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Barrier-Lyapunov Function based Dynamic Surface Control of Quad-Rotorcraft

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Abstract:
A novel framework is proposed in this paper for control of a quad-rotorcraft where hierarchical design is constructed via barrier Lyapunov function (BLF) combined with dynamic surface control (DSC). DSC solves the requirement of higher order differentiability of reference pose and avoiding the complexity that arises due to the "explosion of terms" coming out from repeated derivatives of reference attitude and desired thrust vector. BLF satisfies attitude constraint in real-time and thereby ensures non-singularity of velocity transformation leading to feasible control design. Stability analysis shows that all the signals in the closed-loop system are uniformly ultimately bounded and tracking error converges asymptotically. The performance of the BLF-based DSC is illustrated with a suitable example.

Keywords: Non-linear control systems, mobile and flying robots, autonomous systems, guidance, navigation and control.

1. INTRODUCTION

The area of nonlinear hierarchical control of a quad-rotorcraft has received a great deal of attention in the literature. Classical works on hierarchical design Abdessameud and Tayebi (2010); Roberts and Tayebi (2011); Falconí et al. (2013); Roza and Maggiore (2014), stabilize the vehicle position using attitude and thrust as control variables. In Dasgupta (2018); Dasgupta et al. (2019a,c), the desired attitude and thrust are extracted from virtual force control input of position dynamics by exploiting hierarchical structure of a quad-rotorcraft to achieve the tracking objective. Typically hierarchical control procedure requires position tracking controller to be designed first using the thrust vector as a virtual control input. Thereafter, the desired thrust or the thrust direction is converted into a reference attitude which is used as a desired command for attitude tracking. It is similar to classical backstepping design for strict feedback structure except that there is an additional control input available at the position level. It is well understood that nonlinear hierarchical control is not an exactly backstepping design Dasgupta (2019), however it suffers from the same problem of "explosion of terms" associated with backstepping approach. The control design procedure obtains rate of change of reference attitude Dasgupta (2018); Dasgupta et al. (2019a,c) or angular velocity Roberts and Tayebi (2011); Roza and Maggiore (2014) by differentiating the desired attitude. The derivatives are computed by analytical differentiation Dasgupta (2018); Dasgupta et al. (2019a,c) of the corresponding virtual control inputs. However obtaining such successive derivatives of reference trajectories becomes increasingly difficult from simple to complex flight maneuvers. Moreover the computation is highly sensitive to noise if any modeling uncertainty involved in the dynamics. In case of fast moving quad-rotorcraft, the calculation becomes infeasible for real-time high-order polynomial trajectory planning Mellinger and Kumar (2011); Richter et al. (2016) in 3D slalom courses.

The aim of this work is to resolve the complexity problem of hierarchical control design of a quad-rotorcraft. Motivated by the work Swaroop et al. (1997, 2000); Pan and Yu (2015) of dynamic surface control (DSC) developed for a class of strict-feedback nonlinear systems, an extension to hierarchical control is introduced and applied on an Euler-Lagrange (E-L) vehicle dynamics. The primary design technique is to pass the reference attitude through a second-order low-pass filter so that repeated derivatives of desired attitude and desired thrust vector can be avoided. It not only prevents the explosion of complexity but also relaxes the smoothness requirement of desired linear velocities and vehicle orientation. Unlike previous works Swaroop et al. (1997, 2000); Dasgupta et al. (2019a,c), the proposed control scheme constructs DSC design combined with a symmetric barrier Lyapunov function (BLF) where latter guarantees singularity-free attitude tracking. The stability analysis of the overall system proves that all the signals in the closed-loop system are uniformly ultimately bounded to a neighbourhood of the origin and the tracking error converges asymptotically while ensuring that the attitude constraint is not violated.

2. QUAD-ROTORCRAFT MODEL

The standard E-L dynamics Dasgupta (2018); Dasgupta et al. (2019a,b,c) for a quad-rotorcraft is given by

$$M_q\ddot{\zeta} + \bar{G} = R(\eta)^T e_z T = T_v \quad (1)$$

$$M_\eta(\eta)\ddot{\eta} + V(\eta, \dot{\eta}) = J(\eta)^T \tau \quad (2)$$

Two second order differential equations represent position (1) and rotational dynamics (2) of the vehicle. According to Fig.1, $\zeta = [x\ y\ z]^T \in \mathbb{R}^3$ is linear and $\eta = [\phi\ \theta\ \psi]^T \in \mathbb{R}^3$ is angular position (roll-pitch-yaw) vector of the origin of body-fixed frame $\{F_B\}$ w.r.t inertial frame $\{F_I\}$. The matrix $R(\eta) \in \mathbb{R}^{3\times3}$ denotes rotation in $\{ZYX\}$ Euler sequence between $\{F_B\}$ and $\{F_I\}$, $J(\eta) \in \mathbb{R}^{3\times3}$ is the velocity transformation matrix, $M_q = m_q I \in \mathbb{R}^{3\times3}$ represents mass-matrix (m_q = mass of the quad-rotorcraft, $I \in \mathbb{R}^{3\times3}$ is an identity matrix), $M_\eta(\eta) \in \mathbb{R}^{3\times3}$ is the inertia tensor, $\bar{G} = [0\ 0\ m_q g]^T \in \mathbb{R}^3$ denotes the reduced gravity vector, and $V_\eta(\eta, \dot{\eta}) \in \mathbb{R}^{3\times3}$ is the centripetal-Coriolis matrix. The

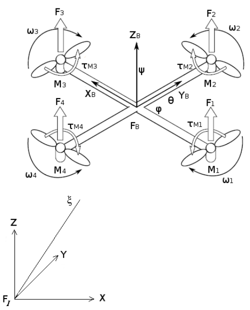


Fig. 1. Quad-Rotorcraft

total thrust $T \in \mathbb{R}$ applied on the vehicle, is produced by four motors where $T_v = \left[T_{v_x}\ T_{v_y}\ T_{v_z}\right]^T \in \mathbb{R}^3$ and $\tau = [\tau_\phi\ \tau_\theta\ \tau_\psi]^T \in \mathbb{R}^3$ define respective force and torque vectors w.r.t $\{F_B\}$.

The set of non-negative-real numbers and the L2-norm of a vector $v \in \mathbb{R}^3$ are given by $\mathbb{R}_+$ and $\|.\|$ respectively. Some fundamental properties of the quad-rotorcraft E-L dynamics, used in the subsequent development, are stated as follows

Property 1. The mass matrix M_q is symmetric, positive definite and lower and upper bounded by

$$\zeta_{M_{q_1}} \leq M_q \leq \zeta_{M_{q_2}} \quad (3)$$

The reduced gravity vector $\bar{G}$ satisfies

$$\left\|\bar{G}\right\| \leq \zeta_{\bar{G}} \quad (4)$$

where $\{\zeta_{M_{q_1}}, \zeta_{M_{q_2}}, \zeta_{\bar{G}}\} \in \mathbb{R}_+$

Property 2. The inertia matrix $M_\eta(\eta)$ and the centripetal-Coriolis matrix $V_\eta(\eta, \dot{\eta})$ together form a skew-symmetric matrix and satisfy the following relationship.

$$v^T\left[\dot{M}_\eta(\eta) - 2V_\eta(\eta, \dot{\eta})\right]v = 0\ \forall v \in \mathbb{R}^3 \quad (5)$$

3. PROBLEM STATEMENT

3.1 Control Objective

The objective is to design a quad-rotorcraft controller, to track a desired position $\zeta_d(t) = [x_d(t)\ y_d(t)\ z_d(t)]^T \in \mathbb{R}^3$ and orientation $\psi_d(t) \in \mathbb{R}$ of the vehicle.

3.2 Motivation

Recently proposed nonlinear hierarchical quad-rotorcraft control technique Das et al. (2009); Yildiz et al. (2015); Dasgupta (2018, 2019) computes commanded roll and pitch trajectories $\{\phi_d(t), \theta_d(t)\} \in \mathbb{R}$ from the designed thrust input $T_d(t) \in \mathbb{R}^3$ to track the desired position. It involves repeated differentiation of desired roll and pitch to design a computed torque control input for attitude tracking. The control design procedure developed in authors' previous works, Dasgupta et al. (2019b) require to derive analytical expressions for command derivatives. They are

$$\dot{\phi}_d(t) = {(\dot{y}_1 x_1 - \dot{x}_1 y_1)}/{(x_1^2 + y_1^2)} = {(\dot{y}_1 x_1 - \dot{x}_1 y_1)}/{T^2} \quad (6)$$

$$\ddot{\phi}_d(t) = {(\ddot{y}_1 x_1 - \ddot{x}_1 y_1)}/{T^2} - 2{(\dot{y}_1 x_1 - \dot{x}_1 y_1)\dot{T}}/{T^3} \quad (7)$$

$$\dot{\theta}_d(t) = {(\dot{y}_2 x_2 - \dot{x}_2 y_2)}/{(x_2^2 + y_2^2)} = {(\dot{y}_2 x_2 - \dot{x}_2 y_2)}/{x_1^2} \quad (8)$$

$$\ddot{\theta}_d(t) = {(\ddot{y}_2 x_2 - \ddot{x}_2 y_2)}/{x_1^2} - 2{(\dot{y}_2 x_2 - \dot{x}_2 y_2)\dot{x}_1}/{x_1^3} \quad (9)$$

where the auxiliary variables $\{x_1(t), x_2(t), y_1(t), y_2(t)\} \in \mathbb{R}$ are the functions of $\psi_d(t)$ and scalar components of $T_d(t)$. It can be inferred from (6)-(9) that $\psi_d(t)$ must be a C^2 function [1]. Since the design of $T_d(t)$ Dasgupta (2018); Dasgupta et al. (2019a,b,c) contains $\ddot{\zeta}_d(t)$, it imposes higher order requirement on smoothness of $\zeta_d(t)$ as compared with $\psi_d(t)$, i.e. it must be a C^4 function. However for a real-time trajectory-planner, which continuously recomputes a time-dependent reference trajectories, satisfying such stringent requirement of differentiability is a major challenge.

In this paper, DSC is developed for an E-L dynamics of a quad-rotorcraft, where the reference attitude $\eta_d(t)$ = $[\phi_d(t)\ \theta_d(t)\ \psi_d(t)]^T \in \mathbb{R}^3$ is passed through a second-order low-pass filter so that the analytical computation of attitude derivatives can be avoided by removing the term involves $\ddot{\eta}_d(t)$, in torque control design. However the thrust design follows authors' earlier works Dasgupta (2019); Dasgupta et al. (2019a,b,c) of nonlinear hierarchical control where saturated position control ensures singularity free computation of $\theta_d(t)$ and tracks $\zeta_d(t)$ via constrained attitude control. The design is based on a symmetric BLF to ensure that the pitch constraint $\theta(t) < {\pi}/{2}$ is not violated.

Moreover assumption made in authors' earlier works Dasgupta (2019); Dasgupta et al. (2019a,b) on continuous differentiability of reference trajectories are relaxed and is stated below.

Assumption 1. The desired orientation and position trajectories and their first and second order time derivatives $\{\zeta_d(t), \dot{\zeta}_d(t), \ddot{\zeta}_d(t), \psi_d(t)\} \in \mathcal{L}_\infty$ where the symbol $\mathcal{L}_\infty$ denotes the space of bounded signals.

4. DESIGN OF CONTROL HIERARCHY

4.1 Reference Thrust Control Design

The position tracking control assumes m_q is known apriory. Like earlier design Dasgupta et al. (2019a,b) it uses a hyperbolic tangent function of position error which is defined subsequently. For any given vector $v \in \mathbb{R}^3$, matrix and vector-valued functions are defined as

[1] In general C^k function has k continuous derivatives

$tanh(v) \triangleq [\tanh(v_1)\ \tanh(v_2)\ \tanh(v_3)]^T \in \mathbb{R}^3$,
$Cosh^2(v) \triangleq diag\{\cosh^2(v_1), \cosh^2(v_2), \cosh^2(v_3)\} \in \mathbb{R}^{3\times3}$,
$Sech^2(v) \triangleq diag\{sech^2(v_1), sech^2(v_2), sech^2(v_3)\} \in \mathbb{R}^{3\times3}$.
The procedure defines position error $e_\zeta(t) \triangleq \zeta_d(t) - \zeta(t) \in \mathbb{R}^3$, filtered tracking error $r_\zeta(t) \in \mathbb{R}^3$

$$r_\zeta = \dot{e}_\zeta + \alpha_\zeta tanh(e_\zeta) + tanh(e_f) \tag{10}$$

and auxiliary filter $e_f(t) \in \mathbb{R}^3$

$$\dot{e}_{f_\zeta} = Cosh^2(e_{f_\zeta})(-k_\zeta r_\zeta + tanh(e_\zeta) - \gamma_\zeta tanh(e_f)) \tag{11}$$

respectively, where $e_f \triangleq \left[e_{f_x}\ e_{f_y}\ e_{f_z}\right] \in \mathbb{R}^3$ with $e_f(0) = 0$, and $\{\alpha_\zeta, \gamma_\zeta\} \in \mathbb{R}_+$ are filter gains.
The time derivatives of position error $e_\zeta(t)$, filtered tracking error (10) followed by substituting (11) and (1) forms the open-loop position error dynamics. Since the quad-rotorcraft system is under-actuated, the position control is achieved through stabilized attitude. Due to the presence of nonlinear coupling between position and rotational dynamics, the actual control force T_v in (1) can be realized with certain attitude error. The errors are rigorously characterized in the proposed controller and accounted for in the thrust design procedure. Consider a desired propulsive force $T_d = [T_{x_d}\ T_{y_d}\ T_{z_d}]^T \in \mathbb{R}^3$ which is free from attitude error, can be designed to stabilize the position dynamics. Performing $\pm T_d$ on open-loop error dynamics Dasgupta (2019); Dasgupta et al. (2019a,c) it yields

$$T_d = M_q\ddot{\zeta}_d + 2M_q tanh(e_\zeta) - k_\zeta M_q tanh(e_f) + \bar{G} \tag{12}$$

where $k_\zeta \in \mathbb{R}_+$ is a position control gain. After defining the error $T_e \triangleq T_v - T_d \in \mathbb{R}^3$ between the actual and desired force, the closed-loop error dynamics using (12) is derived

$$\begin{aligned} M_q\dot{r}_\zeta = &- M_q k_\zeta r_\zeta - M_q tanh(e_\zeta) + k_\zeta M_q tanh(e_f) \\ &+ \chi - T_e \end{aligned} \tag{13}$$

where $\chi = \alpha_\zeta M_q Sech^2(e_\zeta)\dot{e}_\zeta - \gamma_\zeta M_q tanh(e_f)$, contains all disturbance-like terms robustly dominated by high gain feedback. Replace T_v by T_d in R.H.S of (1) and compute the desired roll and pitch angles as

$$\phi_d = \arctan\left(\left(T_{x_d}s_{\psi_d} - T_{y_d}c_{\psi_d}\right)/\sqrt{(T_{x_d}c_{\psi_d} + T_{y_d}s_{\psi_d})^2 + T_{z_d}^2}\right) \tag{14}$$

$$\theta_d = \arctan\left(\left(T_{x_d}c_{\psi_d} + T_{y_d}s_{\psi_d}\right)/T_{z_d}\right) \tag{15}$$

for any given yaw angle $\psi_d(t)$. Here $\{s_{(.)}, c_{(.)}\}$ denote scalar sin(.) and cos(.) functions. Further, according to the proposition [2] stated in earlier works Dasgupta (2019); Dasgupta et al. (2019a,c), stricter constraints should be imposed on bounded desired acceleration $|\ddot{z}_d|$ and position control gain k_ζ

$$|\ddot{z}_d| \leq \epsilon < g \tag{16}$$

$$k_\zeta < g - \epsilon \quad \text{for any } \epsilon \in \mathbb{R}_+ \tag{17}$$

to ensure $T_{z_d} \neq 0$ and $\theta_d(t) \in (-\pi/2, \pi/2)$ for all time.

4.2 Attitude Control Design

The objective is to develop a pitch constrained attitude tracking controller for the quad-rotorcraft dynamics. Despite the simple form of the computed torque control laws Dasgupta (2018, 2019); Dasgupta et al. (2019a,b,c), the design suffers from explosion of complexity while computing successive differentiation of desired attitude $\eta_d(t)$ analytically.

Dynamic Surface Control: The basic idea of dynamic surface technique is to pass $\eta_d(t)$ through a low-pass filter so that it's derivatives can be avoided. Filtered output, defined as $\eta_c(t) = [\phi_c(t)\ \theta_c(t)\ \psi_c(t)]^T \in \mathbb{R}^3$ is obtained using a second order low pass filter given by

$$\tau_1\ddot{\eta}_c + \tau_2\dot{\eta}_c = \alpha_\tau(\eta_d - \eta_c) \qquad \dot{\eta}_c(0) = \eta_c(0) = \eta_d(0) \tag{18}$$

where $\{\tau_1, \tau_2, \alpha_\tau\} \in \mathbb{R}_+$ are filter time constants and filter gain respectively. Moreover to aid control synthesis and analysis, filter time constants are related by

$$\tau_2 = \alpha_\tau\tau_1 + 1 \tag{19}$$

The filter error is defined as

$$\tilde{\eta}(t) \triangleq \eta_d - \eta_c = \left[\tilde{\phi}\ \tilde{\theta}\ \tilde{\psi}\right]^T \in \mathbb{R}^3 \tag{20}$$

where $\tilde{\phi} \triangleq \phi_d$ - ϕ_c, $\tilde{\theta} \triangleq \theta_d$ - θ_c and $\tilde{\psi} \triangleq \psi_d$ - ψ_c. The order of filter dynamics (18) is subsequently reduced by defining a filter error-like variable $r_\tau(t) \in \mathbb{R}^3$

$$r_\tau \triangleq \dot{\tilde{\eta}} + \alpha_\tau\tilde{\eta} \tag{21}$$

Taking the time derivative of (21), pre-multiplying the resulting expression by τ_1, substituting the expression of $\tau_1\ddot{\eta}_c$ from (18) and second time derivative of $\tilde{\eta}(t)$, adding and subtracting the term $\tau_2\dot{\eta}_d$ and finally using (19), the filter error system for $r_\tau(t)$ is formed as

$$\tau_1\dot{r}_\tau = -r_\tau + \tau_1\ddot{\eta}_d + \tau_2\dot{\eta}_d \tag{22}$$

In order to synthesize dynamic surface control, the attitude error is given by

$$e_\eta(t) = \eta_c - \eta = [e_\phi\ e_\theta\ e_\psi]^T \in \mathbb{R}^3 \tag{23}$$

and a filtered tracking error-like variable $r_\eta(t) \in \mathbb{R}^3$ is

$$r_\eta = \dot{e}_\eta + A_\eta e_\eta \tag{24}$$

where $e_\phi{=}\phi_c - \phi$, $e_\theta{=}\theta_c - \theta$, $e_\psi{=}\psi_c - \psi$. The filter gain $A_\eta = \mathrm{diag}\{\alpha_{\eta_{11}}, \alpha_{\eta_{22}}, \alpha_{\eta_{33}}\} \in \mathbb{R}^{3\times3}$, $\alpha_{\eta_{ii}} \in \mathbb{R}_+$ i=1(1)3 [3], is a positive definite constant matrix. Taking the time derivative of (24), pre-multiplying the resulting expression by $M_\eta(\eta)$, substituting (2) and second time derivative of $e_\eta(t)$, the open-loop attitude error system for $r_\eta(t)$ is derived as

$$M_\eta(\eta)\dot{r}_\eta = M_\eta(\eta)\ddot{\eta}_c + M_\eta(\eta)A_\eta\dot{e}_\eta + V_\eta(\eta,\dot{\eta})\dot{\eta} - J(\eta)^T\tau \tag{25}$$

A computed torque control $\tau(t)$ is designed based on the open-loop attitude error system (25) and is given by

$$\begin{aligned} \tau = J(\eta)^{-T}(&M_\eta(\eta)\ddot{\eta}_c + M_\eta(\eta)A_\eta\dot{e}_\eta + V_\eta(\eta,\dot{\eta})\dot{\eta} \\ &+ V_\eta(\eta,\dot{\eta})r_\eta + K_\eta r_\eta + e_\eta + \bar{\eta}) \end{aligned} \tag{26}$$

assuming attitude dynamics is known a-priory. Here $K_\eta \in \mathbb{R}^{3\times3}$ is the attitude control gain represented by a positive definite constant diagonal matrix and the term $\bar{\eta}$ is subsequently defined.

Remark 1. DSC uses second order low pass filter where desired attitude is not differentiated in order to design computed torque control input. It not only avoids the complexity that arises due to the explosion of terms coming out of repeated differentiation of reference attitude $\eta_d(t)$ and desired thrust vector T_d but also relaxes the stringent requirement on higher order differentiability of reference position $\zeta_d(t)$ and orientation $\psi_d(t)$ of the vehicle.

Pitch Constrained Design: In order to satisfy the pitch constraint, an additional term

$$\bar{\eta} \triangleq \left[e_\phi\ \ e_\theta/(k_\theta^2 - e_\theta^2)\ \ e_\psi\right]^T \in \mathbb{R}^3 \tag{27}$$

[2] The detailed proof is provided in Dasgupta (2019); Dasgupta et al. (2019a,c)

[3] The notation "i=a(m)b" implies that "i=a, a+m, a+2m, ..., b".

is introduced in the control design, where $k_\theta \in \mathbb{R}$. It exploits the key technicalities underlying the use of BLF for constraint satisfaction. Using Property 1, Assumption 1, the upper bound of T_d from (12) can be computed as

$$\|T_d\| \le c \tag{28}$$

where $c \in \mathbb{R}_+$. Since T_d is a real vector-valued function and $\{T_{x_d}, T_{y_d}, T_{z_d}\}$ are its scalar components, their supremum and infimum can be defined as

$$S_x \triangleq \sup_{\zeta(t)\in S} \{T_{x_d}\} < c \tag{29}$$

$$S_y \triangleq \sup_{\zeta(t)\in S} \{T_{y_d}\} < c \tag{30}$$

$$L_z \triangleq \inf_{\zeta(t)\in S} \{T_{z_d}\} > 0 \tag{31}$$

within a compact set $S \in \mathbb{R}^3$ of interest [4]. Further, if the constraints on desired vertical acceleration (16) and position control gain (17) are satisfied, it proves (31) to be true $\forall t \ge 0$. Using (29)-(31), $\exists\ \rho \in \mathbb{R}_+$ valid in the same set of interest by which $\theta_d(t)$ can be upper-bounded as

$$|\theta_d| \le \arctan({}^{(S_x+S_y)}/_{L_z}) = \rho \tag{32}$$

Using (18) it can be shown that $\theta_c(t)$ is also bounded from above by the same bound [5] i.e.

$$|\theta_c| \le \rho \tag{33}$$

The constant parameter k_θ is chosen such that

$$k_\theta = {}^{\pi}/_{2} - \rho \tag{34}$$

A constraint on initial pitch error $|e_\theta(0)| < k_\theta$ is imposed so that $\bar{\eta}(0)$ will not be unbounded. Post design analysis establishes that if above initial condition constraint is satisfied, both $|e_\theta(t)| < k_\theta$ and $|\theta(t)| < {}^{\pi}/_{2}$ will be satisfied for all time.
Applying (26) on (25), yields closed-loop tracking error dynamics

$$M_\eta(\eta)\dot{r}_\eta + V_\eta(\eta,\dot{\eta})r_\eta + K_\eta r_\eta + e_\eta + \bar{\eta} = 0 \tag{35}$$

4.3 Thrust Design

The attitude controller is designed to track the reference attitude angles. It doesn't have infinite bandwidth and can't track the desired attitude instantly due to nonlinear coupling between position and rotational dynamics. Therefore T_v is realized with certain attitude errors. Using (20) and (23), vehicle attitude can be re-written as

$$\eta(t) = \eta_d(t) - e_\eta(t) - \tilde{\eta}(t) \tag{36}$$

Resolve (1) into three scalar equations using $R(\eta)$, substitute (36) for $\eta(t)$, yields the following expressions for T_{v_x}, T_{v_y} and T_{v_z}

$$\begin{aligned} T_{v_x} &= \big(c_{(\phi_d-e_\phi-\tilde{\phi})}s_{(\theta_d-e_\theta-\tilde{\theta})}c_{(\psi_d-e_\psi-\tilde{\psi})} \\ &\quad + s_{(\phi_d-e_\phi-\tilde{\phi})}s_{(\psi_d-e_\psi-\tilde{\psi})}\big)T \end{aligned} \tag{37}$$

$$\begin{aligned} T_{v_y} &= \big(c_{(\phi_d-e_\phi-\tilde{\phi})}s_{(\theta_d-e_\theta-\tilde{\theta})}s_{(\psi_d-e_\psi-\tilde{\psi})} \\ &\quad - s_{(\phi_d-e_\phi-\tilde{\phi})}c_{(\psi_d-e_\psi\tilde{\psi})}\big)T \end{aligned} \tag{38}$$

$$T_{v_z} = c_{(\phi_d-e_\phi-\tilde{\phi})}c_{(\theta_d-e_\theta-\tilde{\theta})}T \tag{39}$$

[4] Stability analysis in Section 5 using suitably designed gain conditions ensures the existence of S, which is a positively invariant set for $\zeta(t)$, provided $\zeta(0) \in S$

[5] The detailed proof is not provided here to honor the page limit and will be reported in future publication

Using trigonometric identities [6] (37)-(39) are rewritten as

$$T_{v_x} = T_{x_d} + h_x(\eta_d, e_\eta, \tilde{\eta})T \tag{40}$$

$$T_{v_y} = T_{y_d} + h_y(\eta_d, e_\eta, \tilde{\eta})T \tag{41}$$

$$T_{v_z} = T_{z_d} + h_z(\eta_d, e_\eta, \tilde{\eta})T \tag{42}$$

where $\{h_x, h_y, h_z\}$ are the non-linear coupling terms between position and attitude dynamics. They consist of sine and cosine functions of desired attitude, attitude error and filter error respectively. Using the expressions of desired thrust $\{T_{x_d}, T_{y_d}, T_{z_d}\}$ and $\|T_d\| = T$ obtained during reference attitude computation Dasgupta (2018); Dasgupta et al. (2019a,b,c), actual thrust is

$$T_v = T_d + \|T_d\|\, h \tag{43}$$

where $h = [h_x\ h_y\ h_z]^T \in \mathbb{R}^3$

5. STABILITY ANALYSIS

Stability of the hierarchical system is analyzed considering nonlinear interaction between position and rotational dynamics Dasgupta (2018, 2019); Dasgupta et al. (2019a,b,c). A Lemma developed and proved in authors' earlier works Dasgupta (2019); Dasgupta et al. (2019a,b,c), is briefly stated below and used in the subsequent Lyapunov analysis.

- $\|\chi\| \le \delta\, \|z_\zeta\|$ where $\delta \in \mathbb{R}_+$ and z_ζ is a vector, defined as $z_\zeta \triangleq \left[r_\zeta^T\ tanh^T(e_\zeta)\ tanh^T(e_f)\right]^T \in \mathbb{R}^9$

To prove the main result, the following additional properties and a newly developed Lemma [7], have been applied during stability analysis.

Property 3.

$$ {}^{e_\theta^2}/_{k_\theta^2} < \ln\left(1 + {}^{e_\theta^2}/_{(k_\theta^2-e_\theta^2)}\right) < {}^{e_\theta^2}/_{(k_\theta^2-e_\theta^2)} \tag{44}$$

Property 4.

$$ {}^{1}/_{2}\tanh^2(\|e_\zeta\|) \le \sum \ln\cosh(e_i) \le \|e_\zeta\|^2 \tag{45}$$

where $i = \{x, y, z\}$

Lemma 1. $\|h\| \le \alpha\big(\,\|e_\eta\| + \|r_\tau\|\,\big)$, where $\{e_\eta(t), r_\tau(t)\}$ are the attitude and filter error vector and $\alpha \in \mathbb{R}_+$

The main result is stated in the following theorem

Theorem 1. For the system given by (1)-(2), the position control (12) and attitude control law (26) guarantee that the overall error dynamics $e_v(t) = \left[r_\zeta^T\ e_\zeta^T\ e_f^T\ r_\eta^T\ e_\eta^T\ r_\tau^T\right]^T \in \mathbb{R}^{18}$ is uniformly ultimately bounded (UUB) provided the following gain conditions are satisfied

$$k k_{\zeta_2} > {}^{\delta^2}/_{4\lambda_{min}\{M_q\}} \tag{46}$$

$$k_{\tau_1} k_{\zeta_4} > {}^{\beta^2}/_{4\lambda_{min}\{M_q\}} \tag{47}$$

$$\lambda_{min}\{A_\eta\} > {}^{\beta^2}/_{4k_{\zeta_3}\lambda_{min}\{M_q\}} \tag{48}$$

where λ_{min} represents minimum eigenvalue of the argument matrix and $(k, k_{\zeta_2}, k_{\zeta_3}, k_{\zeta_4}, k_{\tau_1}, \beta)$ are subsequently defined auxiliary positive scalar constants subjected to an initial condition constraint $|e_\theta(0)| < k_\theta$.

[6] The following trigonometric identities are applied to obtain (40), (41) and (42) respectively where $\{x, y\} \in \mathbb{R}$.

$$\cos(x-y) = \cos(x) + 2\sin(x - y/2)\sin(y/2)$$
$$\sin(x-y) = \sin(x) - 2\cos(x - y/2)\sin(y/2)$$

[7] The detailed proof of Lemma 1 is not provided here to honor the page limit and will be reported in future publication

Proof. Consider a Lyapunov function candidate as

$$
\begin{aligned}
L &= L_\zeta + L_\eta + L_\tau \\
&= \big({}^1\!/\!{}_2 r_\zeta^T M_q r_\zeta + m_q \big(\ln\cosh(e_x) + \ln\cosh(e_y) + \ln\cosh(e_z) \big) \\
&+ {}^1\!/\!{}_2 tanh^T(e_f) M_q tanh(e_f) \big) + \big({}^1\!/\!{}_2 r_\eta^T M_\eta(\eta) r_\eta + {}^1\!/\!{}_2 e_\eta^T e_\eta \\
&+ {}^1\!/\!{}_2 \big(e_\phi^2 + \ln \big({}^{k_\theta^2}\!/\!{}_{(k_\theta^2 - e_\theta^2)} \big) + e_\psi^2 \big) + {}^1\!/\!{}_2 r_\tau^T \tau_1 r_\tau \big) \qquad (49)
\end{aligned}
$$

Using Properties 3-4, L is lower and upper bounded by

$$
\begin{aligned}
&\lambda_{min}\{M_q\} \left\| r_\zeta \right\|^2 + m_q \tanh^2(\|e_\zeta\|) \\
&+ \lambda_{min}\{M_q\} \left\| tanh(e_f) \right\|^2 + \lambda_{min}\{M_q\} \left\| r_\eta \right\|^2 + \left\| e_\eta \right\|^2 \\
&+ (e_\phi^2 + {}^{e_\theta^2}\!/\!{}_{k_\theta^2} + e_\psi^2) + \tau_1 \left\| r_\tau \right\|^2 \\
&\le 2L \le \lambda_{max}\{M_q\} \left\| r_\zeta \right\|^2 + 2m_q \left\| e_\zeta \right\|^2 \\
&+ \lambda_{max}\{M_q\} \left\| tanh(e_f) \right\|^2 + \lambda_{max}\{M_q\} \left\| r_\eta \right\|^2 + \left\| e_\eta \right\|^2 \\
&+ (e_\phi^2 + {}^{e_\theta^2}\!/\!{}_{(k_\theta^2 - e_\theta^2)} + e_\psi^2) + \tau_1 \left\| r_\tau \right\|^2 \qquad (50) \\
&\implies L_1(e_v) \le L(e_v, t) \le L_2(e_v) \qquad (51)
\end{aligned}
$$

It indicates L dominates $L_1(e_v)$ and is decrescent because it is dominated by $L_2(e_v)$ $\forall t \ge 0$, where $\{L_1(e_v), L_2(e_v)\}$ are time-invariant positive definite functions.

Differentiate (49), substitute (11), (13), (22), (24), (35) and (43), apply (28), previously and newly defined Lemmas, imposing gain conditions (46)-(48) and finally upper bounding, yields

$$
\begin{aligned}
\dot{L} &\le -k_a \left\| z_\zeta \right\|^2 - \lambda_{min}\{K_\eta\} \left\| r_\eta \right\|^2 \\
&- k_b \left\| e_\eta \right\|^2 - \lambda_{min}\{A_\eta\} \big(e_\phi^2 + {}^{e_\theta^2}\!/\!{}_{(k_\theta^2 - e_\theta^2)} + e_\psi^2 \big) - k_\tau \left\| r_\tau \right\|^2 \\
&+ {}^{\tau_1^2 \|\ddot{\eta}_d\|^2}\!/\!{}_{4k_{\tau_2}} + {}^{\tau_2^2 \|\dot{\eta}_d\|^2}\!/\!{}_{4k_{\tau_3}} \qquad (52)
\end{aligned}
$$

where $k_a \triangleq \big(k - {}^{\delta^2}\!/\!{}_{4k_{\zeta_2}\lambda_{min}\{M_q\}} \big) > 0$, $k_b \triangleq \big(\lambda_{min}\{A_\eta\} - {}^{\beta^2}\!/\!{}_{4k_{\zeta_3}\lambda_{min}\{M_q\}} \big) > 0$, $k_\tau \triangleq \big(k_{\tau_1} - {}^{\beta^2}\!/\!{}_{4k_{\zeta_4}\lambda_{min}\{M_q\}} \big) > 0$, $(k_{\tau_1} + k_{\tau_2} + k_{\tau_3}) = 1$, $k \triangleq min\big(k_{\zeta_1}\lambda_{min}\{M_q\}, \alpha_\zeta \lambda_{min}\{M_q\}, \gamma_\zeta \lambda_{min}\{M_q\} \big)$, $k_\zeta \triangleq (k_{\zeta 1} + k_{\zeta 2} + k_{\zeta 3} + k_{\zeta 4})$, and $\beta \triangleq \alpha c$

Since τ_1 and τ_2 are free parameters, sufficiently small choices of those parameters (depending on the initial conditions) will lead to a uniformly ultimately bounded (UUB) result as follows

$$
\begin{aligned}
\dot{L} &\le -k_a \left\| z_\zeta \right\|^2 - \lambda_{min}\{K_\eta\} \left\| r_\eta \right\|^2 \\
&- k_b \left\| e_\eta \right\|^2 - \lambda_{min}\{A_\eta\} \big(e_\phi^2 + {}^{e_\theta^2}\!/\!{}_{(k_\theta^2 - e_\theta^2)} + e_\psi^2 \big) - k_\tau \left\| r_\tau \right\|^2 \\
&+ \underbrace{({}^{\tau_1^2}\!/\!{}_{4k_{\tau_2}})\epsilon_1^2 + ({}^{\tau_2^2}\!/\!{}_{4k_{\tau_3}})\epsilon_2^2}_{\epsilon} \qquad (53)
\end{aligned}
$$

where ϵ_1 and ϵ_2 are the upper-bounds of first and second derivatives of desired attitude within the invariant set of interest and ϵ characterizes the ultimate-bound. The initial condition dependent $\{\tau_1, \tau_2\}$ must be designed along with $\{k_{\tau_1}, k_{\tau_2}, k_{\tau_3}\}$ such that the negative quadratic terms will dominate ϵ bound near the origin and yield UUB result. In view of (49)

$$
\begin{aligned}
&{}^1\!/\!{}_2 \ln \big({}^{k_\theta^2}\!/\!{}_{(k_\theta^2 - e_\theta^2)} \big) \le L(0) < \infty \\
&\implies \big({}^{k_\theta^2}\!/\!{}_{(k_\theta^2 - e_\theta^2)} \big) \le e^{2L(0)} \implies {}^{e_\theta^2}\!/\!{}_{k_\theta^2} \le 1 - e^{-2L(0)} \\
&i.e.\ {}^{e_\theta^2}\!/\!{}_{k_\theta^2} < 1 \implies |e_\theta(t)| < k_\theta \ \forall t \ge 0 \qquad (54)
\end{aligned}
$$

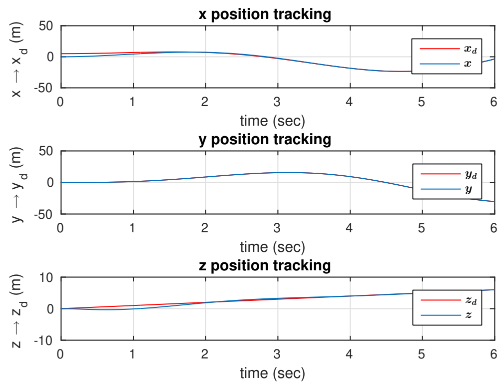
Using the definition of $e_\eta(t)$ and applying (33), (34)

$$
|\theta(t)| \le |\theta_c(t)| + |e_\theta(t)| < \rho + k_\theta = {}^\pi\!/\!{}_2 \qquad \forall t \ge 0 \qquad (55)
$$

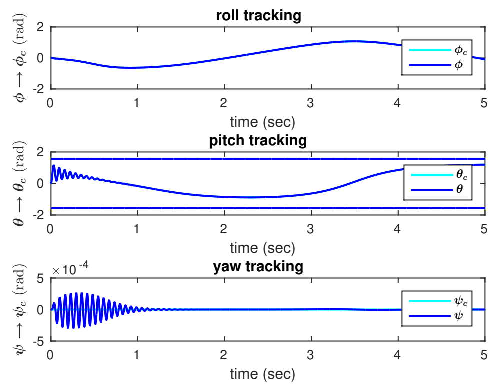
Remark 2. With the help of BLF, (54) proves that $e_\theta(t)$ is strictly less than k_θ for all time provided the initial condition constraint is satisfied. Using above result, (55) restricts pitch angle $|\theta(t)| < {}^\pi\!/\!{}_2$ and singularity is avoided.

6. ILLUSTRATIVE SIMULATIONS

The performance of the BLF-based DSC control law is examined by simulating a complex maneuver of a quadrotorcraft at time-varying depth. The list of vehicle parameters used to evaluate the performance of the control design, are m_q = 0.468 kg, I_{xx} = 4.856 $\times 10^{-3}$kg-m^2, I_{yy} = 4.856 $\times 10^{-3}$kg-m^2, and I_{zz} = 8.801 $\times 10^{-3}$kg-m^2. The aggressive reference flight trajectory is given by $\zeta_d(t) = [5(\cos(t) + t\sin(t))\ 5(t\sin(t) - t\cos(t))\ t]^T$ with a desired orientation $\psi_d(t) = 0$. It is an involute motion. The initial pose is $\zeta(0) = [0\ 0\ 0]^T$ and $\eta(0) = [0\ 1.2\ 0]^T$ which indicates vehicle starts with a pitch $\theta(0)$ close to ${}^\pi\!/\!{}_2$. Simulation result shown in Fig.2 illustrates the performance of the controller where in particular Fig.2*b* shows that the design limits the reference and actual pitch within $\{(-{}^\pi\!/\!{}_2, {}^\pi\!/\!{}_2) -$ blue-line$\}$ during attitude tracking. Control parameters, $k_\zeta = 10$, $K_\eta = \mathrm{diag}\{20, 20, 20\}$, filter design parameters, $\alpha_\zeta = 1$, $\gamma_\zeta = 1$, $A_\eta = \mathrm{diag}\{50, 50, 50\}$, and filter time constants $\tau_1 = 1\times10^{-4}$, $\tau_2 = 1\times10^{-3}$ are used for simulation. Fig.3 exhibits the performance of a second order low pass attitude filter in terms of (1) filter tracking and (2) convergence of filter error appeared in Fig.3*a* and Fig.3*b* respectively. It is observed that the desired and filtered pitch always lie within $\{(-{}^\pi\!/\!{}_2, {}^\pi\!/\!{}_2) -$ blue-line$\}$ which ensures attitude tracking without singularity.

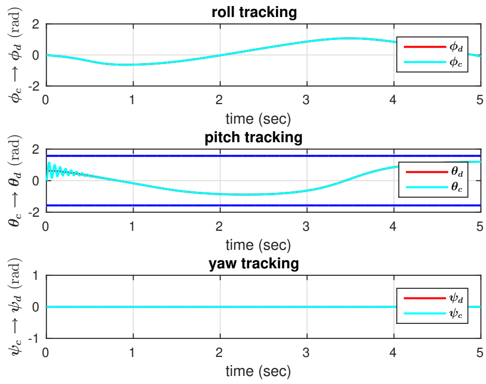


(a) Position Tracking

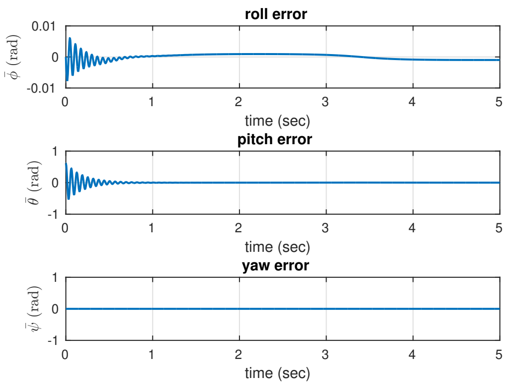


(b) Attitude Tracking

Fig. 2. Aggressive Flight Maneuver



(a) Filter Tracking



(b) Filter Error

Fig. 3. Filter Convergence on Agile Motion

7. CONCLUSION

In this paper, a symmetric barrier Lyapunov function based dynamic surface control has been developed for a quad-rotorcraft system with pitch constraint. The control design based on dynamic surface solves the problem of restrictions on high order differentiability of reference position and orientation and circumvents the complexity that arises due to "explosion of terms" coming from successive differentiation of reference attitude and desired thrust vector. The barrier Lyapunov function guarantees the pitch constraint is not violated and Lyapunov analysis proves that error signals are uniformly ultimately bounded to a neighbourhood of the origin. Using a saturated thrust control technique, the tracking error converges asymptotically without violating the pitch constraint. Two simulation examples demonstrate the effectiveness of the proposed method.

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