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Nonlinear Swing down Control of the Acrobot ⋆

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Abstract: In this paper, we concern swing down control of the Acrobot which is a 2-link planar robot with a single actuator driving the second joint, whose control objective is to stabilize the Acrobot to the downward equilibrium point with the two links in the downward position for all initial states of the Acrobot with the exception of a set of Lebesgue measure zero. To achieve this control objective, we design a nonlinear controller by using *nonnegative* linear feedback of the sine function of the angle of the second joint in addition to the *negative* linear feedback of its angular velocity. By analyzing globally the solution of the closed-loop system consisting of the Acrobot and the presented controller and focusing on the equilibrium points of the closed-loop system and their stability, we prove that the control objective is achieved provided that some conditions on two control gains are satisfied. We design the two control gains such that the real parts of the dominant poles of the linearized model of the closed-loop system around the downward equilibrium point are minimized. We provide simulation results for two Acrobots to show the effectiveness of the presented controller.

Keywords: Underactuated mechanical systems, Acrobot, swing down control, robot control, nonlinear control, passivity, Lyapunov stability, motion analysis.

1. INTRODUCTION

The last two decades have witnessed considerable progress in the study of underactuated robots, which possess fewer actuators than degrees of freedom from the perspectives of lightening weight, increasing reliability and saving energy. One of the important control problems for underactuated systems is the set-point control (regulation or stabilization) of a desired equilibrium point (Su and Stepanenko (1999); De Luca et al. (2001)).

Many researchers studied a particular problem of the set-point control called the swing up control for the Acrobot, which is a 2-link planar robot with a single actuator driving the second joint, see e.g., Spong (1995); Fantoni and Lozano (2001); Ma and Su (2002); Xin and Kaneda (2007). Indeed, the swing up control is to swing the Acrobot to a small neighborhood of the upright equilibrium point (denoted as UEP below), where the two links are in the upright position, and then balance the robot around that point.

In this paper, we study a set-point control named as *swing down control* of the Acrobot; that is, to stabilize the Acrobot about the downward equilibrium point (denoted as DEP below) for all initial states with the exception of a set of Lebesgue measure zero, where the two links are in the downward position.

In Zhang et al. (2013), a time-optimal trajectory for the Acrobot from the downward position all the way to the upright position is constructed by using an artificial friction torque in order to construct a downward trajectory, and rewind it to make an upward trajectory. It is an interesting approach. When the torque for the second joint is designed to be viscous friction; that is, negative linear feedback of the angular velocity of the second joint, which is called D control in this paper for brevity, it is stated in Lemma 1 of Zhang et al. (2013) that the Acrobot will be controlled from any initial state to one of four equilibrium points: the DEP, the UEP, down–up equilibrium point, and up–down

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equilibrium point (where down–up means that the links 1 and 2 of the Acrobot are in the downward and upright positions, respectively), and only the DEP is stable. The above statements imply that the control objective of swing down control is achieved under the proposed control.

Our simulation investigation for two Acrobots in Spong (1995) and Zhang et al. (2013) shows that it takes a long time to drive these Acrobots close to the DEP under the D control (the negative linear feedback of the angular velocity of the second joint) for different linear feedback gains. To achieve a better control performance, we present a nonlinear swing down control by adding a *nonnegative* linear feedback of the sine function of the angle of the second joint to the D control, which is called PsD control in this paper, where P denotes position and s denotes sine.

In this paper, we analyze globally the motion of the Acrobot under the PsD controller. For the Acrobot which is linearly *controllable* around the up–down or down–up equilibrium point, we prove that the solution of the closed-loop system consisting of the Acrobot and the presented controller converges to an equilibrium point of the closed-loop system for all initial states. By analyzing the equilibrium set of the closed-loop system, we show that the equilibrium set only contains the above four equilibrium points with only the DEP being stable when the gain related to the nonnegative linear feedback is restricted to a certain range. In this way, we prove that the control objective of the swing down control can be achieved by the presented PsD controller. For the Acrobot which is linearly *uncontrollable* around the up–down or down–up equilibrium point, our theoretical analysis shows that the Acrobot cannot be stabilized to the DEP under the PsD control from certain initial states beside to the other three equilibrium points. Since the physical parameters for the Acrobot being linearly uncontrollable at the up–down (down–up) equilibrium point are exceptional, we prove that the control objective of the swing down control can be achieved by the presented PsD controller for almost any physical parameters of the Acrobot.

It is known that the dominant poles of a linear stable system give rise to the longest lasting terms in the transient response of the system. Note that the dominant poles are the eigenvalues (of the state space matrix corresponding to the system) whose real parts are maximal (closest to the imaginary-axis). We design the two control gains by minimizing the real parts of the dominant poles of the linearized model of the closed-loop system around the DEP. Our simulation investigation of two Acrobots shows that the PsD control can achieve a better performance than the D control.

2. PRELIMINARY KNOWLEDGE

Consider the Acrobot shown in Fig. 1. The motion equation of the Acrobot (Xin and Kaneda (2007)) is:

$$M(q)\ddot{q} + H(q,\dot{q}) + G(q) = \tau, \tag{1}$$

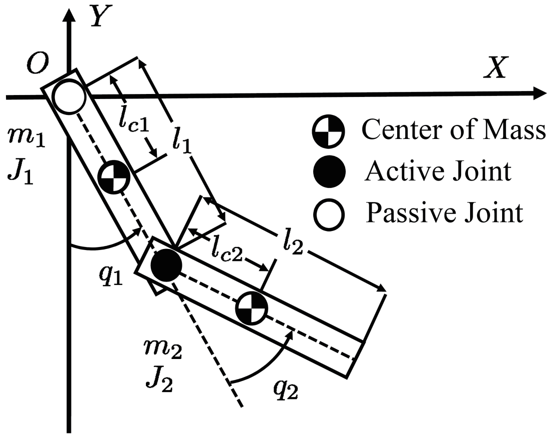

Fig. 1. the Acrobot.

where $q = \begin{bmatrix} q_1, \ q_2 \end{bmatrix}^{\mathrm{T}}$, $\tau = \begin{bmatrix} 0, \ \tau_2 \end{bmatrix}^{\mathrm{T}}$ with τ_2 is a single control torque applied to joint 2, and

$$M(q) = \begin{bmatrix} \alpha_1 + \alpha_2 + 2\alpha_3 \cos q_2 & \alpha_2 + \alpha_3 \cos q_2 \\ \alpha_2 + \alpha_3 \cos q_2 & \alpha_2 \end{bmatrix}, \tag{2}$$

$$H(q,\dot{q}) = \alpha_3 \begin{bmatrix} -2\dot{q}_1\dot{q}_2 - \dot{q}_2^2 \\ \dot{q}_1^2 \end{bmatrix} \sin q_2, \tag{3}$$

$$G(q) = \begin{bmatrix} \beta_1 \sin q_1 + \beta_2 \sin(q_1+q_2) \\ \beta_2 \sin(q_1+q_2) \end{bmatrix}, \tag{4}$$

where

$$\begin{cases} \alpha_1 = J_1 + m_1 l_{c1}^2 + m_2 l_1^2, \\ \alpha_2 = J_2 + m_2 l_{c2}^2, \\ \alpha_3 = m_2 l_1 l_{c2}, \\ \beta_1 = (m_1 l_{c1} + m_2 l_1) g, \\ \beta_2 = m_2 l_{c2} g, \end{cases} \tag{5}$$

where for the ith $(i = 1, 2)$ link, m_i is its mass, l_i is its length, l_{ci} is the distance from joint i to its center of mass (COM), and J_i is the moment of inertia around its COM; and g is the acceleration of gravity. In this paper, we treat $q_i(t)$ $(i = 1, 2)$ in $\mathbb{S}$, where $\mathbb{S}$ denotes a unit circle.

The following D controller is proposed in Zhang et al. (2013):

$$\tau_2 = -k_D \dot{q}_2, \tag{6}$$

where $k_D > 0$ is constant. Consider the closed-loop system consisting of (1) and (6). It has the four equilibrium points defined in the following set:

$$\Omega_s = \{(q^e, 0) \mid q^e = [0,0]^{\mathrm{T}}, [\pi,0]^{\mathrm{T}}, [0,\pi]^{\mathrm{T}}, [\pi,\pi]^{\mathrm{T}}\}. \tag{7}$$

It is mentioned in Lemma 1 of Zhang et al. (2013) that the closed-loop solution $(q(t), \dot{q}(t))$ asymptotically converges to the DEP $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (0, 0, 0, 0)$ as $t \to \infty$ from any initial state, other than the other three equilibrium points in Ω_s. We will discuss this result later.

3. NONLINEAR SWING DOWN CONTROLLER AND MOTION ANALYSIS

In this paper, our goal is to design τ_2 such that

$$\lim_{t\to\infty} q(t) = 0, \ \ \lim_{t\to\infty} \dot{q}(t) = 0 \tag{8}$$

for all initial states of the Acrobot with the exception of a set of Lebesgue measure zero.

We aim to design a feedback controller for achieving this goal with a better performance than (6). We present the following controller:

$$\tau_2 = -k_D \dot{q}_2 + k_P \sin q_2, \tag{9}$$

where $k_D > 0$ and $k_P \geq 0$ are constant.

First, we present the following theorem.

Theorem 1. Consider the closed-loop system consisting of (1) and (9). The following statements hold:

1. Assume that the physical parameters of the Acrobot in (5) *does not* satisfy

$$(\alpha_2 - \alpha_3)\beta_1 = (\alpha_3 - \alpha_1)\beta_2. \tag{10}$$

When $k_D > 0$ and $k_P \geq 0$ hold, the closed-loop solution $(q(t), \dot{q}(t))$ approaches an equilibrium point of the closed-loop system as $t \to \infty$.

2. Assume that the physical parameters of the Acrobot in (5) satisfy (10). For any initial state with $\dot{q}_1(0) \neq 0$ and $(q_2(0), \dot{q}_2(0)) = (\pi, 0)$, the Acrobot moves like a single link robot with $q_2(t) = \pi$ and its mechanical energy being invariant for all $t \geq 0$.

Second, we characterize the equilibrium points of the closed-loop system consisting of (1) and (9). Let $q^e = \left[\, q_1^e,\ q_2^e \,\right]^{\mathrm{T}}$ be a closed-loop equilibrium configuration. Putting $\ddot{q} = 0$, $\dot{q} = 0$, $q = q^e$, and $\tau_2 = \tau_2^e = k_P \sin q_2^e$ into (1), we obtain

$$\beta_1 \sin q_1^e + \beta_2 \sin(q_1^e + q_2^e) = 0, \tag{11}$$

$$k_P \sin q_2^e - \beta_2 \sin(q_1^e + q_2^e) = 0. \tag{12}$$

Define the following set which contains all equilibrium points of the closed-loop system:

$$\Omega = \{(q^e, 0) \mid q^e \ \text{ satisfies } \ (11) \text{ and } (12)\}. \tag{13}$$

Clearly, Ω_s in (7) is a subset of Ω in (13) for any k_P. We present the following theorem.

Theorem 2. For Ω in (13) and Ω_s in (7), if

$$0 \leq k_P < \frac{\beta_1 \beta_2}{\beta_1 + \beta_2}, \tag{14}$$

or

$$|\beta_1 - \beta_2| k_P > \beta_1 \beta_2, \tag{15}$$

then

$$\Omega = \Omega_s. \tag{16}$$

Third, by linearizing the closed-loop system consisting of the Acrobot in (1) and the controller (9) around each of four equilibrium points in Ω_s and using Routh–Hurwitz criterion to check its stability, we present the following theorem.

Theorem 3. Consider the closed-loop system consisting of (1) and (9). Assume that the physical parameters of the Acrobot in (5) do not satisfy (10). Assume $k_D > 0$. If k_P satisfies (14), then only the DEP $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (0, 0, 0, 0)$ in Ω_s is stable, and the other three equilibrium points in Ω_s are unstable.

Finally, from Theorems 2 and 3, since the DEP is stable, and the other three equilibrium points in Ω_s are unstable, according to Ortega et al. (2002) (p. 1225), the Lebesgue measure of the set of initial states converging to one of these unstable equilibrium points is zero. To summarize the above results, we present the following main result of this paper.

Theorem 4. Consider the closed-loop system consisting of (1) and (9). Assume that the physical parameters of the Acrobot in (5) do not satisfy (10). If $k_D > 0$ and k_P satisfies (14), then as $t \to \infty$ the closed-loop solution $(q(t), \dot{q}(t))$ approaches the DEP $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (0, 0, 0, 0)$ for all initial states with exception of a set of Lebesgue measure zero.

4. DISCUSSION

We have the following remark for Theorem 1. In Xin and Kaneda (2007), an example is given to show that there do exist $\alpha_1, \alpha_2, \alpha_3, \beta_1$, and β_2 in (5) satisfying (10). Moreover, Lemma 1 of Liu and Xin (2015) states that (10) is a necessary and sufficient condition such that the Acrobot is linearly uncontrollable around the down–up equilibrium point or up–down equilibrium point. Thus, for the Acrobot which is linearly uncontrollable around the up–down or down–up equilibrium point, our theoretical analysis shows that the Acrobot cannot be stabilized to the DEP under the PsD control from certain initial states beside to the other three equilibrium points. Moreover, since Theorem 1 holds for $k_P \geq 0$, Lemma 1 of Zhang et al. (2013) (the case of $k_P = 0$) is true for any Acrobot except the one being linearly uncontrollable around the up–down or down–up equilibrium point.

We have the following remark for Theorem 4. Since the physical parameters for the Acrobot being linearly uncontrollable at the up–down (down–up) equilibrium point (that is, the physical parameters satisfy (10)) are exceptional, we prove that the control objective of the swing down control can be achieved by the D control in Zhang et al. (2013) and the presented PsD controller in this paper for almost any physical parameters of the Acrobot.

We discuss the controller in (9) further. Let us replace $\sin q_2$ by q_2; that is,

$$\tau_2 = -k_D \dot{q}_2 + k_P q_2. \tag{17}$$

Consider the closed-loop system consisting of (1) and (17). Assume that the physical parameters of the Acrobot in (5) do not satisfy (10). Similar to Theorem 3, we can claim that if $k_D > 0$ and k_P satisfies (14), then the DEP $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (0, 0, 0, 0)$ in Ω_s is stable locally. This is different from the result of almost global stabilization expressed in Theorem 4. This shows the difference between two controllers (9) and (17). Please refer the numerical validation in Section 6.

5. DESIGN OF TWO CONTROL GAINS

Let A_{dd} be the state space matrix of the linearized model of the closed-loop system consisting (1) and (9) around the DEP. The poles of the linearized model are the eigenvalues of A_{dd} denoted as $\lambda_i(A_{dd})$ $(1 \leq i \leq 4)$.

Since the dominant poles which are closest to the imaginary-axis give rise to the longest lasting terms in the transient response of a linear stable system, to improve the control performance of the PsD controller, we consider the following optimization problem of minimizing the real parts of the dominant poles (denoted as RDP) to design the two control gains k_D and k_P. Define

$$\mathrm{RDP}(k_D, k_P) = \max_{1 \leq i \leq 4} \mathrm{Re}\,[\lambda_i(A_{dd})], \tag{18}$$

where Re $[\lambda_i]$ denotes the real part of λ_i. The minimum of $\mathrm{RDP}(k_D, k_P)$ for the PsD controller is defined as:

$$\mathrm{RDP}^*_{\mathrm{psd}} = \min_{k_D>0,\ k_P \text{ satisfies (14)}} \mathrm{RDP}(k_D, k_p), \tag{19}$$

and k_D^* and k_P^* are corresponding optimal control gains.

Similarly, for the D controller (that is, the PsD controller with $k_P = 0$), the minimum of $\mathrm{RDP}(k_D, 0)$ is defined as:

$$\mathrm{RDP}^*_{\mathrm{d}} = \min_{k_D>0} \mathrm{RDP}(k_D, 0). \tag{20}$$

Clearly,

$$\mathrm{RDP}^*_{\mathrm{psd}} \leq \mathrm{RDP}^*_{\mathrm{d}}. \tag{21}$$

Since $\lambda_i(A_{dd})$ is a complicated function k_D and k_P, in this paper, the optimization problems (19) and (20) are solved numerically by using the “fminsearch” function in MATLAB.

6. SIMULATION RESULTS

We provide simulation results for two Acrobots to validate the effectiveness of the presented PsD control in comparison with the D control.

6.1 Example 1

Consider the Acrobot in Spong (1995) with the following physical parameters: $m_1 = 1$ kg, $m_2 = 1$ kg, $l_1 = 1$ m, $l_2 = 2$ m, $l_{c1} = 0.5$ m, $l_{c2} = 1$ m, $J_1 = 0.083$ kg·m^2, $J_2 = 0.33$ kg·m^2. We take $g = 9.81$ m/s^2. We obtain $\alpha_1 = 1.333$, $\alpha_2 = 1.330$, $\alpha_3 = 1$, $\beta_1 = 14.72$ and $\beta_2 = 9.81$, which do not satisfy the condition (10).

For the PsD controller, by solving the optimization problem (19), we obtain $\mathrm{RDP}^*_{\mathrm{psd}} = -0.606$ with $k_D^* = 0.402$ and $k_P^* = 5.01$. The four eigenvalues of A_{dd} with $k_D = k_D^*$ and $k_P = k_P^*$ are $-0.606 \pm 2.22j$ and $-0.607 \pm 2.21j$.

For the D controller, by solving the optimization problem (20), we obtain $\mathrm{RDP}^*_{\mathrm{d}} = -0.027$ with $k_D^* = 2.32$. The four eigenvalues of A_{dd} with $k_D = k_D^*$ and $k_P = 0$ are $-0.0271 \pm 2.27j$, -3.48, and -10.5.

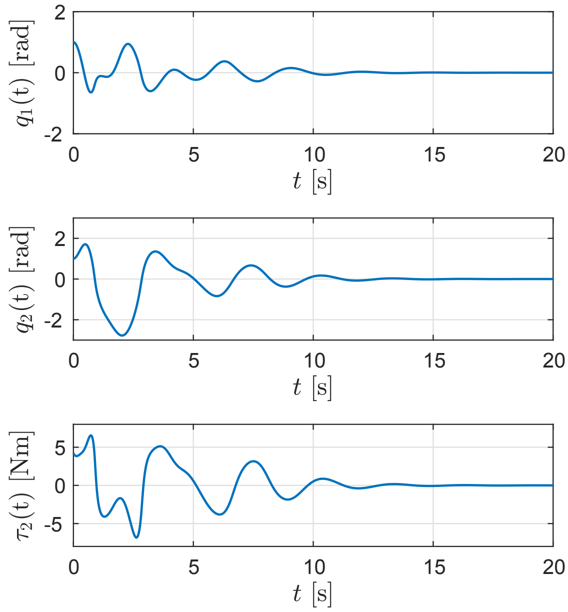

Fig. 2. Time responses of $q(t)$ and $\tau_2(t)$ under the PsD controller (9) for Example 1.

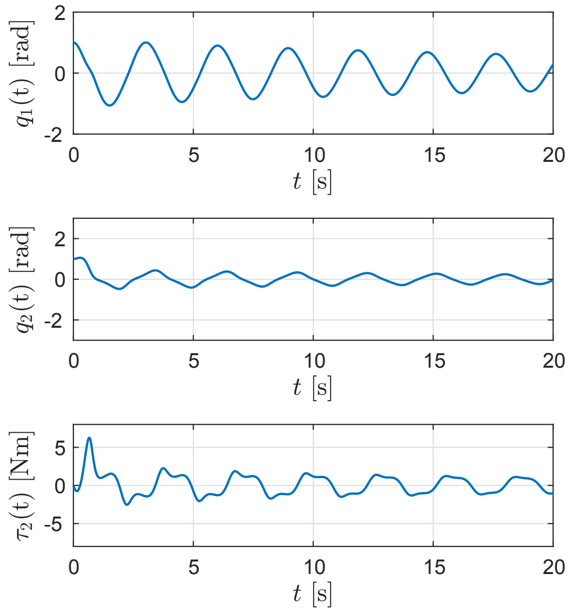

Fig. 3. Time responses of $q(t)$ and $\tau_2(t)$ under the D controller (6) for Example 1.

Thus, we obtain $\mathrm{RDP}^*_{\mathrm{psd}} < \mathrm{RDP}^*_{\mathrm{d}}$ with the ratio being:

$$\frac{\mathrm{RDP}^*_{\mathrm{psd}}}{\mathrm{RDP}^*_{\mathrm{d}}} = 22.4.$$

For an initial state $(q_1(0), q_2(0), \dot{q}(0), \dot{q}_2(0)) = (1.0, 1.0, 0, 0)$, the time responses of $q(t)$ and $\tau_2(t)$ of the Acrobot under the PsD and D controllers are depicted in Fig. 2 and Fig. 3, respectively. These two figures show that it takes much shorter time to swing down the Acrobot to the DEP by the PsD controller than by the D controller. Note that there is no much difference in the maximal values of the $|\tau_2(t)|$ of these two controllers.

6.2 *Example 2*

Consider the Acrobot in Zhang et al. (2013) with the following physical parameters: $m_1 = 0.105$ kg, $m_2 = 0.080$ kg, $l_1 = 0.109$ m, $l_2 = 0.215$ m, $l_{c1} = 0.073$ m, $l_{c2} = 0.1075$ m, $J_1 = 1.0396 \times 10^{-4}$ kg·m^2, $J_2 = 3.0817 \times 10^{-4}$ kg·m^2. We obtain $\alpha_1 = 0.0016$, $\alpha_2 = 0.0012$, $\alpha_3 = 9.374 \times 10^{-4}$, $\beta_1 = 0.1607$, and $\beta_2 = 0.0844$, which do not satisfy the condition (10).

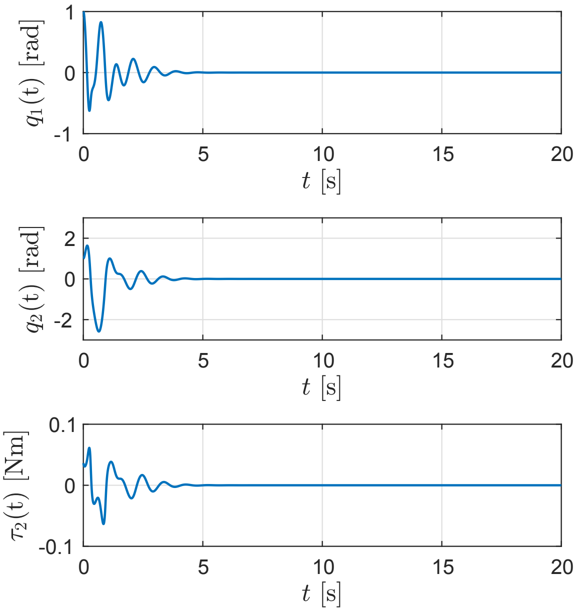

Fig. 4. Time responses of $q(t)$ and $\tau_2(t)$ under the PsD controller (9) for Example 2.

The optimal gains of the PsD controller for this robot are $k_D^* = 0.00175$ and $k_P^* = 0.0431$, under which we obtain $\text{RDP}_{\text{psd}}^* = -1.86$. The corresponding four eigenvalues of A_{dd} with these gains are $-1.86 \pm 6.84j$ and $-1.86 \pm 7.08j$.

The optimal gain of the D controller for this robot is $k_D^* = 0.0066$, under which we obtain $\text{RDP}_{\text{d}}^* = -0.1303$. The four eigenvalues of A_{dd} with $k_D = k_D^* = 0.00663$ and $k_P = 0$ are $-0.130 \pm 7.08j$ and $-13.8 \pm 7.26j$.

Thus, we have $\text{RDP}_{\text{psd}}^* < \text{RDP}_{\text{d}}^*$ with the ratio being:

$$\frac{\text{RDP}_{\text{psd}}^*}{\text{RDP}_{\text{d}}^*} = 14.3.$$

For an initial state $(q_1(0), q_2(0), \dot{q}(0), \dot{q}_2(0)) = (1.0, 1.0, 0, 0)$, the time responses of $q(t)$ and $\tau_2(t)$ of the Acrobot under the PsD and D controllers are depicted in Fig. 4 and Fig. 5, respectively. Similar to the simulation results described in Example 1, these two figures show that it takes much

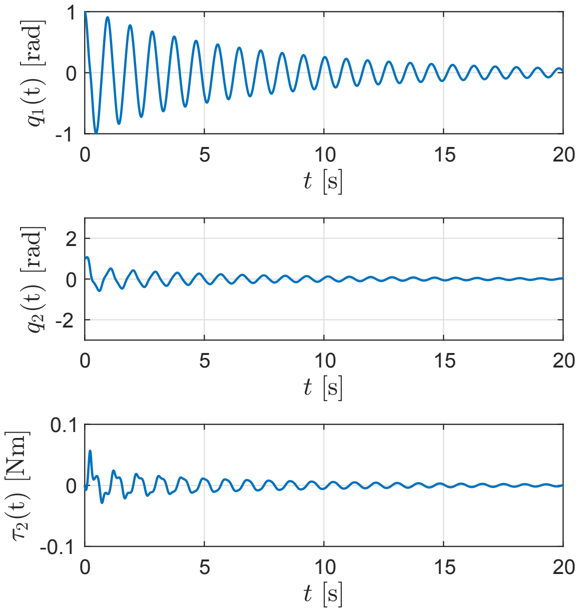

Fig. 5. Time responses of $q(t)$ and $\tau_2(t)$ under the D controller (6) for Example 2.

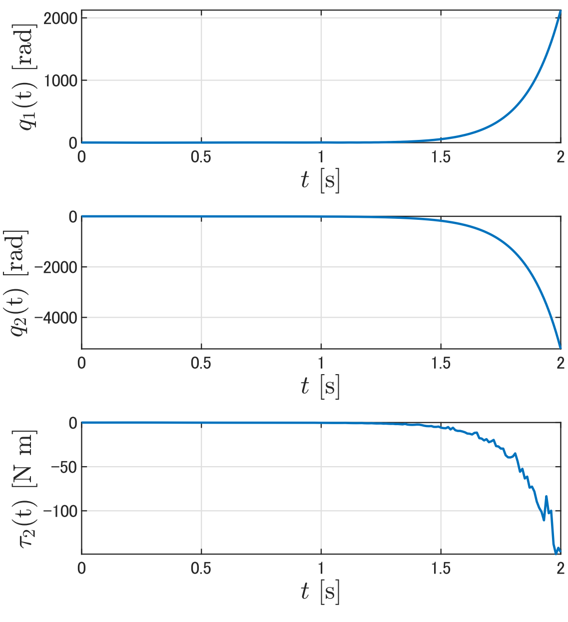

Fig. 6. Time responses of $q(t)$ and $\tau_2(t)$ in (17) for Example 2.

shorter time to swing down the Acrobot to the DEP by the PsD controller than by the D controller. Moreover, there is no much difference in the maximal values of the $|\tau_2(t)|$ of these two controllers.

If we use controller (17) with the same gains k_P and k_D used by the PsD controller for this example, then we find that we cannot stabilize the Acrobot for the same initial state $(q_1(0), q_2(0), \dot{q}_1(0), \dot{q}_2(0)) = (1, 1, 0, 0)$, see Fig. 6. This validated the statement in Section 4.

7. CONCLUSION

In this paper, we studied the swing down control of the Acrobot, whose control objective is to stabilize the Acrobot to the downward equilibrium point for almost all initial states of the Acrobot. To achieve this control objective, we designed a nonlinear controller which combines the nonnegative linear feedback of the sine function of the angle of the second joint and the negative linear feedback of its angular velocity. We analyzed the solution of the closed-loop system consisting of the Acrobot and the presented controller by characterizing the equilibrium points of the closed-loop system and investigating their stability. For the Acrobot being linearly controllable around the down–up equilibrium point or up–down equilibrium point, we proved that the control objective is achieved for all initial states of the Acrobot with the exception of a set of Lebesgue measure zero provided the two control gains satisfy the conditions shown in Theorem 4. We designed the two control gains such that the real parts of the dominant poles of the linearized model of the closed-loop system around the downward equilibrium point are minimized. We provided simulation results for two Acrobots to show that the presented PsD controller can swing down the Acrobot to the equilibrium point faster than the existing D controller.

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