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On the Reachable Set of Uncertain Closed Loop Discrete-Time Linear Systems

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Abstract: This work tackles the reachability problem of uncertain discrete-time linear systems controlled by estimated state feedback regulators. First an augmented system is considered in order to include the effect of the estimation error on the behavior of the closed loop system. Then, based on interval analysis, an interval predictor is proposed to compute tight trajectory tubes that contain in a guaranteed way the actual state trajectory of the augmented system. Moreover, under the standard controllability and observability assumptions of linear systems, the convergence of the width of these trajectory tubes is shown.

Keywords: Uncertain linear systems, closed-loop systems, estimated state feedback control, reachability analysis, interval analysis, interval estimators.

1. INTRODUCTION

Reachability problems have been received a great attention these last years, for example (Chisci et al., 1996; Daryin et al., 2006; Guernic and Girard, 2010; Kühn, 1998; Kurzhanski and Varaiya, 2002, 2007; Pico and Aliprantis, 2016, 2018; Rakovic et al., 2006; Ramdani et al., 2009, 2010). In fact, reachability is a fundamental issue that appears in different engineering fields where the safety requirement is crucial, since safety properties specify the (unsafe/bad) state regions that a system should not enter. In the context of dynamical systems, the reachability problem consists in characterizing all the possible state trajectories generated from a given set of initial states and governed by a (possible uncertain) dynamics. In the literature, this problem has been tackled for continuous and hybrid systems [1], in both continuous-time and discrete-time domains, where various set representations (for example [2], polytopes in (Bak et al., 2019), parallelotopes in (Dreossi et al., 2017), ellipsoids in (Kurzhanski and Varaiya, 2007), zonotopes in (Pico and Aliprantis, 2018), boxes in (Ramdani et al., 2009; Chen et al., 2013), polynomial sub-level set in (Rodríguez-Carbonell and Tiwari, 2005), level sets in (Mitchell and Templeton, 2005)) have been used in specific set-membership reachability algorithms. Notice that most of these approaches are based on numerical methods to compute outer approximations of reachable sets; however other methods using SMT and theorem proving have also been developed by (Platzer and Quesel, 2008; Kong et al., 2015). Among the numerical algorithms, we can further classify them into two categories: the algorithms that incrementally approximate the reachable set for each time step, and the algorithms that aim at approximating the whole reachable set using invariance conditions. While the second category enables us to prove unbounded-time safety properties, approximation accuracy may not be a stringent requirement since only an approximation sufficient to prove a given safety property is needed. Concerning the algorithms of the first category, a major problem they face is conservatism linked to the *wrapping effect* (Alefeld and Mayer, 2000; Corliss, 1989; Moore, 1966). More precisely, in order to reduce the computational effort, at every step of these methods, the exact reachable set of the system is overestimated by super-sets that are easy to design and computationally tractable. Repeating this procedure for a large number of steps may lead to significant error accumulation that returns overly conservative state enclosures.

Recently, a new approach to characterize tight enclosures of the reachable set of a class of discrete-time dynamical systems has been introduced in (Meslem and Martinez, 2020). To face the *wrapping effect*, their approach proposes to compute enclosures of the exact reachable set at every time instant directly from the initial set. The main idea brought by their approach is the use of a pre-processed interval version of the analytic expression of the state response of discrete-time linear systems, where interval assessments of the state enclosures are not carried out in a recursive way. The same idea has been also applied in (Meslem et al., 2017, 2018) to characterize the reachable set of the estimation error of discrete-time linear set-membership state estimators. In this work, we extend the use of their approach to closed loop uncertain discrete-time linear systems, where the system outputs are not required to obtain guaranteed enclosures of the actual state vector. More precisely, we consider the case of uncertain discrete-time linear systems that are controlled by estimated state feedback regulators. Thus, the effect of the estimation error on the behavior of the closed loop system has

[1] This problem for discrete systems is traditionally treated in model checking originated in computer science.
[2] The reader is referred to a survey in the chapter "Verification of Hybrid Systems" of the book entitled "Handbook of Model Checking" Doyen et al. (2018).

to be considered while computing outer approximations of its reachable set. To achieve that, we propose an augmented system that includes the dynamics of the estimation error and the bounds of the feasible domains of the measurement noise and the state disturbance. It is worth pointing out that, to the best of our knowledge, this reachability problem of this class of uncertain systems has not been yet treated in the literature. However, the error accumulation problem is addressed in (Guernic and Girard, 2010) for uncertain affine systems by separating the deterministic linear part (for which the reachable set can be approximated without error accumulation) and the uncertain input part. However, their approach uses zonotopes of increasing geometric complexity of which (measured by the number of generators) grows step after step, and if additional over-approximation or reduction (as in (Combastel, 2003; Guernic and Girard, 2010)) is used to keep a reasonable number of generators, the resulting error accumulation is hard to control. Our proposed interval approach can reduce error accumulation and provides a good compromise between complexity and accuracy. Furthermore, the approach using interval approximation can be extended to nonlinear systems as future work pointed out in the conclusion.

The remainder of this paper is structured as follows. Section 2 is devoted to the problem statement, where the considered reachability problem is formulated. A brief introduction on interval analysis is presented in Section 3. The main novelty of this work is stated in Section 4, where an interval predictor for the augmented closed loop system is introduced, and the convergence of the width of the computed state enclosures is examined. Section 5 illustrates the performance of the proposed reachability approach.

2. PROBLEM STATEMENT

Consider a discrete-time linear system subject to additive state disturbances and measurement noises that is described by

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{w}_k \\ \mathbf{y}_k \quad\;\; = \mathbf{C}\mathbf{x}_k + \mathbf{v}_k \end{cases} \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ stands for the state vector, $\mathbf{u}_k \in \mathbb{R}^{n_u}$ is the input vector, $\mathbf{w}_k \in \mathbb{R}^n$ stands for an unknown but bounded state disturbance vector, $\mathbf{y}_k \in \mathbb{R}^{n_y}$ is the output vector and $\mathbf{v}_k \in \mathbb{R}^{n_y}$ stands for the unknown but bounded measurement noise vector. The matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times n_u}$ and $\mathbf{C} \in \mathbb{R}^{n_y\times n}$ are known and constant. Throughout this paper, system (1) is assumed to be controllable and observable.

Thus, based on this standard assumption, estimated state-feedback controllers can be designed to drive system (1). These controllers are based on the Luenberger observer and have the following dynamical structure:

$$\begin{cases} \hat{\mathbf{x}}_{k+1} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k + \mathbf{L}\mathbf{y}_k \\ \mathbf{u}_k \quad\;\; = -\mathbf{K}\hat{\mathbf{x}}_k + \mathbf{G}\mathbf{r}_k \end{cases} \tag{2}$$

where $\hat{\mathbf{x}}_k \in \mathbb{R}^n$ is the estimated state vector and $\mathbf{r}_k \in \mathbb{R}^{n_y}$ stands for the desired setpoint. The gain matrices $\mathbf{L} \in \mathbb{R}^{n\times n_y}$, $\mathbf{K} \in \mathbb{R}^{n_u\times n}$ and $\mathbf{G} \in \mathbb{R}^{n_y\times n_y}$ stand respectively for the observer gain, the state-feedback gain and the feedforward gain. Notice, in the ideal case where system (1) is free from the state disturbances and measurement noises ($\mathbf{w}_k = 0$, $\mathbf{v}_k = 0$), the linear controller (2) succeeds to impose the desired performance (settling time, tracking error,...) to the controlled system.

Now, a legitimate question that arises is about the assessment of the robustness of the proposed controller against state disturbances and measurements noises. That is, does the controller keep its theoretical performance in the presence of state disturbances and measurement noises that could affect both system dynamics and state estimation algorithm?

Based on the reachability analysis, this issue could be tackled in the bounded error context. That is, the state disturbance and the measurement noises are considered unknown but bounded with known bounds.

Hence, relying on this assumption, a reliable decision about the expected performance of the linear controller (2) could be taken by the use of the reachability analysis. In this work, we introduce a set-membership algorithm that yields an outer approximation of the reachable set of the closed-loop system described by (1) and (2). First, we define by $\mathbf{e}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$ the estimation error. Then, under the control law (2), the dynamics in (1) can be rewritten as:

$$\mathbf{x}_{k+1} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}_k + \mathbf{B}\mathbf{K}\mathbf{e}_k + \mathbf{B}\mathbf{G}\mathbf{r}_k + \mathbf{w}_k \tag{3}$$

On the other hand, the dynamics of the estimation error is described as follows:

$$\mathbf{e}_{k+1} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}_k + (\mathbf{I}_n \ \ -\mathbf{L})\begin{pmatrix} \mathbf{w}_k \\ \mathbf{v}_k \end{pmatrix} \tag{4}$$

where $\mathbf{I}_n$ stands for an identity matrix of dimension n. Thus, from (3) and (4) the closed-loop system (1)-(2) is equivalent to the following coupled (augmented) system:

$$\begin{aligned} \begin{pmatrix} \mathbf{x}_{k+1} \\ \mathbf{e}_{k+1} \end{pmatrix} &= \begin{pmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0}_{n,n} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{pmatrix}\begin{pmatrix} \mathbf{x}_k \\ \mathbf{e}_k \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{n,n_y} \\ \mathbf{I}_n & -\mathbf{L} \end{pmatrix}\begin{pmatrix} \mathbf{w}_k \\ \mathbf{v}_k \end{pmatrix} + \begin{pmatrix} \mathbf{B}\mathbf{G} \\ \mathbf{0}_{n,n_y} \end{pmatrix}\mathbf{r}_k \end{aligned} \tag{5}$$

where by $\mathbf{0}_{l,s}$ one denotes a zero matrix of dimension $l \times s$.

Thereby, the reachable set of the closed-loop system can be defined as follows.

Definition 1. Let $\mathbf{x}_0 \in \mathcal{X}_0 \subset \mathbb{R}^n$ and $\mathbf{e}_0 \in \mathcal{E}_0 \subset \mathbb{R}^n$. The reachable set of the system (5) at the time instant k is the set $\Omega_k = \{(\mathbf{x}_k, \mathbf{e}_k) \mid \mathbf{e}_0 \in \mathcal{E}_0, \mathbf{x}_0 \in \mathcal{X}_0\}$ of all the solutions to (5) that are:

- generated from the initial sets $\mathcal{X}_0$ and $\mathcal{E}_0$,
- driven by the desired set-point input vector $\mathbf{r}_k$ and
- affected by the exogenous bounded signals $\mathbf{w}_k$, $\mathbf{v}_k$.

Moreover, the reachable set of the system (5) over a finite period $k \in \{k_0, \dots k_m\}$ is the trajectory set that can be defined by:

$$\Omega_{\{k_0,\dots,k_m\}} = \{\Omega_{k_0}, \dots, \Omega_{k_m}\} \tag{6}$$

In what follows, interval analysis (Alefeld and Mayer, 2000; Jaulin et al., 2001; Moore, 1966) will be used to compute an outer approximation of the trajectory tube defined in (6).

3. INTERVAL ANALYSIS

One way to represent and manipulate sets of real values is interval arithmetic. By definition, a real interval denoted by $[x] = [\underline{x}, \overline{x}]$ is a connected and closed subset of $\mathbb{R}$,

$$[x] = \{a \mid \underline{x} \leq a \leq \overline{x}, \quad \underline{x}, \overline{x} \in \mathbb{R}\}$$

where $\underline{x}$ and $\overline{x}$ stand respectively for its lower and upper endpoints. The set of all real intervals of $\mathbb{R}$ is denoted by $\mathbb{IR}$. Moreover, the basic arithmetic operations $\circ \in \{+, -, \times, \div\}$ on real numbers are extended to the interval case, over $\mathbb{IR}$, according to the following set-membership rule

$$\forall [x], [y] \in \mathbb{IR} \quad [x] \circ [y] = \{a \circ b \mid a \in [x],\ b \in [y]\} \tag{7}$$

More formally, the set-membership definition (7) can be detailed as follows for $\circ \in \{+, -\}$:

$$\begin{aligned} [x] + [y] &= [\underline{x} + \underline{y},\ \overline{x} + \overline{y}], \\ [x] - [y] &= [\underline{x} - \overline{y},\ \overline{x} - \underline{y}] \end{aligned} \tag{8}$$

On the other hand, the width of an interval is defined by $w([x]) = \overline{x} - \underline{x}$.

Note that, interval analysis are also extended to vector and matrix expressions in (Alefeld and Mayer, 2000; Moore, 1966). In this work, only addition and subtraction of two interval vectors of the same dimension and the multiplication between a real matrix and an interval vector of appropriate dimensions are needed. Thus, the arithmetic operation $\circ \in \{+, -\}$ between two interval vectors $[\mathbf{x}], [\mathbf{y}] \in \mathbb{IR}^n$ provides an interval vector defined as follows:

$$[\mathbf{z}] = [\mathbf{x}] \circ [\mathbf{y}] = \{\mathbf{a} \circ \mathbf{b} \mid \mathbf{a} \in [\mathbf{x}],\ \mathbf{b} \in [\mathbf{y}]\} \tag{9}$$

where the two first interval equations in (8) are applied element-wise, in both cases, to compute the lower $\underline{\mathbf{z}} \in \mathbb{R}^n$ and upper $\overline{\mathbf{z}} \in \mathbb{R}^n$ bounds of $[\mathbf{z}]$.

The parallelotope $\mathcal{P}$ defined by

$$\mathcal{P} = \{\mathbf{M}\mathbf{x} \mid \mathbf{M} \in \mathbb{R}^{m \times n},\ \mathbf{x} \in [\mathbf{x}] \subset \mathbb{IR}^n\} \tag{10}$$

can be outer approximated by the following interval vector

$$[\mathbf{p}] = \mathbf{M}[\mathbf{x}] \tag{11}$$

Notice, the upper and lower bounds of the box $[\mathbf{p}]$ can be computed directly by:

$$\underline{\mathbf{p}} = \mathbf{M}^+ \underline{\mathbf{x}} - \mathbf{M}^- \overline{\mathbf{x}} \text{ and } \overline{\mathbf{p}} = \mathbf{M}^+ \overline{\mathbf{x}} - \mathbf{M}^- \underline{\mathbf{x}} \tag{12}$$

where the element-wise non-negative matrices $\mathbf{M}^+$ and $\mathbf{M}^-$ are determined by

$$\mathbf{M}^+ = \max\{\mathbf{M}, \mathbf{0}_{m,n}\} \text{ and } \mathbf{M}^- = \mathbf{M}^+ - \mathbf{M} \tag{13}$$

In (13), the max operator is applied component-wise. Finally, the maximal diameter of an interval vector $[\mathbf{x}] \in \mathbb{IR}^n$ is given by,

$$W([\mathbf{x}]) = \max_{1 \leq i \leq n} w([x_i]) \tag{14}$$

4. MAIN RESULTS

Before introducing the main contribution of this work, let us state our assumptions.

Assumption 1. The initial state $\mathbf{x}_0$ of the system (1) belongs to a bounded box: $\mathbf{x}_0 \in [\mathbf{x}_0] = [\underline{\mathbf{x}}_0\ \overline{\mathbf{x}}_0]$, where the lower and upper bounds respectively $\underline{\mathbf{x}}_0$, $\overline{\mathbf{x}}_0$ are assumed to be known.

Remark 1. To reduce the impact of the estimation error at the transient regime, the initial state $\hat{\mathbf{x}}_0$ of the Luenberger observer in (2) has to be picked from the box $[\mathbf{x}_0]$.

Assumption 2. The state disturbances $\mathbf{w}_k$ and the measurement noises $\mathbf{v}_k$ are assumed to be unknown signals but bounded with known bounds:

$$\forall k \geq 0,\ \mathbf{w}_k \in [\underline{\mathbf{w}}\ \ \overline{\mathbf{w}}] \text{ and } \mathbf{v}_k \in [\underline{\mathbf{v}}\ \ \overline{\mathbf{v}}] \tag{15}$$

Assumption 3. The pairs $(\mathbf{A}, \mathbf{B})$ and $(\mathbf{A}, \mathbf{C})$ of system (1) are respectively controllable and observable. That is, there exist an observer gain and a state feedback gain with appropriate dimensions such that the matrices $\mathbf{A} - \mathbf{LC}$ and $\mathbf{A} - \mathbf{BK}$ are Schur stable.

For the sake of simplicity, let us introduce the following matrices and vectors before presenting the main contribution of this work:

$$\begin{gathered} \mathbf{N} = \begin{pmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0}_{n,n} & \mathbf{A} - \mathbf{LC} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{n,n_y} \\ \mathbf{I}_n & -\mathbf{L} \end{pmatrix} \\ \mathbf{H} = \begin{pmatrix} \mathbf{BG} \\ \mathbf{0}_{n,n_y} \end{pmatrix}, \quad \mathbf{g}_k = \begin{pmatrix} \mathbf{x}_k \\ \mathbf{e}_k \end{pmatrix} \end{gathered} \tag{16}$$

Notice, based on the definition of $\mathbf{g}_k$ in (16), in what follows the state vector $\mathbf{x}_k$ and the estimation error vector $\mathbf{e}_k$ will be re-denoted by ${}^1\mathbf{g}_k$ and ${}^2\mathbf{g}_k$, respectively. That is,

$$\mathbf{g}_k = \begin{pmatrix} {}^1\mathbf{g}_k \\ {}^2\mathbf{g}_k \end{pmatrix} \tag{17}$$

and

$$\mathbf{g}_{k+1} = \mathbf{N}\mathbf{g}_k + \mathbf{F}\mathbf{d}_k + \mathbf{G}\mathbf{r}_k \tag{18}$$

where $\mathbf{d}_k = (\mathbf{w}_k, \mathbf{v}_k)^T \in ([\mathbf{w}], [\mathbf{v}])^T = [\mathbf{d}]$.

Proposition 1. Let Assumptions 1, 2 and 3 be satisfied and define:

$$[\mathbf{f}_0] = \mathbf{F}[\mathbf{d}],\ [\mathbf{g}_0] = \begin{pmatrix} [\mathbf{x}_0] \\ [\mathbf{e}_0] \end{pmatrix} \text{ and } \mathbf{b}_0 = \mathbf{H}\mathbf{r}_0 \tag{19}$$

Then, there exist a state feedback gain $\mathbf{K}$ and an observer gain $\mathbf{L}$ such that, for all $k \geq 1$, the following interval predictor:

$$\begin{aligned} [\mathbf{g}_k] &= \mathbf{N}^k[\mathbf{g}_0] + [\mathbf{f}_{k-1}] + \mathbf{b}_{k-1} \\ [\mathbf{f}_k] &= \mathbf{N}^k[\mathbf{f}_0] + [\mathbf{f}_{k-1}] \\ \mathbf{b}_k &= \mathbf{N}\mathbf{b}_{k-1} + \mathbf{H}\mathbf{r}_k, \end{aligned} \tag{20}$$

yields a converging outer approximation of the reachable set of the closed-loop system (5). That is, $\forall\ \mathbf{x}_0 \in [\mathbf{x}_0]$ and $\forall\ \mathbf{e}_0 \in [\mathbf{e}_0]$:

$$\Omega_{\{0,\ldots,s\}} \subseteq \{[\mathbf{g}_0], \ldots, [\mathbf{g}_s]\} \tag{21}$$

and

$$\lim_{k \to \infty} W([\mathbf{g}_k]) = c \tag{22}$$

where c is a positive constant.

Proof. Note that (20) is obtained by iterating successively the interval extension of the dynamic equation (18). That is, (20) can be rewritten as follows,

$$[\mathbf{g}_k] = \mathbf{N}^k[\mathbf{g}_0] + \sum_{i=0}^{k-1}(\mathbf{N}^i[\mathbf{f}_0]) + \sum_{i=0}^{k-1}\mathbf{N}^{k-1-i}\mathbf{H}\mathbf{r}_i \tag{23}$$

On the other hand, based on Assumption 2 and the definition of $[\mathbf{f}_0]$, we have $\forall k \geq 1$:

$$\sum_{i=0}^{k-1}\mathbf{N}^{k-1-i}\mathbf{F}\begin{pmatrix} \mathbf{w}_k \\ \mathbf{v}_k \end{pmatrix} \in \sum_{i=0}^{k-1}(\mathbf{N}^i[\mathbf{f}_0]) \tag{24}$$

Therefore, we can affirm that the exact solution of the augmented system (5) given by,

$$\mathbf{g}_k = \mathbf{N}^k \mathbf{g}_0 + \sum_{i=0}^{k-1} \mathbf{N}^{k-1-i} \mathbf{F} \begin{pmatrix} \mathbf{w}_i \\ \mathbf{v}_i \end{pmatrix} + \sum_{i=0}^{k-1} \mathbf{N}^{k-1-i} \mathbf{H} \mathbf{r}_i \quad (25)$$

at each time instant k, is included inside the box $[\mathbf{g}_k]$. Moreover, the first part of the box $[\mathbf{g}_k]$ represents an outer approximation of the reachable set of the augmented system. That is, $\forall k \geq 1 \; \mathbf{x}_k \in [^1\mathbf{g}_k]$ and

$$\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_s\} \subseteq \{[^1\mathbf{g}_0], [^1\mathbf{g}_1], \ldots, [^1\mathbf{g}_s]\} \quad (26)$$

This completes the framing proof of this proposition. Now, let us consider the convergence proof. Since system (1) is assumed controllable and observable, there exist a state feedback gain $\mathbf{K}$ and an observer gain $\mathbf{L}$ such that the matrix $\mathbf{N}$ is Schur stable. This implies that,

$$\lim_{k \to \infty} \mathbf{N}^k = 0 \quad (27)$$

Therefore, we can claim that the second interval equation in (20) has a fixed point. That means,

$$\lim_{k \to \infty} [\mathbf{f}_k] = \lim_{k \to \infty} [\mathbf{f}_{k-1}] = [\mathbf{f}_*] \quad (28)$$

Thus, according to the first equation in (20), we can affirm that

$$\begin{aligned} \lim_{k \to \infty} [\mathbf{g}_k] &= \lim_{k \to \infty} W\big(\mathbf{N}^k [\mathbf{g}_0] + [\mathbf{f}_{k-1}]\big) \\ &= W([\mathbf{f}_*]) = c \end{aligned} \quad (29)$$

This ends the convergence proof. □

Hereafter some useful remarks about Proposition 1.

Remark 2.

- Note that the convergence of the interval predictor (20) is ensured by the stability of the augmented matrix $\mathbf{N}$, while iterative interval methods required the stability of the element-wise non-negative matrix $|\mathbf{N}|$ to provide a non-diverging outer approximation of the reachable set of the augmented system (5). This highlight the efficiency of the proposed method against the *wrapping effect*.
- The contribution of this work is the new numerical scheme (20) that allows avoiding the wrapping effect. Moreover, to obtain tighter outer approximation of the reachable set of the augmented system (5), we suggest to use zonotopic sets in (20) instead of interval vectors.
- The proposed reachability method can be extended at a moderate effort to other types of controllers described by state space models.
- Algorithm (20) can be also applied to detectable and stabilizable systems.

5. ILLUSTRATIVE EXAMPLE

Consider the following discrete-time uncertain system,

$$\begin{aligned} \mathbf{x}_{k+1} &= \begin{pmatrix} 1 & 0.63 \\ 0 & 0.37 \end{pmatrix} \mathbf{x}_k + \begin{pmatrix} 0.37 \\ 0.63 \end{pmatrix} \mathbf{u}_k + \mathbf{w}_k \\ \mathbf{y}_k &= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}_k + \mathbf{v}_k \end{aligned} \quad (30)$$

where, the state disturbances $\mathbf{w}_k$ and the measurement noise $\mathbf{v}_k$ are assumed unknown but bounded with well known bounds:

$$\begin{aligned} (-0.02,\ -0.02)^T &\leq \mathbf{w}_k \leq (0.02,\ 0.02)^T \\ -0.01 &\leq \mathbf{v}_k \leq 0.01 \end{aligned} \quad (31)$$

In addition, the initial state of (30) is poorly-known but belongs into a bounded box:

$$\mathbf{x}_0 \in [0.8 \ 1.5] \times [1.2 \ 3] \quad (32)$$

Since this system is observable and controllable, an estimated state-feedback controller can be designed to drive its output to a desired set-point trajectory,

$$\mathbf{r}_k = 1.5 \sin(0.2k) \quad (33)$$

To do that, first the uncertainties of the system are neglected and LQR (Linear-Quadratic Regulator) method is applied to design both state feedback and observer gains:

$$\begin{aligned} \mathbf{K} &= (0.8743 \ \ 0.7237) \\ \mathbf{L} &= (1.0841 \ \ 0.1278)^T \end{aligned} \quad (34)$$

and then the feed-forward gain $\mathbf{G}$ is obtained by

$$\mathbf{G} = (\mathbf{C}(\mathbf{I}_2 - \mathbf{A} + \mathbf{B}\mathbf{K})^{-1}\mathbf{B})^{-1} = 0.8743 \quad (35)$$

The considered initial state for the Luenberger observer is $\hat{\mathbf{x}}_0 = (0,\ 0)^T$. Thus, the initial box of the estimation error is:

$$[\mathbf{e}_0] = [\mathbf{x}_0] - \hat{\mathbf{x}}_0 = [\mathbf{x}_0] \quad (36)$$

Now, it is of great interest to be able to evaluate a priori the impact of the neglected uncertainties on the behavior of the closed-loop system. To do that, the result of Proposition 1 is useful. In fact, based on the outer approximation of the reachable set of the closed loop-system all possible behaviors of the system can be observed and analyzed a priori.

For this example, the design matrices $\mathbf{N}$, $\mathbf{H}$ and $\mathbf{F}$ of the interval predictor (20) are:

$$\begin{aligned} \mathbf{N} &= \begin{pmatrix} 0.6765 & 0.3622 & 0.3235 & 0.2678 \\ -0.5508 & -0.0859 & 0.5508 & 0.4559 \\ 0 & 0 & -0.0841 & 0.63 \\ 0 & 0 & -0.1278 & 0.37 \end{pmatrix}, \\ \mathbf{F} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1.0841 \\ 0 & 1 & -0.1278 \end{pmatrix}, \ \mathbf{H} = \begin{pmatrix} 0.3235 \\ 0.5508 \\ 0 \\ 0 \end{pmatrix} \end{aligned} \quad (37)$$

and the initial box $[\mathbf{f}_0]$ is given by the following interval Cartesian product:

$$\begin{aligned} [\mathbf{f}_0] = & [-0.02 \ -0.02] \times [-0.02 \ -0.02] \ldots \\ & \times [-0.0308 \ \ 0.0308] \times [-0.0213 \ \ 0.0213] \end{aligned} \quad (38)$$

5.1 Simulation results

In this subsection we have applied the interval predictor (20) to compute an outer approximation of the reachable set of the uncertain system (30) driven by the proposed estimated state feedback controller (2),(34),(35).

Figure 1 shows the time evolution of both state variables $x_1(k)$ and $x_2(k)$. The solid lines in the top pictures stand for the outer enclosure of each state variable while the dashed lines represent the actual state variables of the system generated from the initial point $\mathbf{x}_0 = (1,\ 2)^T$. On the other hand, the bottom pictures show an enclosure of

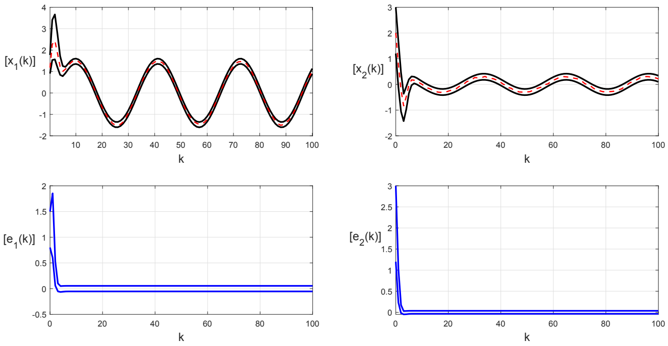

Fig. 1. Top figures show guaranteed enclosures for each state variables x_1 and x_2. Bottom figures represent all possible estimation errors considered in the augmented system.

the reachable set of the estimation error generated by the augmented system.

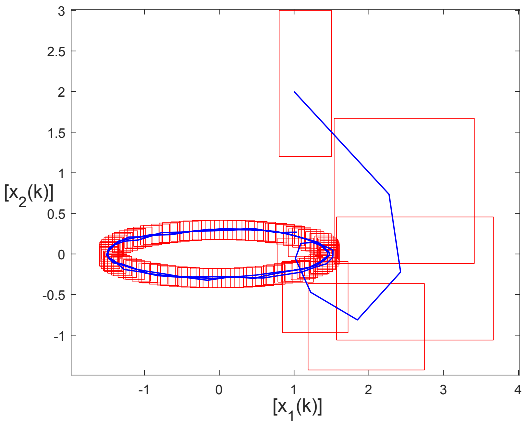

Fig. 2. The outer approximation of the reachable set of the closed-loop system plotted in the state plan (x_1, x_2).

Figure 2 shows the same outer approximation of the reachable set of the controlled system (30) in the plan phase (x_1, x_2). The rectangles represent the boxes $[\mathbf{x}_k]$ of the state vector $\mathbf{x}_k$ at each time instant k and the solid line stands for the actual state trajectory of the controlled system.

5.2 Comparison with other methods

To show the interest of the proposed algorithm (20), the results of four reachability methods are considered in this subsection. The efficiency of each method is measured by the width of its computed outer approximation of the reachable set of the augmented system. The first method is a direct application of algorithm (20) (with interval computation). The second one is a zonotopic version of this algorithm, where zonotope sets are used instead of interval vectors. In this case the maximal number of the segments of each zonotope is $d = 5$. The third and the fourth methods are respectively iterative interval and zonotopic methods applied directly to the dynamics of the augmented system. Note that, for the zonotopic iterative method the maximal number of the segments of each zonotope is $d = 5$.
As shown in Figure 3, algorithm (20) provides more accurate outer approximations than that computed by the

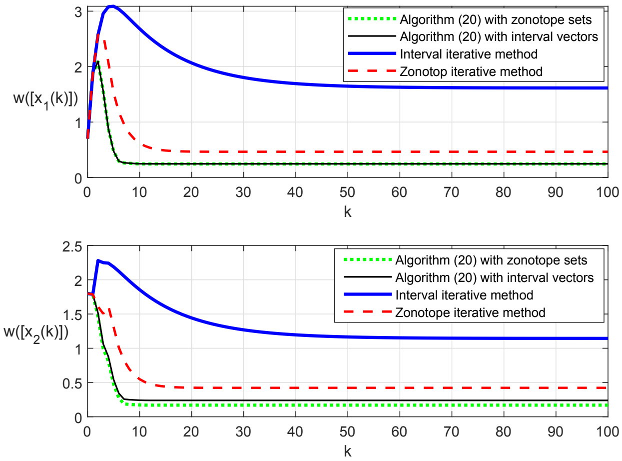

Fig. 3. The width of each component of the outer approximations of the reachable set computed by different methods.

iterative methods. Indeed, for this example, the smallest width of the computed outer approximations is given by the zonotopic version of algorithm (20). Note that, with a high maximal number of segments $d = 90$, the zonotopic iterative method provides similar results than that obtained by the zonotopic version of algorithm (20). However, using a high value of d leads to a huge oneline computational effort.

6. CONCLUSION

In this paper, an interval predictor has been proposed to compute tight outer approximations of the reachable set of a class of uncertain discrete-time linear systems driven by linear estimated state feedback controllers. The considered closed loop systems are rewritten as augmented systems that encompass the dynamics of the estimation error. This result is mainly based on a recent non-conservative interval scheme developed to reconstruct all the possible behaviors of a class of uncertain dynamical systems. Moreover, under the standard observability and controllability assumptions of linear systems, the convergence of the size of the computed state trajectory tubes is proven.

In forthcoming work, we will investigate two directions. One is combining this interval predictor with the consistency techniques in order to certificate numerically the performance of linear controllers. The other is to extend our method to some class of nonlinear systems.

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