tilde{w}_4 = 0.03$, this does not happen and $\mathbb{E}[\|x(t)\|^2]$ stays bounded.

5. CONCLUSION

We have investigated discrete-time linear stochastic systems with the goal of identifying the cases where stabilization of these systems is not possible with control policies that have bounded time-averaged second moments. First, we obtained a general result that provides conditions, under which the second moment of the system state diverges. These conditions are given in terms of the system dynamics and the bounding constant of the control constraint. Then we obtained conditions based on the eigenstructures of system matrices and applied our results to networked control systems. In networked control systems, our results provide limits for average successful packet transmission probabilities below which stabilization is not possible.

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APPENDIX

Lemma A.1. Let $f\colon \mathbb{N}_0 \to [0,\infty)$, $g\colon \mathbb{N}_0 \to [0,\infty)$ be nonnegative functions. If f is nonincreasing and g satisfies

$$\sum_{i=0}^{t-1} g(i) \leq \overline{g}t, \quad t \in \mathbb{N}, \tag{25}$$

where $\overline{g} \geq 0$, then for every $t \in \mathbb{N}$, we have

$$\sum_{i=0}^{t-1} \left(f(i) - f(t)\right) g(i) \leq \overline{g} \sum_{i=0}^{t-1} \left(f(i) - f(t)\right), \tag{26}$$

and moreover,

$$\sum_{i=0}^{t-1} f(i)g(i) \leq \overline{g} \sum_{i=0}^{t-1} f(i), \quad t \in \mathbb{N}. \tag{27}$$