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Profit Optimization for Zero Ending Inventories Dynamic Pricing Model under Stochastic Demand and Fixed Lifetime Product

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Abstract. We consider a single perishable product under a compound Poisson demand with a price sensitive intensity and a continuous batch size distribution. A model of a dynamic retail price control with an adjustable coefficient is proposed providing almost surely zero ending inventories at the end of the product's lifetime. To obtain probabilistic characteristics of the selling process and the expected profit a diffusion approximation of the demand process is used. The task of the expected profit optimization with respect to the coefficient and lot size for a linear intensity-of-price dependence is solved.

Keywords: Dynamic Pricing, Zero Ending Inventories, Adjustable Coefficient, Price Sensitive Demand, Compound Poisson Demand, Diffusion Process, Expected Profit Maximization

1. INTRODUCTION AND PROBLEM STATEMENT

The influence of dynamic pricing on revenue and spoilage as well as environmental and social impacts of perishable products management have been discussed recently by Adenso-Díaz et al. (2017), and Tekin and Erol (2017), where more references concerned the problem can be found. Options for waste reduction have the highest priority, since even disregarding possible negative social effects, waste disposal of perishables can be expensive and can cause environmental problems; see Curran and Williams (2012); Singh, Ramakrishna, and Gupta (2017); and Hannon and Zaman (2018).

In this paper, we consider a generalization of the dynamic pricing control model for perishable items proposed in Kitaeva et al. (2019). We introduce a scale coefficient into the model, which allows us to optimize the sales process.

Let us consider a supply chain consisting of a vendor and customers. A vendor buys lot Q_0. The duration of the product's lifetime is T. The demand is price-sensitive, and the product would be sold for sure, if the price is low enough. This assumption holds, for example, for essential goods. Above all, the vendor wants to avoid leftovers. The demand is modeled as a compound Poisson process with intensity $\lambda(c)$, where $c = c(t)$ is a dynamic retail price per unit, the values of orders are i.i.d. continuous random variables with the first and second moments a_1 and a_2 respectively.

We assume that stock level process $Q(\cdot)$ can be described approximately by the following stochastic differential equation:

$$dQ(t) = -a_1\lambda(c(t))dt - \sqrt{a_2\lambda(c(t))}dw(t), \tag{1}$$

where $w(\cdot)$ is the Wiener process.

In Kitaeva et al. (2014) a theoretical justification of the diffusion approximation with some numerical results is given. The Brownian motion process is one of the most commonly used process for demand modeling in the inventory literature; see Kitaeva et al. (2019) for references.

We are going to consider the following model of the retail price control

$$a_1\lambda\big(c(t)\big) = \kappa\frac{Q(t)}{T-t}, \tag{2}$$

that is, we require that the average rate of a product's sale at $\big[0, T-t\big]$ and the instantaneous rate of its sale at time t (left hand side of the equation) be proportional to each other; coefficient $\kappa > 0$.

From (1) and (2) it follows that the stock level process satisfies the following equation:

$$dQ(t) = -\kappa \frac{Q(t)}{T-t} dt + \sqrt{\kappa \frac{a_2}{a_1} \frac{Q(t)}{T-t}} dw(t) . \qquad (3)$$

Consider the effect of coefficient κ on probabilistic characteristics of process $Q(\cdot)$. Extending the previously obtained results to the new model is straightforward, but for completeness we should briefly present them.

2. PROBABILISTIC CHARACTERICTICS OF THE STOCK LEVEL PROCESS

2.1 Expectation and variance of process $Q(\cdot)$

Let us denote expectation $E\{Q(t)\} = \bar{Q}(t) = \bar{Q}$. From (3) we have

$$d\bar{Q}(t) = -\kappa \frac{\bar{Q}(t)}{T-t} dt \qquad (4)$$

with the initial condition $\bar{Q}(0) = Q_0$. It follows that $\bar{Q}(t) = Q_0 \left(1 - t/T\right)^{\kappa}$.

Applying Ito's formula, we get from (3)

$$d\left(Q^2\right) = \left(-2\frac{\kappa Q^2}{T-t} + \frac{\kappa a_2}{a_1} \frac{Q}{T-t}\right) dt + 2Q\sqrt{\frac{\kappa a_2}{a_1} \frac{Q}{T-t}} dw(t) .$$

It follows that

$$\frac{d\overline{Q^2}}{dt} + 2\kappa \frac{\overline{Q^2}}{T-t} = \kappa \frac{a_2}{a_1} \frac{\bar{Q}}{T-t} , \qquad (5)$$

where $\overline{Q^2} = \overline{Q^2}(t) = E\{Q^2(t)\}$.

From (4) we get

$$\frac{d\left(\bar{Q}^2\right)}{dt} + 2\kappa \frac{\bar{Q}^2}{T-t} = 0 . \qquad (6)$$

Subtracting (6) from (5) we get equation for variance of $Q(\cdot)$

$$\frac{dV_Q}{dt} + 2\kappa \frac{V_Q}{T-t} = \frac{a_2}{a_1} \kappa \frac{\bar{Q}}{T-t} = \frac{a_2}{a_1} \kappa \frac{Q_0}{T} \left(1 - \frac{t}{T}\right)^{\kappa-1} , \quad (7)$$

where $V_Q = V_Q(t) = Var\{Q(t)\}$, subject to $V_Q(0) = 0$.

Solution of (7) has the following form:

$$V_Q(t) = \frac{a_2}{a_1} Q_0 \left(1 - \frac{t}{T}\right)^{\kappa} \left(1 - \left(1 - \frac{t}{T}\right)^{\kappa}\right).$$

Figure 1 shows the graphs of functions $f_2(x) = (1-x)^{\kappa}\left(1-(1-x)^{\kappa}\right)$ for $\kappa = 0.5, 1, 2$. These functions describe ratio $a_1 V_Q(\cdot)/\left(a_2 Q_0\right)$ dependence of normalized time *t/T*. Note, that the larger coefficient κ value, the closer maximum value of the variance to the beginning of the time interval. For $\kappa = 1$ the dependence is parabolic with a vertex in the centre of the interval.

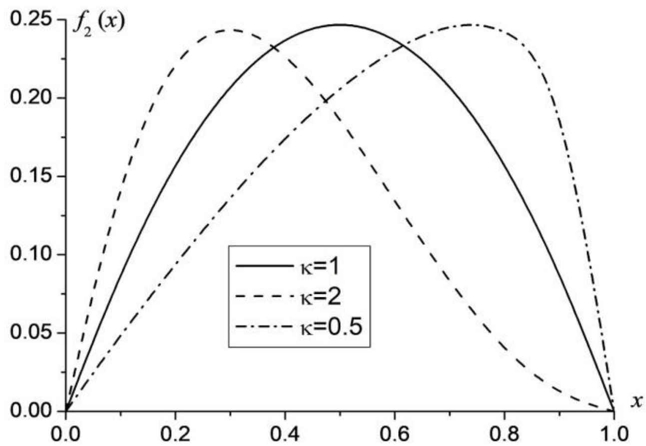

Fig. 1. $a_1 V_Q(\cdot)/\left(a_2 Q_0\right) = f_2(\cdot)$ dependence of *t/T = x*, $\kappa = 0.5, 1, 2$.

The second initial moment of $Q(\cdot)$

$$\overline{Q^2} = \frac{a_2}{a_1} Q_0 \left(1 - \frac{t}{T}\right)^{\kappa} \left[1 - \left(1 - \frac{t}{T}\right)^{\kappa}\right] + Q_0^2 \left(1 - \frac{t}{T}\right)^{2\kappa} .$$

2.2 Covariance function of process $Q(\cdot)$

The covariance function of process $Q(\cdot)$ $R_0(t_1, t_2) = R(t_1, t_2) - E\{Q(t_1)\}E\{Q(t_2)\}$, where $R(t_1, t_2) = E\{Q(t_1)Q(t_2)\}$.

From (3) we get $\frac{\partial R(t_1,t_2)}{\partial t_2} = -\kappa \frac{R(t_1,t_2)}{T-t_2}$, and it follows that

$$R(t_1,t_2) = C(t_1)(T-t_2)^{\kappa},$$

where $C(t) = \frac{R(t,t)}{(T-t)^{\kappa}} = \frac{1}{(T-t)^{\kappa}}\left(V_Q(t) + \frac{Q_0^2}{T^{2\kappa}}(T-t)^{2\kappa}\right)$.

Thus, for $t_2 > t_1$

$$R_0(t_1,t_2) = \frac{a_2}{a_1} Q_0 \left(1-\left(1-\frac{t_1}{T}\right)^{\kappa}\right)\left(1-\frac{t_2}{T}\right)^{\kappa}.$$

2.3 Probability density function of process $Q(\cdot)$

Consider the Laplace transform of the probability density function (PDF) of $Q(\cdot)$ $\Phi(p,t) = E\{\exp(-pQ(t))\}$. According to Ito's formula, we get from (3)

$$d\exp(-pQ(t)) = p\kappa \frac{Q(t)}{T-t}\exp(-pQ(t))\left(1+\frac{a_2}{2a_1}p\right)dt -$$

$$-p\exp(-pQ(t))\sqrt{\kappa \frac{a_2}{a_1}\frac{Q(t)}{T-t}}dw(t). \quad (8)$$

After averaging (8), we get

$$(T-t)\frac{\partial \Phi}{\partial t} + \kappa p\left(1+\frac{a_2}{2a_1}p\right)\frac{\partial \Phi}{\partial p} = 0. \quad (9)$$

Solution of (9) has the form $\Phi(p,t) = \varphi\left(\frac{p(T-t)^{\kappa}}{p+\beta}\right)$, where $\varphi(\cdot)$ is an unknown function and $\beta = 2a_1/a_2$.

From the initial condition it follows that $\Phi(p,0) = = \varphi\left(\frac{pT^{\kappa}}{p+\beta}\right) = \exp(-pQ_0)$, and $\varphi(z) = \exp\left(-\frac{\beta z Q_0}{T^{\kappa}-z}\right)$.

Finally, we get

$$\Phi(p,t) = \varphi\left(\frac{p(T-t)^{\kappa}}{p+\beta}\right) = \exp\left(-\frac{p\beta(1-t/T)^{\kappa}}{p+\beta-p(1-t/T)^{\kappa}}Q_0\right).$$

Using inverse Laplace transform we obtain PDF of $Q(\cdot)$

$$f(q,t) = \exp\left(\frac{-\beta Q_0(1-t/T)^{\kappa}}{1-(1-t/T)^{\kappa}}\right)\left[\delta(q) + \beta\exp\left(\frac{-\beta Q_0}{1-(1-t/T)^{\kappa}}\right)\times\right.$$

$$\left.\times\frac{\sqrt{(1-t/T)^{\kappa}Q_0/q}}{1-(1-t/T)^{\kappa}} I_1\left(\frac{2\beta\sqrt{(1-t/T)^{\kappa}Q_0 q}}{1-(1-t/T)^{\kappa}}\right)\right], \quad (10)$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind and first order, $\delta(\cdot)$ is the Dirac delta function. The term containing the delta function in (10) arises because at some point of time a customer can buy all inventories on hand.

2.4 Distribution of the selling duration

From (10) it follows that the cumulative distribution function (CDF) of the length of time it takes to sell lot Q_0

$$F_{\tau}(t) = \exp\left(-\beta Q_0 \frac{(1-t/T)^{\kappa}}{1-(1-t/T)^{\kappa}}\right). \quad (11)$$

Thus, $F_{\tau}(T) = 1$, that is, the lot will be sold by the end of the product's lifetime almost surely. On the other hand, the service level during the period is high enough for $\beta Q_0 >> 1$.

The expectation of the selling duration τ

$$E\{\tau\} = \int_0^T (1-F_{\tau}(t))dt =$$

$$= T\left(1-\int_0^1 \exp\left(-\beta Q_0 \frac{z^k}{1-z^k}\right)dz\right). \quad (12)$$

For $\beta Q_0 >> 1$

$$E\{\tau\} \approx T\left(1-\int_0^1 \exp\left(-\beta Q_0 z^k\right)dz\right)=$$

$$=T\left(1-\frac{1}{k\left(\beta Q_0\right)^{1/k}}\int_0^\infty u^{(1-k)/k}e^{-u}du\right)=T\left(1-\frac{\Gamma\left((k+1)/k\right)}{\left(\beta Q_0\right)^{1/k}}\right),$$

where $\Gamma(\cdot)$ is the gamma function.

In Figures 2-3 graphs of $F_\tau(\cdot)$ are shown for different values of βQ_0 and κ.

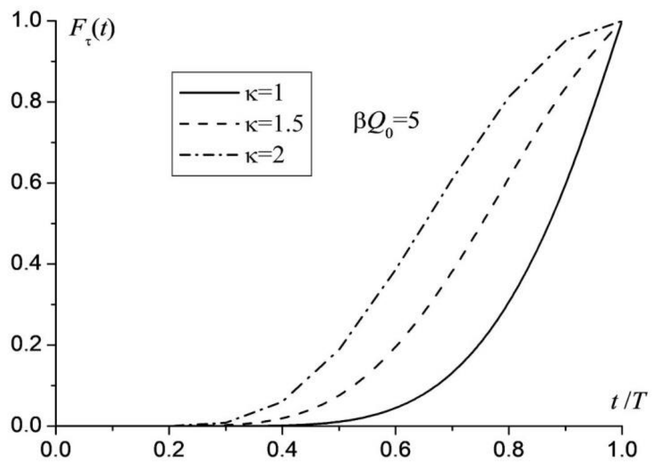

Fig. 2. CDF of the selling duration for $\beta Q_0 = 5;\ \kappa = 1, 1.5, 2$.

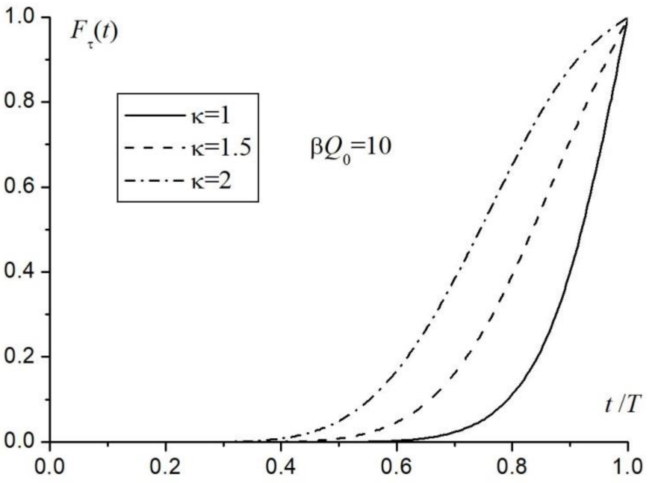

Fig. 3. CDF of the selling duration for $\beta Q_0 = 10;\ \kappa = 1, 1.5, 2$.

A stock out duration is $T-\tau$, and it follows from (11) that

$$P(T-\tau \le s) = 1-\exp\left(-\beta Q_0 \frac{\left(1-(T-s)/T\right)^\kappa}{1-\left(1-(T-s)/T\right)^\kappa}\right).$$

We can also obtain the conditional probability density function of the stock level process analogously as it has been done in Kitaeva et al. (2019). Using this density we can find the variance of the revenue.

3. THE EXPECTED PROFIT AND ITS OPTIMIZATION

Let us consider linear intensity-of-price dependence

$$\lambda(c) = \lambda_0 - \lambda_1 \frac{c(t)-c_0}{c_0}, \tag{13}$$

where c_0 is an “usual” price corresponding “usual” intensity λ_0 and parameter $\lambda_1 > 0$ characterizes the sensitivity of $\lambda(\cdot)$ to relative price‘s deviations from c_0.

From (2) we get

$$c(t) = c_0\left(1+\frac{\lambda_0}{\lambda_1}-\frac{\kappa Q(t)}{a_1\lambda_1(T-t)}\right).$$

The average revenue at time unit

$$E\{c(t)a_1\lambda(c)\} = c_0\kappa E\left\{\left(1+\frac{\lambda_0}{\lambda_1}-\frac{\kappa}{a_1\lambda_1}\frac{Q(t)}{T-t}\right)\frac{Q(t)}{T-t}\right\}=$$

$$=c_0\left(1+\frac{\lambda_0}{\lambda_1}\right)\kappa\frac{\bar{Q}}{T-t}-c_0\frac{\kappa^2}{a_1\lambda_1}\frac{\overline{Q^2}}{(T-t)^2}.$$

Finally, taking into account the results of subsection 2.1 we get

$$E\{c(t)a_1\lambda(c)\} = c_0\left(1+\frac{\lambda_0}{\lambda_1}\right)\kappa\frac{Q_0}{T-t}\left(1-\frac{t}{T}\right)^\kappa -$$

$$-\frac{c_0\kappa^2 Q_0}{a_1\lambda_1 (T-t)^2}\left(\frac{a_2}{a_1}\left(1-\frac{t}{T}\right)^{\kappa}\left(1-\left(1-\frac{t}{T}\right)^{\kappa}\right)+\left(1-\frac{t}{T}\right)^{2\kappa} Q_0^2\right).$$

3.1 The adjustable parameter optimization

Let us consider the expected revenue over the whole cycle

$$\overline{S} = a_1\int_0^T E\{c(t)\lambda(t)\}\,dt,$$

and find the range of possible parameter κ values for the linear model under consideration.

We should calculate three integrals:

$$\int_0^T \left(1-\frac{t}{T}\right)^{\kappa}\frac{dt}{T-t} = \frac{1}{\kappa} \text{ for } \kappa > 0;$$

$$\int_0^T \left(1-\frac{t}{T}\right)^{\kappa}\frac{dt}{(T-t)^2} = \frac{1}{T(\kappa-1)} \text{ for } \kappa > 1;$$

$$\int_0^T \left(1-\frac{t}{T}\right)^{2\kappa}\frac{dt}{(T-t)^2} = \frac{1}{T(2\kappa-1)} \text{ for } \kappa > 0.5.$$

Thus, for $\kappa > 1$

$$\overline{S} = c_0 Q_0\left(1+\frac{\lambda_0}{\lambda_1}\right) - \frac{c_0 a_2 Q_0\kappa^2}{a_1^2\lambda_1 T}\left(\frac{1}{\kappa-1}-\frac{1}{2\kappa-1}\right)-\frac{c_0 Q_0^2\kappa^2}{a_1\lambda_1 T(2\kappa-1)}. \tag{14}$$

To maximize $\overline{S}$ with respect to κ under fixed Q_0 we should solve the following cubic equation:

$$\frac{a_1 Q_0}{a_2} = -\frac{\kappa\left(2\kappa^2-6\kappa+3\right)}{2(\kappa-1)^3}. \tag{15}$$

It is easy to see that we are interested in only the roots that belong to the interval $\left(1,\frac{3+\sqrt{3}}{2}\right)$ because the right hand side of (15) should be positive.

In Figure 4 the dependence of optimal κ value of ratio $a_1 Q_0 / a_2$ is shown.

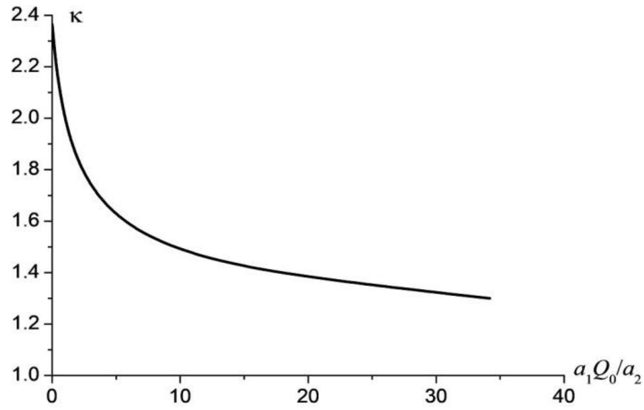

Fig. 4. Optimal κ value for a fixed lot size.

For large values of $a_1 Q_0 / a_2$ optimal κ value is close to 1.

Let $\kappa = 1 + \varepsilon$, where ε is a small value, then approximate equation holds

$$-\frac{\kappa\left(2\kappa^2-6\kappa+3\right)}{2(\kappa-1)^3} \approx \frac{1}{2\varepsilon^3}.$$

It follows that for a fixed large lot size optimal

$$\kappa \approx 1+\sqrt[3]{\frac{a_2}{2a_1 Q_0}}.$$

3.2 Lot size optimization

Denote d the buying price per unit of the product. Taking (14) into account, the average profit during the cycle

$$\mathrm{P} = \overline{S} - Q_0 d = \frac{\lambda_0}{\lambda_1}c_0 Q_0 + Q_0(c_0-d) - \frac{c_0 Q_0\kappa^2}{a_1\lambda_1 T}\left(\frac{a_2}{a_1}\left(\frac{1}{\kappa-1}-\frac{1}{2\kappa-1}\right)-\frac{Q_0}{2\kappa-1}\right).$$

It follows that the optimal lot size value can be written as

$$Q_0 = \frac{a_1\lambda_1 T}{2}\left(1+\frac{\lambda_0}{\lambda_1}-\frac{d}{c_0}\right)\frac{2\kappa-1}{\kappa^2}-\frac{a_2}{2a_1}\frac{\kappa}{(\kappa-1)}$$

for a fixed κ value.

Solving the optimization problem simultaneously for both lot size and coefficient κ we get system of equations

$$\begin{cases} \dfrac{a_1}{a_2} Q_0 = -\dfrac{\kappa\left(2\kappa^2 - 6\kappa + 3\right)}{2\left(\kappa - 1\right)^3}, \\ \\ Q_0 = \dfrac{a_1 \lambda_1 T}{2}\left(1 + \dfrac{\lambda_0}{\lambda_1} - \dfrac{d}{c_0}\right)\dfrac{2\kappa - 1}{\kappa^2} - \dfrac{a_2 \kappa}{2a_1\left(\kappa - 1\right)}. \end{cases} \tag{16}$$

It follows that the optimal κ value satisfies the following equation:

$$\frac{a_1^2 \lambda_1 T}{a_2}\left(1 + \frac{\lambda_0}{\lambda_1} - \frac{d}{c_0}\right) = -\frac{\kappa^3\left(\kappa^2 - 4\kappa + 2\right)}{\left(\kappa - 1\right)^3\left(2\kappa - 1\right)}. \tag{17}$$

Inequalities $\kappa^2 - 4\kappa + 2 > 0$ and $\kappa > 1$, that is, $1 < \kappa < 2 + \sqrt{2}$ also should be hold.

Equation (17) has a unique solution in this interval. Figure 5 shows the graph of function

$$\psi(\kappa) = -\frac{\kappa^3\left(\kappa^2 - 4\kappa + 2\right)}{\left(\kappa - 1\right)^3\left(2\kappa - 1\right)}$$

for $\kappa \in \left(1, 2 + \sqrt{2}\right)$.

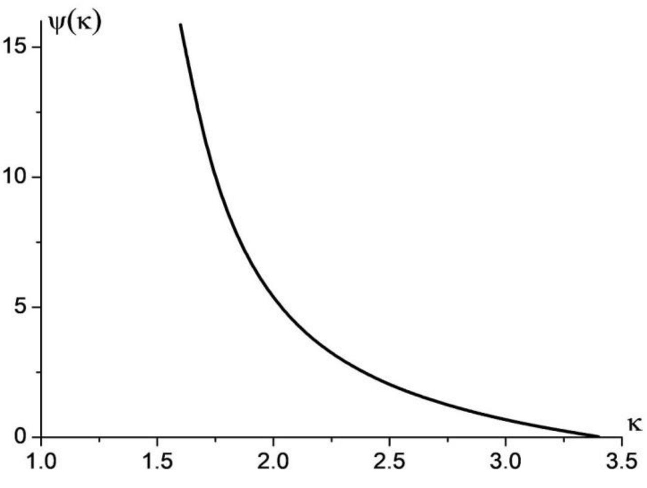

Fig. 5. Function defying optimal κ value for simultaneous optimization with respect to Q_0 and κ.

By setting the parameters a_1, a_2, λ_0, λ_1, T, d and c_0, we can obtain the optimal value of the adjustable coefficient within interval $\left(1, 2 + \sqrt{2}\right)$ from (17); and then the optimal lot size is defined by (16).

For large values of $\lambda_1 T$ the optimal κ value is close to 1, and for $0 < \varepsilon << 1$

$$\kappa = 1 + \varepsilon \approx 1 + \sqrt[3]{\frac{a_2 c_0}{a_1^2 T\left(\lambda_1\left(c_0 - d\right) + \lambda_0 c_0\right)}}.$$

The optimal lot size in this case

$$Q_0 \approx \frac{a_2}{2a_1 \varepsilon^3} = \frac{a_1 T}{2c_0}\left(\lambda_1\left(c_0 - d\right) + \lambda_0 c_0\right).$$

4. CONCLUSION

The proposed dynamic price control model not only almost surely solves the problem of leftovers for the stochastic demand but also allows us to optimize the sales process using the introduced adjustable coefficient.

In Kitaeva et al. (2019) we deal with small deviations of the dynamic price from the stationary one. Here, such an approach leads to too complicated dependence of revenue on the coefficient and lot size. So, in this paper, we solve the optimization problem for a linear dependence of the intensity of the price. The presence of the adjustable coefficient allows us to consider this dependence.

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