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Compositional Construction of Finite MDPs for Continuous-Time Stochastic Systems: A Dissipativity Approach ⋆

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Abstract: This paper provides a compositional scheme based on dissipativity approaches for constructing finite abstractions of continuous-time continuous-space stochastic control systems. The proposed framework enjoys the structure of the interconnection topology and employs a notion of *stochastic storage functions*, that describe joint dissipativity-type properties of subsystems and their abstractions. By utilizing those stochastic storage functions, one can establish a relation between continuous-time continuous-space stochastic systems and their finite counterparts while quantifying probabilistic distances between their output trajectories. Consequently, one can employ the finite system as a suitable substitution of the continuous-time one in the controller design process with a guaranteed error bound. In this respect, we first leverage dissipativity-type compositional conditions for the compositional quantification of the distance between the interconnection of continuous-time continuous-space stochastic systems and that of their discrete-time (finite or infinite) abstractions. We then consider a specific class of stochastic affine systems and construct their finite abstractions together with their corresponding stochastic storage functions. We illustrate the effectiveness of the proposed techniques by applying them to a physical case study.

Keywords: Compositional Abstraction-based Synthesis, Continuous-Time Stochastic Systems, Finite Markov Decision Processes, Dissipativity Reasoning, Formal Synthesis.

1. INTRODUCTION

Motivations. Automated controller synthesis for continuous time continuous-space stochastic systems against high-level logical properties such as those expressed as linear temporal logic (LTL) formulae (Pnueli, 1977) is naturally a difficult task mainly due to continuous state sets. To deal with this problem, one potential direction is to first abstract the given system by a simpler one, *i.e.,* discrete in time and potentially in space, then synthesize a desired controller for the abstract system, and finally transfer the controller back to the original one while quantifying probabilistic error bounds.

Unfortunately, *curse of dimensionality* is the main problem in the construction of finite abstractions (a.k.a. finite Markov decision processes (MDPs)) for large-scale systems: the complexity of constructing finite abstractions increases exponentially with the dimension of the state set. Compositional techniques play significant roles to alleviate this complexity. In this regard, one can consider the large-scale stochastic system as an interconnected system composed of several smaller subsystems, and then develop a compositional scheme for the construction of finite abstractions for the given complex system via abstractions of smaller subsystems.

Related Literature. There have been some results, proposed in the past few years, on the construction of finite abstractions for *continuous-time* continuous-space stochastic systems. A reachability analysis for continuous-time stochastic systems by constructing Markov chain with quantified error bounds is proposed by (Laurenti et al., 2017). Abstraction approaches for incrementally stable stochastic control systems without discrete dynamics, incrementally stable stochastic switched systems, and randomly switched stochastic systems are respectively studied by (Zamani et al., 2014), (Zamani et al., 2015), and (Zamani and Abate, 2014). Although original systems in (Zamani et al., 2014), (Zamani et al., 2015), and (Zamani and Abate, 2014) are stochastic, their abstractions are constructed as finite labeled transition systems while finite abstractions in this work are presented as finite Markov decision processes. Finite labeled transition systems in this context are useful only if the noise in the system is small. An approximation scheme for the construction of infinite abstractions for jump-diffusion processes is developed by (Julius and Pappas, 2009). An (in)finite abstraction-based technique for synthesis of continuous-time stochastic control systems is recently discussed by (Nejati et al., 2019).

For *discrete-time* stochastic systems with continuous-state sets, there also exist several results. Finite abstractions for formal synthesis of discrete-time stochastic control systems are proposed by (Abate et al., 2008). An adaptive and sequential gridding approach is proposed by (Soudjani and Abate, 2013). Moreover, formal abstraction-based policy synthesis is discussed by (Tkachev et al., 2013), and (Kamgarpour et al., 2013). Compositional construction of infinite abstractions via dissipativity conditions is proposed by (Lavaei et al., 2019). Compositional construction of finite abstractions utilizing dynamic Bayesian networks and dissipativity conditions is studied by (Soudjani et al., 2017) and (Lavaei et al., 2018), respectively. Although

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the proposed compositional approach by (Lavaei et al., 2018) is based on dissipativity conditions, their results are provided for discrete-time systems. In comparison, we deal with continuous-time systems here and the ultimate goal is to develop a compositional approach to construct finite MDPs from *continuous-time* stochastic systems.

Compositional construction of (in)finite abstractions via max-type small-gain conditions is proposed by (Lavaei et al., 2020). Compositional construction of finite abstractions for networks of stochastic systems via *relaxed* dissipativity approaches is presented by (Lavaei et al., 2020). A notion of approximate simulation relation for stochastic systems based on a lifting probabilistic evolution of systems is proposed by (Haesaert et al., 2017). Compositional construction of finite abstractions for networks of stochastic *switched* systems is presented by (Lavaei et al., 2020).

Contributions. In this paper, we provide a compositional scheme for constructing finite MDPs from continuous-time continuous-space stochastic systems. We derive dissipativity type conditions to propose compositionality results which are established based on relations between continuous-time subsystems and that of their abstract counterparts utilizing notions of so-called *stochastic storage functions*. The provided compositionality conditions can enjoy the structure of interconnection topology and be potentially fulfilled independently of the interconnection or gains of the subsystems (cf. the case study).

To this end, we first compositionally quantify the probabilistic distance between the interconnection of continuous-time continuous-space stochastic subsystems and their discrete-time (finite or infinite) abstractions. We then focus on a particular class of stochastic affine systems and construct their finite abstractions together with their corresponding stochastic storage functions. Finally, we illustrate the effectiveness of the proposed techniques by applying them to a physical case study. Due to lack of space, proofs of all statements as well as details of the case study are provided in an arXiv version of the paper (Nejati and Zamani, 2020).

Recent Works. Compositional abstraction-based synthesis of continuous-time stochastic systems is also proposed by (Nejati et al., 2020), but using a different compositionality scheme based on *small-gain* conditions. Our proposed compositionality approach here can be potentially less conservative than the one presented by (Nejati et al., 2020) for some classes of systems. The dissipativity-type compositional reasoning proposed here can enjoy the structure of the interconnection topology and may not require any constraint on the number or gains of subsystems (cf. the case study). Consequently, the proposed approach here can provide a scale-free compositionality condition which is independent of the number of subsystems compared to the proposed results in (Nejati et al., 2020).

2. NOTATIONS AND MODEL CLASSES

2.1 Notations

A probability space in this work is defined as $(\Omega, \mathcal{F}_\Omega, \mathbb{P}_\Omega)$, where Ω is the sample space, $\mathcal{F}_\Omega$ is a sigma-algebra on Ω comprising subsets of Ω as events, and $\mathbb{P}_\Omega$ is a probability measure that assigns probabilities to events. We assume that triple $(\Omega, \mathcal{F}_\Omega, \mathbb{P}_\Omega)$ denotes a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0}$ satisfying the usual conditions of completeness and right continuity.

Sets of nonnegative and positive integers are respectively denoted by $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{N}_{\geq 1} := \{1, 2, 3, \ldots\}$. Symbols $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ respectively denote sets of real, positive and nonnegative real numbers. We use $x = [x_1; \ldots; x_N]$ to denote the corresponding vector of dimension $\sum_i n_i$, given N vectors $x_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}_{\geq 1}$, and $i \in \{1, \ldots, N\}$. Given functions $f_i : X_i \to Y_i$, for any $i \in \{1, \ldots, N\}$, their Cartesian product $\prod_{i=1}^N f_i : \prod_{i=1}^N X_i \to \prod_{i=1}^N Y_i$ is defined as $(\prod_{i=1}^N f_i)(x_1, \ldots, x_N) = [f_1(x_1); \ldots; f_N(x_N)]$. We denote by $\|\cdot\|$ the Euclidean norm. Given a function $f : \mathbb{N} \to \mathbb{R}^n$, the supremum of f is denoted by $\|f\|_\infty := (\text{ess})\sup\{\|f(k)\|, k \geq 0\}$. The identity matrix in $\mathbb{R}^{n\times n}$ is denoted by $\mathbb{I}_n$. Column vectors in $\mathbb{R}^{n\times 1}$ with all elements equal to zero and one are respectively denoted by $\mathbf{0}_n$ and $\mathbb{1}_n$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\gamma(0) = 0$. A class $\mathcal{K}$ function $\gamma(\cdot)$ is said to be a class $\mathcal{K}_\infty$ if $\gamma(s) \to \infty$ as $s \to \infty$.

2.2 Continuous-Time Stochastic Control Systems

Definition 1. A continuous-time stochastic control system (ct-SCS) in this paper is defined by the tuple

$$\Sigma = (X, U, W, \mathcal{U}, \mathcal{W}, f, \sigma, Y_1, Y_2, h_1, h_2), \tag{1}$$

where:

- $X \subseteq \mathbb{R}^n$ is the state set of the system;
- $U \subseteq \mathbb{R}^m$ is the *external* input set of the system;
- $W \subseteq \mathbb{R}^p$ is the *internal* input set of the system;
- $\mathcal{U}$ and $\mathcal{W}$ are subsets of the sets of all $\mathbb{F}$-progressively measurable processes taking values respectively in $\mathbb{R}^m$ and $\mathbb{R}^p$;
- $f : X \times U \times W \to X$ is the drift term which is globally Lipschitz continuous: there exist constants $\mathscr{L}_x, \mathscr{L}_\nu, \mathscr{L}_w \in \mathbb{R}_{\geq 0}$ such that $\|f(x, \nu, w) - f(x', \nu', w')\| \leq \mathscr{L}_x\|x - x'\| + \mathscr{L}_\nu\|\nu - \nu'\| + \mathscr{L}_w\|w - w'\|$ for all $x, x' \in X$, for all $\nu, \nu' \in U$, and for all $w, w' \in W$;
- $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times \mathsf{b}}$ is the diffusion term which is globally Lipschitz continuous with the Lipschitz constant $\mathscr{L}_\sigma$;
- $Y_1 \subseteq \mathbb{R}^{q_1}$ is the *external* output set of the system;
- $Y_2 \subseteq \mathbb{R}^{q_2}$ is the *internal* output set of the system;
- $h_1 : X \to Y_1$ is the *external* output map;
- $h_2 : X \to Y_2$ is the *internal* output map.

A continuous-time stochastic control system Σ satisfies

$$\Sigma : \begin{cases} \mathsf{d}\xi(t) = f(\xi(t), \nu(t), w(t))\,\mathsf{d}t + \sigma(\xi(t))\,\mathsf{d}\mathbb{W}_t, \\ \zeta_1(t) = h_1(\xi(t)), \\ \zeta_2(t) = h_2(\xi(t)), \end{cases} \tag{2}$$

$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.) for any $\nu \in \mathcal{U}$ and $w \in \mathcal{W}$, where $(\mathbb{W}_t)_{t\geq 0}$ is a b-dimensional Brownian motion, and stochastic processes $\xi : \Omega \times \mathbb{R}_{\geq 0} \to X$, $\zeta_1 : \Omega \times \mathbb{R}_{\geq 0} \to Y_1$, and $\zeta_2 : \Omega \times \mathbb{R}_{\geq 0} \to Y_2$ are respectively called the *solution process* and the external and internal *output trajectories* of Σ. We also use $\xi_{a\nu w}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_{\geq 0}$ under input trajectories ν and w from an initial condition $\xi_{a\nu w}(0) = a$ $\mathbb{P}$-a.s., where a is a random variable that is $\mathcal{F}_0$-measurable. We also denote by $\zeta_{1_{a\nu w}}$ and $\zeta_{2_{a\nu w}}$ the external and internal *output trajectories* corresponding to the *solution process* $\xi_{a\nu w}$.

Remark 2. Note that in this article, the term "internal" is used for inputs and outputs of subsystems that are affecting each other in the interconnection topology while properties of interest are defined over "external" outputs. The ultimate goal is to synthesize "external" inputs to fulfill desired properties over "external" outputs.

In this paper, we are interested in investigating interconnected continuous-time stochastic systems, defined later in Subsection 4.1, without internal signals. Then the tuple (1) reduces to $(X, U, \mathcal{U}, f, \sigma, Y, h)$ with $f : X \times U \to X$, and ct-SCS (2) can be re-written as

$$\Sigma:\begin{cases}\mathsf{d}\xi(t)=f(\xi(t),\nu(t))\,\mathsf{d}t+\sigma(\xi(t))\,\mathsf{d}\mathbb{W}_t,\\ \varsigma(t)=h(\xi(t)).\end{cases}$$

2.3 Finite Abstractions of ct-SCS

In order to construct finite abstractions of continuous-time stochastic systems, we first need to provide a *time-discretized* version of ct-SCS in (2) as in the following definition.

Definition 3. A *time-discretized* version of ct-SCS Σ is defined by the tuple

$$\widetilde{\Sigma}=\left(\tilde{X},\tilde{U},\tilde{W},\varsigma,\tilde{f},\tilde{Y}_1,\tilde{Y}_2,\tilde{h}_1,\tilde{h}_2\right),\qquad(3)$$

where:

- $\tilde{X}\subseteq\mathbb{R}^n$ is a Borel space as the state set of the system. We denote by $(\tilde{X},\mathcal{B}(\tilde{X}))$ the measurable space with $\mathcal{B}(\tilde{X})$ being the Borel sigma-algebra on the state space;
- $\tilde{U}\subseteq\mathbb{R}^m$ is a Borel space as the *external* input set;
- $\tilde{W}\subseteq\mathbb{R}^p$ is a Borel space as the *internal* input set;
- ς is a sequence of independent and identically distributed (i.i.d.) random variables from a sample space Ω to the set $\mathcal{V}_\varsigma$,
$$\varsigma:=\{\varsigma(k):\Omega\rightarrow\mathcal{V}_\varsigma,\ k\in\mathbb{N}\};$$
- $\tilde{f}:\tilde{X}\times\tilde{U}\times\tilde{W}\times\mathcal{V}_\varsigma\rightarrow\tilde{X}$ is a measurable function characterizing the state evolution of the system;
- $\tilde{Y}_1\subseteq\mathbb{R}^{q_1}$ is a Borel space as the *external* output set;
- $\tilde{Y}_2\subseteq\mathbb{R}^{q_2}$ is a Borel space as the *internal* output set;
- $\tilde{h}_1:\tilde{X}\rightarrow\tilde{Y}_1$ is the *external* output map;
- $\tilde{h}_2:\tilde{X}\rightarrow\tilde{Y}_2$ is the *internal* output map.

The evolution of $\widetilde{\Sigma}$, for given initial state $\tilde{x}(0)\in\tilde{X}$ and input sequences $\{\tilde{\nu}(k):\Omega\rightarrow\tilde{U},\ k\in\mathbb{N}\}$ and $\{\tilde{w}(k):\Omega\rightarrow\tilde{W},\ k\in\mathbb{N}\}$, can be written as

$$\widetilde{\Sigma}:\begin{cases}\tilde{\xi}(k+1)=\tilde{f}(\tilde{\xi}(k),\tilde{\nu}(k),\tilde{w}(k),\varsigma(k)),\\ \tilde{\zeta}_1(k)=\tilde{h}_1(\tilde{\xi}(k)),\\ \tilde{\zeta}_2(k)=\tilde{h}_2(\tilde{\xi}(k)),\end{cases}\quad k\in\mathbb{N}.\qquad(4)$$

The sets $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{W}}$ are associated to $\tilde{U}$ and $\tilde{W}$ to be the collections of sequences $\{\tilde{\nu}(k):\Omega\rightarrow\tilde{U},\ k\in\mathbb{N}\}$ and $\{\tilde{w}(k):\Omega\rightarrow\tilde{W},\ k\in\mathbb{N}\}$, in which $\tilde{\nu}(k)$ and $\tilde{w}(k)$ are independent of $\varsigma(z)$ for any $k,z\in\mathbb{N}$ and $z\geq k$. For any initial state $\tilde{a}\in\tilde{X}$, $\tilde{\nu}(\cdot)\in\tilde{\mathcal{U}}$ and $\tilde{w}(\cdot)\in\tilde{\mathcal{W}}$, the random sequences $\tilde{\xi}_{\tilde{a}\tilde{\nu}\tilde{w}}:\Omega\times\mathbb{N}\rightarrow\tilde{X}$, $\tilde{\zeta}_{1_{\tilde{a}\tilde{\nu}\tilde{w}}}:\Omega\times\mathbb{N}\rightarrow\tilde{Y}_1$, and $\tilde{\zeta}_{2_{\tilde{a}\tilde{\nu}\tilde{w}}}:\Omega\times\mathbb{N}\rightarrow\tilde{Y}_2$ fulfilling (4) are respectively called the *solution process*, and external and internal *output trajectories* of $\widetilde{\Sigma}$ under an external input $\tilde{\nu}$, an internal input $\tilde{w}$, and an initial state $\tilde{a}$.

Remark 4. Note that the discrete-time system $\widetilde{\Sigma}$ in (4) is presented independently of ct-SCS Σ for now. In particular, in order to construct finite abstractions of continuous-time stochastic systems Σ (*i.e.,* $\widehat{\Sigma}$) as proposed in Definition 6, one first needs to provide a time-discretized version of ct-SCS (*i.e.,* $\widetilde{\Sigma}$) as a middle stage. In Section 5, we focus on a particular class of continuous-time stochastic affine systems Σ and discuss the best choice for $\widetilde{\Sigma}$ to acquire the least approximation error between Σ and $\widehat{\Sigma}$.

The discrete-time stochastic control system $\widetilde{\Sigma}$ can be *equivalently* reformulated as a Markov decision process (Kallenberg, 1997, Proposition 7.6)

$$\widetilde{\Sigma}=\left(\tilde{X},\tilde{U},\tilde{W},\tilde{T}_{\tilde{\mathsf{x}}},\tilde{Y}_1,\tilde{Y}_2,\tilde{h}_1,\tilde{h}_2\right),$$

where the map $\tilde{T}_{\tilde{\mathsf{x}}}:\mathcal{B}(\tilde{X})\times\tilde{X}\times\tilde{U}\times\tilde{W}\rightarrow[0,1]$, is a conditional stochastic kernel that assigns to any $\tilde{x}\in\tilde{X}$, $\tilde{\nu}\in\tilde{U}$, and $\tilde{w}\in\tilde{W}$, a probability measure $\tilde{T}_{\tilde{\mathsf{x}}}(\cdot|\tilde{x},\tilde{\nu},\tilde{w})$ on the measurable space $(\tilde{X},\mathcal{B}(\tilde{X}))$ so that for any set $\mathcal{A}\in\mathcal{B}(\tilde{X})$,

$$\mathbb{P}(\tilde{x}(k+1)\in\mathcal{A}|\tilde{x}(k),\tilde{\nu}(k),\tilde{w}(k))=\int_{\mathcal{A}}\tilde{T}_{\tilde{\mathsf{x}}}(\mathsf{d}\tilde{x}'|\tilde{x}(k),\tilde{\nu}(k),\tilde{w}(k)).$$

For given inputs $\tilde{\nu}(\cdot),\tilde{w}(\cdot)$, the stochastic kernel $\tilde{T}_{\tilde{\mathsf{x}}}$ captures the evolution of the state of $\widetilde{\Sigma}$ and can be uniquely specified by the pair $(\varsigma,\tilde{f})$ from (3). We now define *Markov policies* in order to control the system.

Definition 5. For the discrete-time stochastic control system $\widetilde{\Sigma}$ in (4), a Markov policy is a sequence $\bar{\mu}=(\bar{\mu}_0,\bar{\mu}_1,\bar{\mu}_2,\ldots)$ of universally measurable stochastic kernels $\bar{\mu}_n$ (Bertsekas and Shreve, 1996), each defined on the input space $\tilde{U}$ given $\tilde{X}\times\tilde{W}$ such that for all $(\tilde{\xi}_n,\tilde{w}_n)\in\tilde{X}\times\tilde{W}$, $\bar{\mu}_n(\tilde{U}|(\tilde{\xi}_n,\tilde{w}_n))=1$. The class of all such Markov policies is denoted by $\mathcal{P}_{\mathcal{M}}$.

Now we construct finite MDPs $\widehat{\Sigma}$ as finite abstractions of *discrete-time* stochastic systems $\widetilde{\Sigma}$ in (4). The abstraction algorithm is based on finite partitions of sets $\tilde{X}=\cup_i\mathsf{X}_i$, $\tilde{U}=\cup_i\mathsf{U}_i$, and $\tilde{W}=\cup_i\mathsf{W}_i$ and the selection of representative points $\bar{\xi}_i\in\mathsf{X}_i$, $\bar{\nu}_i\in\mathsf{U}_i$, and $\bar{w}_i\in\mathsf{W}_i$ as abstract states and inputs as formalized in the following definition.

Definition 6. Given a discrete-time system $\widetilde{\Sigma}=(\tilde{X},\tilde{U},\tilde{W},\varsigma,\tilde{f},\tilde{Y}_1,\tilde{Y}_2,\tilde{h}_1,\tilde{h}_2)$, its finite abstraction $\widehat{\Sigma}$ can be characterized as

$$\widehat{\Sigma}=(\hat{X},\hat{U},\hat{W},\varsigma,\hat{f},\hat{Y}_1,\hat{Y}_2,\hat{h}_1,\hat{h}_2),\qquad(5)$$

where $\hat{X}=\{\bar{\xi}_i,i=1,\ldots,n_{\tilde{\xi}}\}$, $\hat{U}=\{\bar{\nu}_i,i=1,\ldots,n_{\tilde{\nu}}\}$, and $\hat{W}=\{\bar{w}_i,i=1,\ldots,n_{\tilde{w}}\}$ are sets of selected representative points. Function $\hat{f}:\hat{X}\times\hat{U}\times\hat{W}\times\mathcal{V}_\varsigma\rightarrow\hat{X}$ is defined as

$$\hat{f}(\hat{\xi},\hat{\nu},\hat{w},\varsigma)=\Pi_{\tilde{\xi}}(\tilde{f}(\hat{\xi},\hat{\nu},\hat{w},\varsigma)),\qquad(6)$$

where $\Pi_{\tilde{\xi}}:\tilde{X}\rightarrow\hat{X}$ is a map that assigns to any $\tilde{\xi}\in\tilde{X}$, the representative point $\bar{\xi}\in\hat{X}$ of the corresponding partition set containing $\tilde{\xi}$. The output maps $\hat{h}_1$, $\hat{h}_2$ are the same as $\tilde{h}_1$, $\tilde{h}_2$ with their domain restricted to the finite state set $\hat{X}$ and the output sets $\hat{Y}_1$, $\hat{Y}_2$ are just the image of $\hat{X}$ under $\tilde{h}_1$, $\tilde{h}_2$. The initial state of $\widehat{\Sigma}$ is also selected according to $\hat{\xi}_0:=\Pi_{\tilde{\xi}}(\tilde{\xi}_0)$ with $\tilde{\xi}_0$ being the initial state of $\widetilde{\Sigma}$.

The abstraction map $\Pi_{\tilde{\xi}}$ defined in (6) satisfies the inequality

$$\|\Pi_{\tilde{\xi}}(\tilde{\xi})-\tilde{\xi}\|\leq\delta,\quad\forall\tilde{\xi}\in\tilde{X},\qquad(7)$$

where δ is the *state* discretization parameter defined as $\delta:=\sup\{\|\tilde{\xi}-\tilde{\xi}'\|,\ \tilde{\xi},\tilde{\xi}'\in\mathsf{X}_i,\ i=1,2,\ldots,n_{\tilde{\xi}}\}$.

Remark 7. Note that to construct finite abstractions as in Definition 6, we assume the state and input sets of the discrete-time system $\widetilde{\Sigma}$ are restricted to compact regions.

3. STOCHASTIC STORAGE AND SIMULATION FUNCTIONS

In this section, we first define a notion of stochastic storage functions (SStF) for ct-SCS with both internal and external signals. We then define a notion of stochastic simulation functions (SSF) for ct-SCS with only external signals. We utilize these two definitions to quantify the probabilistic closeness of interconnected *continuous-time* stochastic systems and that of their *discrete-time* (finite or infinite) abstractions.

Definition 8. Consider a ct-SCS $\Sigma = (X, U, W, \mathcal{U}, \mathcal{W}, f, \sigma, Y_1, Y_2, h_1, h_2)$ and its (in)finite abstraction $\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \varsigma, \hat{f}, \hat{Y}_1, \hat{Y}_2, \hat{h}_1, \hat{h}_2)$. A function $S : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a stochastic storage function (SStF) from $\widehat{\Sigma}$ to Σ if

- $\exists \alpha \in \mathcal{K}_\infty$ such that
$$\forall x \in X, \forall \hat{x} \in \hat{X}, \ \alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|) \leq S(x, \hat{x}), \quad (8)$$
- $\forall k \in \mathbb{N}$, $\forall \xi := \xi(k\tau) \in X, \forall \hat{\xi} := \hat{\xi}(k) \in \hat{X}$, and $\forall \hat{\nu} := \hat{\nu}(k) \in \hat{U}$, $\forall w := w(k\tau) \in W$, $\forall \hat{w} := \hat{w}(k) \in \hat{W}$, $\exists \nu := \nu(k\tau) \in U$ such that
$$\begin{aligned}&\mathbb{E}\Big[S(\xi((k+1)\tau), \hat{\xi}(k+1)) \,|\, \xi, \hat{\xi}, \nu, \hat{\nu}, w, \hat{w}\Big] \quad (9)\\ &\leq \kappa S(\xi, \hat{\xi}) + \rho_{\text{ext}}(\|\hat{\nu}\|) + \psi\\ &+ \begin{bmatrix} w - \hat{w} \\ h_2(x) - \hat{h}_2(\hat{x}) \end{bmatrix}^T \underbrace{\begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22} \end{bmatrix}}_{\bar{X}:=} \begin{bmatrix} w - \hat{w} \\ h_2(x) - \hat{h}_2(\hat{x}) \end{bmatrix},\end{aligned}$$
for some chosen sampling time $\tau \in \mathbb{R}_{>0}$, $0 < \kappa < 1$, $\rho_{\text{ext}} \in \mathcal{K}_\infty$, $\psi \in \mathbb{R}_{>0}$, and a symmetric matrix $\bar{X}$ with conformal block partitions $\bar{X}^{ij}$, $i, j \in \{1, 2\}$.

We call the control system $\widehat{\Sigma}$ a *discrete-time* (in)finite abstraction of concrete (original) system Σ if there exists an SStF S from $\widehat{\Sigma}$ to Σ. Abstraction $\widehat{\Sigma}$ could be finite or infinite depending on cardinalities of sets $\hat{X}, \hat{U}, \hat{W}$. Since the above definition does not put any restriction on the state set of abstract systems, it can be also used to define a stochastic storage function from discrete-time system $\widetilde{\Sigma}$ presented in (3) to Σ (cf. the case study).

Remark 9. Note that one can rewrite the left-hand side of (9) using Dynkin's formula (Dynkin, 1965) as
$$\begin{aligned}&\mathbb{E}\Big[S(\xi((k+1)\tau), \hat{\xi}(k+1)) \,|\, \xi(k\tau), \hat{\xi}(k), \nu(k\tau), \hat{\nu}(k), w(k\tau),\\ &\hat{w}(k)\Big] = \mathbb{E}_\varsigma\Big[S(\xi(k\tau), \hat{\xi}(k+1)) + \mathbb{E}\Big[\int_{k\tau}^{(k+1)\tau} \mathcal{L}S(\xi(t),\\ &\hat{\xi}(k+1))\mathsf{d}t\Big] \,|\, \hat{\xi}(k), \hat{\nu}(k), \hat{w}(k)\Big],\end{aligned}$$
where $\mathcal{L}S$ is the *infinitesimal generator* of the stochastic process applying on the function S, and $\mathbb{E}_\varsigma$ is the *conditional expectation* acting only on the noise of the abstract system. The above Dynkin's formula is utilized later in Section 5 to show the results of Theorem 19.

Now, we write the above notion for the interconnected ct-SCS as the following definition.

Definition 10. Consider a ct-SCS $\Sigma = (X, U, \mathcal{U}, f, \sigma, Y, h)$ and its finite abstraction $\widehat{\Sigma} = (\hat{X}, \hat{U}, \varsigma, \hat{f}, \hat{Y}, \hat{h})$ without internal signals. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a stochastic simulation function (SSF) from $\widehat{\Sigma}$ to Σ if

- $\exists \alpha \in \mathcal{K}_\infty$ such that $\forall x \in X, \forall \hat{x} \in \hat{X}$, one has
$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}), \quad (10)$$
- $\forall k \in \mathbb{N}$, $\forall \xi := \xi(k\tau) \in X, \forall \hat{\xi} := \hat{\xi}(k) \in \hat{X}$, and $\forall \hat{\nu} := \hat{\nu}(k) \in \hat{U}$, $\exists \nu := \nu(k\tau) \in U$ such that
$$\begin{aligned}&\mathbb{E}\Big[V(\xi((k+1)\tau), \hat{\xi}(k+1)) \,|\, \xi, \hat{\xi}, \nu, \hat{\nu}\Big]\\ &\leq \kappa V(\xi, \hat{\xi}) + \rho_{\text{ext}}(\|\hat{\nu}\|) + \psi, \quad (11)\end{aligned}$$
for some chosen sampling time $\tau \in \mathbb{R}_{>0}$, $0 < \kappa < 1$, $\rho_{\text{ext}} \in \mathcal{K}_\infty$, and $\psi \in \mathbb{R}_{>0}$.

The next theorem is borrowed from (Lavaei et al., 2017, Theorem 3.3) and shows how SSF can be useful in providing the probabilistic closeness between output trajectories of original interconnected *continuous-time* stochastic systems and that of their *discrete-time* (finite or infinite) abstractions.

Theorem 11. Let $\Sigma = (X, U, \mathcal{U}, f, \sigma, Y, h)$ be a ct-SCS and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \varsigma, \hat{f}, \hat{Y}, \hat{h})$ its *discrete-time* abstraction. Suppose V is an SSF from $\widehat{\Sigma}$ to Σ. For any input trajectory $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ that preserves Markov property for the closed-loop $\widehat{\Sigma}$, and for any random variables a and $\hat{a}$ as initial states of the ct-SCS and its *discrete-time* abstraction, there exists an input trajectory $\nu(\cdot) \in \mathcal{U}$ of Σ such that the following inequality holds over the finite-time horizon T_d:
$$\begin{aligned}&\mathbb{P}\left\{\sup_{0 \leq k \leq T_d} \|\zeta_{a\nu}(k\tau) - \hat{\zeta}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,|\, a, \hat{a}\right\} \quad (12)\\ &\leq \begin{cases} 1-(1-\dfrac{V(a,\hat{a})}{\alpha(\varepsilon)})(1-\dfrac{\hat{\psi}}{\alpha(\varepsilon)})^{T_d}, & \text{if } \alpha(\varepsilon) \geq \dfrac{\hat{\psi}}{\kappa}, \\ (\dfrac{V(a,\hat{a})}{\alpha(\varepsilon)})(1-\kappa)^{T_d} + (\dfrac{\hat{\psi}}{\kappa\alpha(\varepsilon)})(1-(1-\kappa)^{T_d}), & \text{if } \alpha(\varepsilon) < \dfrac{\hat{\psi}}{\kappa}, \end{cases}\end{aligned}$$
where $\hat{\psi} > 0$ satisfies $\hat{\psi} \geq \rho_{\text{ext}}(\|\hat{\nu}\|_\infty) + \psi$.

4. COMPOSITIONAL ABSTRACTIONS FOR INTERCONNECTED SYSTEMS

In this section, we analyze networks of stochastic control subsystems, $i \in \{1, \ldots, N\}$,
$$\Sigma_i = (X_i, U_i, W_i, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, Y_{1_i}, Y_{2_i}, h_{1_i}, h_{2_i}), \quad (13)$$
and discuss how to construct their finite abstractions together with an SSF based on corresponding SStF of their subsystems.

4.1 Interconnected Stochastic Control Systems

Definition 12. Consider $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems $\Sigma_i = (X_i, U_i, W_i, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, Y_{1_i}, Y_{2_i}, h_{1_i}, h_{2_i})$, $i \in \{1, \ldots, N\}$, and a matrix M defining the coupling between these subsystems. We require the condition $M \prod_{i=1}^N Y_{2i} \subseteq \prod_{i=1}^N W_i$ to establish a well-posed interconnection. The interconnection of Σ_i, $\forall i \in \{1, \ldots, N\}$, is the ct-SCS $\Sigma = (X, U, \mathcal{U}, f, \sigma, Y, h)$, denoted by $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, such that $X := \prod_{i=1}^N X_i$, $U := \prod_{i=1}^N U_i$, $f := \prod_{i=1}^N f_i$, $\sigma := [\sigma_1(x_1); \cdots; \sigma_N(x_N)]$, $Y := \prod_{i=1}^N Y_{1i}$, and $h = \prod_{i=1}^N h_{1i}$, with the internal inputs constrained according to:
$$[w_1; \cdots; w_N] = M[h_{21}(x_1); \cdots; h_{2N}(x_N)].$$

Remark 13. Note that we do not have any restrictions on the interconnected matrix M and its entries can take any values depending on the forms of interconnection topologies.

4.2 Compositional Abstractions of Interconnected Systems

We consider $\Sigma_i = (X_i, U_i, W_i, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, Y_{1_i}, Y_{2_i}, h_{1_i}, h_{2_i})$ as an original ct-SCS and $\widehat{\Sigma}_i$ as its discrete-time finite abstraction given by the tuple $\widehat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, \hat{W}_i, \varsigma_i, \hat{f}_i, \hat{Y}_{1_i}, \hat{Y}_{2_i}, \hat{h}_{1_i}, \hat{h}_{2_i})$. We also assume that there exist an SStF S_i from $\widehat{\Sigma}_i$ to Σ_i with the corresponding functions, constants, and matrices denoted by α_i, $\rho_{\text{ext}i}$, κ_i, ψ_i, $\bar{X}_i$, $\bar{X}_i^{11}$, $\bar{X}_i^{12}$, $\bar{X}_i^{21}$, and $\bar{X}_i^{22}$. In the next theorem, we quantify the error between the interconnection of *continuous-time* stochastic subsystems and that of their *discrete-time* abstractions in a compositional fashion.

Theorem 14. Consider an interconnected stochastic control system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems Σ_i and the coupling matrix M. Let each subsystem Σ_i admit an abstraction $\widehat{\Sigma}_i$ with the corresponding SStF S_i. Then

$$V(x,\hat{x}) := \sum_{i=1}^{N} \mu_i S_i(x_i,\hat{x}_i), \tag{14}$$

is a stochastic simulation function from the interconnected system $\widehat{\Sigma} = \mathcal{I}(\widehat{\Sigma}_1,\ldots,\widehat{\Sigma}_N)$, with coupling matrix $\hat{M}$, to $\Sigma = \mathcal{I}(\Sigma_1,\ldots,\Sigma_N)$ if there exist $\mu_i > 0$, $i \in \{1,\ldots,N\}$, and

$$\begin{bmatrix} M \\ \mathbb{I}_{\tilde{q}} \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} M \\ \mathbb{I}_{\tilde{q}} \end{bmatrix} \preceq 0, \tag{15}$$

$$M = \hat{M}, \tag{16}$$

$$\hat{M} \prod_{i=1}^{N} \hat{Y}_{2i} \subseteq \prod_{i=1}^{N} \hat{W}_i, \tag{17}$$

where

$$\bar{X}_{cmp} := \begin{bmatrix} \mu_1 \bar{X}_1^{11} & & & \mu_1 \bar{X}_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N \bar{X}_N^{11} & & & \mu_N \bar{X}_N^{12} \\ \mu_1 \bar{X}_1^{21} & & & \mu_1 \bar{X}_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N \bar{X}_N^{21} & & & \mu_N \bar{X}_N^{22} \end{bmatrix}, \tag{18}$$

and $\tilde{q} = \sum_{i=1}^{N} q_{2i}$ with q_{2i} being dimensions of the internal output of subsystems Σ_i.

5. CONSTRUCTION OF STOCHASTIC STORAGE FUNCTIONS FOR A CLASS OF SYSTEMS

In this section, we focus on a special class of continuous-time stochastic affine systems and impose conditions enabling us to establish an SStF from its finite abstraction $\widehat{\Sigma}$ to Σ. The model of the system is given by

$$\Sigma : \begin{cases} \mathsf{d}\xi(t) = (A\xi(t) + B\nu(t) + Dw(t) + \mathbf{b})\mathsf{d}t + G\mathsf{d}W_t, \\ \zeta_1(t) = C_1\xi(t), \\ \zeta_2(t) = C_2\xi(t), \end{cases} \tag{19}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{n\times p}, C_1 \in \mathbb{R}^{q_1\times n}, C_2 \in \mathbb{R}^{q_2\times n}$, $G \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^n$. We employ the tuple

$$\Sigma = (A, B, C_1, C_2, D, G, \mathbf{b}),$$

to refer to the class of stochastic affine systems in (19). The *time-discretized* version of Σ is proposed as

$$\widetilde{\Sigma} : \begin{cases} \tilde{\xi}(k+1) = \tilde{\xi}(k) + \tilde{\nu}(k) + \tilde{D}\tilde{w}(k) + \tilde{R}\varsigma(k), \\ \tilde{\zeta}_1(k) = \tilde{C}_1\tilde{\xi}(k), \\ \tilde{\zeta}_2(k) = \tilde{C}_2\tilde{\xi}(k), \end{cases} \quad k \in \mathbb{N}, \tag{20}$$

where $\tilde{D}$ and $\tilde{R}$ are matrices chosen arbitrarily, and $\tilde{C}_1 = C_1P, \tilde{C}_2 = C_2P$ with P as chosen in (22) (cf. Theorem 19). Our main target here is to employ $\widetilde{\Sigma}$ as the discrete-time version of Σ in order to establish an SStF from $\widehat{\Sigma}$ to Σ through $\widetilde{\Sigma}$ while quantifying the *best approximation error*. Later, in Remark 20, we show that $\tilde{R} = \mathbf{0}_n$ and $\tilde{D} = \mathbf{0}_{n\times p}$ result in the least approximation error in our settings. Now, we describe the *finite* abstraction of $\widetilde{\Sigma}$ as

$$\widehat{\Sigma} : \begin{cases} \hat{\xi}(k+1) = \Pi_{\hat{\xi}}(\hat{\xi}(k) + \hat{\nu}(k) + \tilde{D}\hat{w}(k) + \tilde{R}\varsigma(k)), \\ \hat{\zeta}_1(k) = \hat{C}_1\hat{\xi}(k), \\ \hat{\zeta}_2(k) = \hat{C}_2\hat{\xi}(k), \end{cases} \quad k \in \mathbb{N}, \tag{21}$$

where map $\Pi_{\hat{\xi}} : \hat{X} \to \hat{X}$ satisfies the inequality (7). Now we candidate the following quadratic stochastic storage function

$$S(x,\hat{x}) = (x - P\hat{x})^T \bar{\mathcal{M}}(x - P\hat{x}), \tag{22}$$

where P is a square matrix and $\bar{\mathcal{M}}$ is a positive-definite matrix of an appropriate dimension. In order to show that S in (22) is an SStF from $\widehat{\Sigma}$ to Σ, we need the following key assumptions over Σ.

Assumption 15. Assume that there exists a *concave* function $\gamma \in \mathcal{K}_\infty$ such that S satisfies

$$S(x,x') - S(x,x'') \le \gamma(\|x' - x''\|), \ \forall x, x', x'' \in X. \tag{23}$$

Note that Assumption 15 is always fulfilled for the function S in (22) as long as it is restricted to a compact subset of $X \times X$.

Assumption 16. Let $\Sigma = (A, B, C_1, C_2, D, G, \mathbf{b})$. Assume that for some constant $\tilde{\kappa} \in \mathbb{R}_{>0}$, there exist matrices $\bar{\mathcal{M}} \succ 0$, K, Q and H of appropriate dimensions such that the following matrix (in)equalities hold:

$$(A + BK)^T \bar{\mathcal{M}} + \bar{\mathcal{M}}(A + BK) \le -\tilde{\kappa}\bar{\mathcal{M}}, \tag{24a}$$

$$BQ = AP, \tag{24b}$$

$$D = BH. \tag{24c}$$

Note that stabilizability of the pair (A, B) is necessary and sufficient to satisfy condition (24a). Moreover, there exist matrices Q and H satisfying conditions (24b) and (24c) if and only if im $AP \subseteq$ im B and im $D \subseteq$ im B, respectively.

Assumption 17. Let $\Sigma = (A, B, C_1, C_2, D, G, \mathbf{b})$. Assume that for some constants $\pi > 0$ and $0 < \bar{\kappa} < 1 - e^{-\tilde{\kappa}\tau}$ with a sampling time τ, there exist matrices $\bar{X}^{11}$, $\bar{X}^{12}$, $\bar{X}^{21}$, and $\bar{X}^{22}$ of appropriate dimensions such that

$$\begin{bmatrix} \pi e^{-\tilde{\kappa}\tau}\tau B^T\bar{\mathcal{M}}B & 0 \\ 0 & \pi e^{-\tilde{\kappa}\tau}\tau D^T\bar{\mathcal{M}}D \end{bmatrix} \preceq \begin{bmatrix} \bar{\kappa}\bar{\mathcal{M}} + C_2^T\bar{X}^{22}C_2 & C_2^T\bar{X}^{21} \\ \bar{X}^{12}C_2 & \bar{X}^{11} \end{bmatrix}. \tag{25}$$

Remark 18. Note that in Assumption 17, matrices B, D, C_2 are those in the system dynamics, constant and matrix $\tilde{\kappa}, \bar{\mathcal{M}}$ are the same as those satisfying the condition (24a), and constants and matrices $\pi, \tau, \bar{\kappa}, \bar{X}^{11}, \bar{X}^{12}, \bar{X}^{21}, \bar{X}^{22}$ are our decision variables to be designed. One can readily satisfy this assumption via semi-definite programing toolboxes and then check the compositionality condition (15) with obtained conformal block partitions $\bar{X}^{ij}$, $i, j \in \{1, 2\}$ of subsystems (cf. the case study).

Now we provide another main result of the paper showing that under which conditions S in (22) is an SStF from $\widehat{\Sigma}$ to Σ.

Theorem 19. Let $\Sigma = (A, B, C_1, C_2, D, G, \mathbf{b})$ and $\widehat{\Sigma}$ be its *finite* MDP with discretization parameter δ. Suppose Assumptions 15, 16 and 17 hold, $\hat{C}_1 = \tilde{C}_1 = C_1P$, and $\hat{C}_2 = \tilde{C}_2 = C_2P$. Then the quadratic function S in (22) is an SStF from $\widehat{\Sigma}$ to Σ.

The functions and constants $\alpha, \rho_{\text{ext}} \in \mathcal{K}_\infty$, $0 < \kappa < 1$, and $\psi \in \mathbb{R}_{>0}$ in Definition 8 associated with S in (22) are computed as

$$\alpha(s) := \frac{\lambda_{\min}(\bar{\mathcal{M}})}{\lambda_{\max}(C_1^TC_1)}s^2, \quad \forall s \in \mathbb{R}_{\ge 0},$$

$$\kappa := \bar{\kappa} + e^{-\tilde{\kappa}\tau},$$

$$\rho_{\text{ext}}(s) := \gamma((1 + \frac{1}{\bar{\eta}})(1 + \bar{\eta}')(1 + \bar{\eta}'')s), \quad \forall s \in \mathbb{R}_{\ge 0},$$

$$\begin{aligned} \psi := {} & e^{-\tilde{\kappa}\tau}\tau(G^T\bar{\mathcal{M}}G + \pi\|\sqrt{\bar{\mathcal{M}}}\mathbf{b}\|^2) \\ & + \gamma((1 + \bar{\eta})\delta) + \gamma((1 + \frac{1}{\bar{\eta}})(1 + \frac{1}{\bar{\eta}'})\sqrt{\text{Tr}(\tilde{R}^T\tilde{R})}) \\ & + \gamma((1 + \frac{1}{\bar{\eta}})(1 + \bar{\eta}')(1 + \frac{1}{\bar{\eta}''})\,\|\tilde{D}\|\,\|\hat{w}\|), \end{aligned}$$

where $\bar{\eta}, \bar{\eta}', \bar{\eta}'' > 0$ are some positive constants chosen arbitrarily.

Remark 20. Note that for the discrete-time system $\widetilde{\Sigma}$ in (20), $\rho_{\rm ext}$, and ψ defined above reduce to

$$\begin{aligned}\rho_{\rm ext}(s) :=&\ \gamma((1+\bar{\eta})(1+\bar{\eta}')s), \quad \forall s \in \mathbb{R}_{\geq 0},\\ \psi :=&\ e^{-\tilde{\kappa}\tau}\tau(G^T\bar{\mathcal{M}}G + \pi\|\sqrt{\bar{\mathcal{M}}}\mathbf{b}\|^2)\\ &+ \gamma((1+\frac{1}{\bar{\eta}})\sqrt{\mathrm{Tr}(\tilde{R}^T\tilde{R})})\\ &+ \gamma((1+\bar{\eta})(1+\frac{1}{\bar{\eta}'})\,\|\tilde{D}\|\,\|\hat{w}\|).\end{aligned}$$

Moreover, if the abstraction $\widetilde{\Sigma}$ is *non-stochastic* (*i.e.,* $\tilde{R} = \mathbf{0}_n$) with $\tilde{D} = \mathbf{0}_{n\times p}$, then

$$\begin{aligned}\rho_{\rm ext}(s) :=&\gamma(s), \quad \forall s \in \mathbb{R}_{\geq 0},\\ \psi :=& e^{-\tilde{\kappa}\tau}\tau(G^T\bar{\mathcal{M}}G + \pi\|\sqrt{\bar{\mathcal{M}}}\mathbf{b}\|^2).\end{aligned}$$

This simply means if the concrete system satisfies some stability property (cf. (24a)), it is better to pick non-stochastic discrete-time system rather than stochastic ones since the non-stochastic systems provide smaller approximation errors (cf. the case study).

Note that $\tilde{D} = \mathbf{0}_{n\times p}$ (*i.e.,* not having any internal input in the abstract systems in (21)) will result in less approximation errors. In fact, a smart choice of the interface map (9.3) in Appendix of (Nejati et al., 2020) still ensures that the output trajectories of abstract systems follow those of the original ones with a quantified probabilistic error bound which gets smaller if $\tilde{D} = \mathbf{0}_{n\times p}$.

6. CASE STUDY

To illustrate the effectiveness of the proposed results, we apply our approaches to a temperature regulation in a circular network containing 100 rooms and construct compositionally a discrete-time system from its original continuous-time dynamic. We then employ the constructed discrete-time abstractions as substitutes to compositionally synthesize policies regulating the temperature of each room in a comfort zone. By employing Theorem 11, we guarantee that the distance between outputs of continuous-time system Σ and discrete-time system $\widetilde{\Sigma}$ will not exceed $\varepsilon = 0.5$ during the time horizon $T_d = 12$ with a probability at least 91%. Due to lack of space, we provide all details of the case study in the arXiv version (Nejati and Zamani, 2020).

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