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Coordination Control of Double-Integrator Systems with Time-Varying Weighted Inputs

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Abstract: This paper considers coordination control of double-integrator systems and proposes general control laws involving time-varying inputs. The nominal control input is weighted by time-varying (time-dependent or state-dependent) positive definite matrices, providing more freedoms in defining the control tasks. We present sufficient conditions to ensure the asymptotic convergence of double-integrator networked systems in this context, and support the theoretical results by several application examples. This includes distance-based multi-agent formation control and power network systems with unknown inertia matrices.

Keywords: Coordination control, double integrator, weighted input, nonlinear control

1. INTRODUCTION

As a canonical example of second-order control systems, the double-integrator is one of the most fundamental models in control theory. It has been used extensively in describing motion properties and design of controllers for acceleration-driven control systems. Examples include robot manipulator dynamics Spong and Vidyasagar (2008), vehicle dynamics Ren and Beard (2008), single-axis spacecraft dynamics Rao and Bernstein (2001) and power system dynamics Machowski et al. (2011); Chiang and Alberto (2015). As compared to the single-integrator system considered in Zhao and Sun (2017), double-integrator models better reflect the dynamics of physical control systems in which the control input directly relates to forces that generate permissible accelerations. The double-integrator systems have been studied extensively since the early days of control theory, and a review of results on the stability and convergence properties for such systems are available in the literature (see Rao and Bernstein (2001)).

In recent years, double-integrator models have been adopted in modeling networked control systems, such as multi-agent consensus Ren (2008) and multi-robotics rendezvous control Dong and Huang (2013). In particular, double-integrator models are popular and have been studied extensively for flocking control of multi-agent systems, partly originated by the pioneering work of Olfati-Saber (2006); Tanner et al. (2007) and subsequent works, e.g., Su et al. (2009); Ren and Cao (2010); Deghat et al. (2015); Sun et al. (2017). Inspired by these prior works on rich applications on double-integrator systems, we revisit the conventional control design for coordination control (in e.g., Olfati-Saber (2006); Tanner et al. (2007); Deghat et al. (2015); Sun et al. (2017)), and aim to propose a more general feedback law that allows time-varying weighting of control input. In many applications involving double-integrator systems, the control input typically consists of a velocity damping term, and a gradient-like control term that regulates the system state to reach a control task described by a potential function. In this paper we generalize this conventional control law, by designing a time-varying input for individual systems that renders more motion freedoms and better transient behaviors.

The main contribution of this paper lies in the introduction of a time-varying positive definite matrix weighting the input signals, granting the systems extra motion freedoms to locally adapt the control law and permit extra control tasks while retaining certain stability properties. With the time-varying input, sufficient conditions are derived to ensure the (global) asymptotic stability of the closed-loop system. We also propose update laws on the eigenvalues of the time-varying weight matrix, guaranteeing asymptotic stability of the system states and global convergence of the spectrum of the time-varying matrix to a desired spectrum. Therefore, we generalize the standard double-integrator control design methodology in prior works, facilitating its use in a wide class of second-order systems. This is demonstrated in a set of simulations, where the input weighting is generalized to typical networked systems, including double-integrator formation systems, and power system networks with unknown inertia matrix.

Outline This paper is organized as follows. Section 2 defines the problem and gives the necessary mathematical preliminaries. Section 3 addresses this control problem by

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generalizing the single-integrator ideas in Zhao and Sun (2017) to the double integrator case with weighted inputs. Section 4 demonstrates the utility of the weighted input control in a set of networked control systems, including multi-agent distance-based formation shape control, and synchronization control of networks of power generators with unknown time-varying inertia tensors. Finally, Section 5 closes the paper with a set of conclusions.

2. PROBLEM SETUP

We consider a set of n systems with state vectors $x_i = [p_i^\top, v_i^\top]^\top \in \mathbb{R}^{2d}$ and control input $u_i \in \mathbb{R}^d$, satisfying the double integrator dynamics $\ddot{p}_i = u_i$, or in state-space form

$$\dot{x}_i = Ax_i + Bu_i = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ I \end{bmatrix} u_i. \quad (1)$$

The interaction between each individual system is described by a graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ with a vertex set $\mathcal{V} = \{1, \cdots, n\}$ and the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. By this, we mean that if $(i, j) \in \mathcal{E}$, the system i has knowledge of the state of system j. Furthermore, the number of edges in $\mathcal{G}$ is written $m = |\mathcal{E}|$, and we denote the set of neighbouring nodes to system i by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. In the application examples, we shall make use to both directed and undirected graphs, where in the latter $(i, j) \in \mathcal{E} \Rightarrow (j, i) \in \mathcal{E}$.

For the stability proofs, we will use standard Lyapunov theory, as well as the Lemma of Barbalat in the slightly modified form, originally in Micaelli and Samson (1993).

Lemma 1. (Micaelli and Samson (1993)). Let $f : \mathbb{R}_+ \to \mathbb{R}$ be any differentiable function. If $\lim_{t\to\infty} f(t) = 0$, and

$$\dot{f}(t) = f_0(t) + \eta(t), \qquad t \geq 0, \quad (2)$$

where $f_0(t)$ is a uniformly continuous function and $\lim_{t\to\infty} \eta(t) = 0$, then $\lim_{t\to\infty} \dot{f}(t) = \lim_{t\to\infty} f_0(t) = 0$.

With this brief background, the problem addressed in this paper can be stated in the following general form.

Problem Formulation Assume a set of initial conditions, $x_i(t_0) = x_{0,i},\ i = 1, ..., n$, and a graph, $\mathcal{G}$, describing the system interactions. Find a decentralized feedback law to for the system (1) minimizing a potential function $V(e(p))$ of the position p, expressed in terms of an error $e(p)$, while permitting the control action for each system to be weighted locally and dynamically, such that despite the input weighting, $(e, v) \to (0, 0)$ for all $i \in 1, ..., n$ as $t \to \infty$.

3. CONTROL WITH WEIGHTED INPUTS

Before considering a multi-agent problem formulated in the previous section, we address the problem of controlling the double integrator by generalizing the single-integrator framework in Zhao and Sun (2017) to a non-autonomous double integrator setting, where a time-varying positive definite matrix is acting on the input signals. The goal is to establish sufficient conditions on this weight matrix for asymptotic stability in the resulting closed-loop system, thereby enabling a richer class of application examples.

3.1 Main results

The first main result of this paper is presented as follows.

Theorem 1. Consider a control task defined in terms of a potential function $V(e)$, a positional error $e(p)$ and some time-varying positive definite matrix $0 \prec D(t) = D(t)^\top$. Assume that

(A1) The potential function $V(e)$ is positive definite, continuously differentiable and radially unbounded, its level sets are compact, and $\nabla_p V(e) = 0 \Leftrightarrow e = 0$;
(A2) The gradient of the potential function, $\nabla_p V(e)$, is uniformly continuous if e and v are bounded;
(A3) The matrix D satisfies $0 \prec 2kDD + \dot{D}$, $\dot{D}$ is uniformly continuous, and the spectral radius $\rho(D) \leq C$ for tow constants $k, C > 0$.

Given these assumptions, the feedback law

$$u(t) = -D(t)(\nabla_p V(e(t)) + kv(t)) \quad (3)$$

applied to the double integrator model

$$\dot{p} = v, \quad (4a)$$
$$\dot{v} = u, \quad (4b)$$

yields a globally asymptotically stable (GAS) equilibrium point in $(e, v) = (0, 0)$.

Proof. With the proposed feedback, we obtain

$$\dot{p} = v, \quad (5a)$$
$$\dot{v} = -D(\nabla_p V(e) + kv). \quad (5b)$$

Consider the Lyapunov function candidate

$$W = V(e) + (1/2)v^\top D^{-1} v. \quad (6)$$

Differentiation of W along the trajectories of (5) yields

$$\begin{aligned} \dot{W} =& (\nabla_p V(e))^\top \dot{p} + v^\top D^{-1}\dot{v} + (1/2)v \left(d(D^{-1})/dt\right) v \\ =& (\nabla_p V(e))^\top v + v^\top(-\nabla_p V(e) - kv) - (1/2)vD^{-1}\dot{D}D^{-1}v \\ =& -v^\top(kI + (1/2)D^{-1}\dot{D}D^{-1})v \leq 0, \end{aligned} \quad (7)$$

for all

$$0 \preceq kI + (1/2)D^{-1}\dot{D}D^{-1} \Leftrightarrow 0 \preceq 2kDD + \dot{D}. \quad (8)$$

Since $\dot{W}$ is negative semi-definite and D is bounded,

$$W(t_0) \geq V(e) + (1/2)v^\top D^{-1} v \quad (9a)$$
$$\geq V(e) + (1/2)v^\top \rho(D)^{-1} v \quad (9b)$$
$$\geq (1/2C)v^\top v, \quad (9c)$$

or equivalently $\|v(t)\|_2^2 \leq 2CW(t_0)\ \forall t \geq t_0$. Furthermore,

$$\begin{aligned} \ddot{W} =& 2v^\top(kI + (1/2)D^{-1}\dot{D}D^{-1})D(\nabla_p V(e) + kv) \\ &+ v^\top D^{-1}(\dot{D}D^{-1}\dot{D} - (1/2)\ddot{D})D^{-1}v. \end{aligned} \quad (10)$$

Consequently, uniform continuity of $\dot{D}$ implies that $\ddot{W}$ is bounded when differentiated along the trajectories of (5), since v is bounded by (9). Therefore, $\dot{W}$ is uniformly continuous along the trajectories of (5). Subsequent application of Barbalat's lemma in Barbălat (1959), yields

$$\lim_{t\to\infty} \dot{W} = 0 \Rightarrow \lim_{t\to\infty} v = 0 \Rightarrow \lim_{t\to\infty} \dot{v} = 0. \quad (11)$$

Furthermore, since $0 \prec D$, there exists no solution $0 = Dx$ unless $x = 0$, which means that $Dv = 0 \Leftrightarrow v = 0$ and $D\nabla_p V(e) = 0 \Leftrightarrow \nabla_p V(e) = 0 \Leftrightarrow e = 0$ by (A1). Invoking Theorem 1, one has $\lim_{t\to\infty} \eta(t) = \lim_{t\to\infty} \dot{f}(t) = 0$ by (11), and uniform continuity of $\nabla_p V(e)$ then implies,

$$\lim_{t\to\infty} D\nabla_p V(e) = 0 \Leftrightarrow \lim_{t\to\infty} \nabla_p V(e) = 0 \Leftrightarrow \lim_{t\to\infty} e = 0$$

and (5) is GAS at the equilibrium point $(e, v) = (0, 0)$. □

Remark 2. Note that it is possible to retain stability while adapting D, provided that the constraints posed in (A3) of Theorem 1 are satisfied. Consequently, we can steer the eigenvalues of D by feedback, so as to always satisfy these conditions, granting substantial freedoms in the design of the transient behavior of the error dynamics.

Proposition 3. Let $D(t) = Q\Lambda(t)Q^\top \in \mathbb{R}^{d\times d}$ where $\Lambda(t) = \mathrm{diag}(\lambda_1(t), \cdots, \lambda_d(t))$. Consider a target eigenvalue matrix, with a set of target eigenvalues as $\Lambda_f = \mathrm{diag}(\lambda_{f1}, \cdots, \lambda_{fd})$. Assume that $0 \prec \Lambda(t_0)$, $0 \prec \Lambda_f$, and $\tau, M > 0$. Granted these assumptions, the feedback law

$$\dot{\lambda}_j(t) = (1/\tau)\tanh(M[\lambda_{fj} - \lambda_j(t)])[\lambda_{fj} - \lambda_j(t)]^2, \quad (12)$$

yields $D(t) \to Q\Lambda_f Q^\top$ as $t \to \infty$, while ensuring that (A3) in Theorem 1 is met at all times provided $2k\tau > 1$.

Proof. Let $e_j(t) = \lambda_{fj} - \lambda_j(t)$ and consider the Lyapunov function candidates $V_j(e_j) = (1/2)e_j^2$, for $j = 1, \cdots, d$. When differentiated along the solutions of (12), we obtain

$$\dot{V}_j = -(1/\tau)\tanh(Me_j)e_j^3 < 0, \quad (13)$$

for all $e_j \neq 0$, as $\tanh(Me_j)e_j^3 \geq 0$. Consequently, each subsystem in (12) has a globally asymptotically stable equilibrium point in $e_j = 0 \Rightarrow \lambda_j(t) = \lambda_{fj}$ by Theorem 4.2 in Khalil (2002), and $D(t) \to Q\Lambda_f Q^\top$ as $t \to \infty$.

To show that the conditions (A3) in Theorem 1 are satisfied, note that $\dot{V}_j \leq 0 \Rightarrow V_j(t) \leq V_j(t_0)\ \forall t \geq t_0$. Therefore, $\rho(D(t)) \leq \max\{\rho(\Lambda(t_0)), \rho(\Lambda_f)\}\ \forall t \geq t_0$, and

$$\begin{aligned} |\ddot{\lambda}_j| &= |-(1/\tau)M\mathrm{sech}^2(Me_j)e_j^3 - (3/\tau)\tanh(Me_j)e_j^2|, \\ &\leq (1/\tau)(|Me_j^3| + |3e_j^2|), \quad (14a)\\ &\leq (1/\tau)(|M(2V_j(t_0))^{3/2}| + |6V_j(t_0)|), \quad (14b)\end{aligned}$$

for all $t \geq t_0$, implying that $\ddot{D}(t) = Q\ddot{\Lambda}(t)Q^\top$ is bounded in the spectral norm, and that $\dot{D}$ is uniformly continuous.

Finally, we consider the matrix inequality in (A3), where

$$0 \preceq 2kDD + \dot{D} \Leftrightarrow 0 \leq 2k\lambda_j^2 + \dot{\lambda}_j \quad \forall j = 1, ..., d, t \geq t_0. \quad (15)$$

Note that the conditions $0 \prec \Lambda_f$ and $0 \prec \Lambda(t_0)$ can be equivalently stated as $0 < \lambda_j(t_0)$, $0 < \lambda_{fj}$ $\forall j = 1, ..., d, t \geq t_0$. Now, if $\lambda_{fj} \geq \lambda_j$, $\dot{\lambda}_j \geq 0$ and the matrix inequalities in (A3) hold trivially at all times $t \geq t_0$. If instead $\lambda_{fj} < \lambda_j(t)$, then $\dot{\lambda}_j \leq 0\ \forall\ t \geq t_0$, and

$$\dot{\lambda}_j \geq -(1/\tau)(\lambda_{fj} - \lambda_j)^2 \geq -(1/\tau)\lambda_j^2 \geq -2k\lambda_j^2, \quad (16)$$

if $1 \leq 2k\tau$. Given this relation between k and τ, we note that $0 \leq 2k\lambda_j^2 + \dot{\lambda}_j$ for all initial conditions $\lambda_j(t_0) > 0$ and all terminal configurations $\lambda_{fj} > 0$, hence the assumptions in (A3) of Proposition 1 hold at all times by (15). □

3.2 Numerical simulations

To demonstrate the eigenvalue update law in Proposition 3, consider a single eigenvalue system (one of the eigenvalues of D) driven between various set-points as depicted in Fig. 1. Note that $\dot{\lambda}(t) \geq -(1/\tau)\lambda_j(t)^2$ at all times, and that the bound is close to tight on $t \in [9, 10]$, on which $\lambda_{fj} = 0 \Rightarrow e_j = -\lambda_j$.

With this eigenvalue update law, we can consider dynamically updating the input weighing matrix in a double integrator context. To illustrate this, consider a positional error defined in terms of a stationary reference position, $e = p - p_r$, and a potential function $V(e) = \|e\|_2^4 + \|e\|_2^2$,

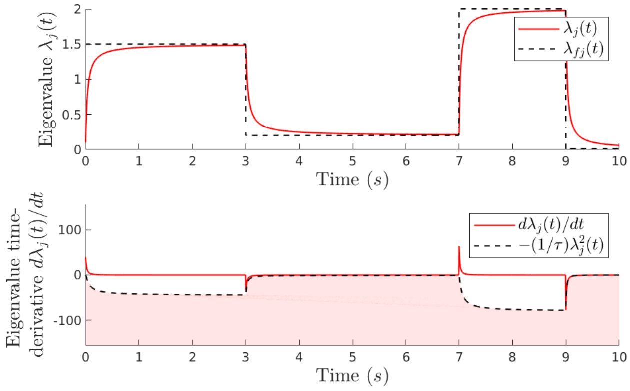

Fig. 1. *Top*: One of the eigenvalue trajectories, $\lambda_j(t)$, when using the general feedback in Proposition 3 and switching the reference eigenvalue λ_{fj} in time. *Bottom*: Illustration of the associated matrix inequality.

which meets the conditions (A1) of Theorem 1. Furthermore, the gradient is $\nabla_p V(e) = (4\|e\|_2^2 + 2)e$, whereby

$$\frac{d\nabla_p V(e)}{dt} = (4\|e\|_2^2 + 2)v + 8(e^\top v)e, \quad (17)$$

which is bounded if e and v are bounded. Hence, the gradient is uniformly continuous and condition (A2) in Theorem 1 is also satisfied. Finally, the third condition (A3) can be met by using any weight matrix D computed by the proposed eigenvalue update law in Proposition 3.

To demonstrate the input weighting, the initial conditions of the double integrator system are randomized. The eigenvalues defining $D(t_0)$ are randomized on the interval $\lambda_j(D(t_0)) \in (0, 1]$ and a random orthogonal basis Q is chosen. Given this initialization, the system response, Lyapunov function $W(e, v)$ as defined in (6) and the matrix elements of $D(t)$ are depicted in Fig. 2. The Lyapunov function is strictly decreasing and the system states converge to the equilibrium point $(p, v) = (p_r, 0)$.

4. NETWORKED DOUBLE-INTEGRATOR CONTROL SYSTEMS WITH WEIGHTED INPUTS

Next we demonstrate the practicality of the input weighting and the freedom it provides in a distributed controller design and networked control systems modelled by double-integrator dynamics. Two main application examples will be considered. The main results in the last section are first generalized to input-weighted distance-based formation control of multiple double integrator systems, where the time-varying weight matrix D can be used to locally modify the control law so as to incorporate other tasks such as obstacle avoidance. Then, the approach is generalized to the design of synchronizing a set of networked power generators with unknown time-varying inertia tensors.

4.1 Distance-based formation control with weighted inputs

The main result in Theorem 1 can be applied to distance-based formation control, where a set of n double-integrator agents, $\ddot{p}_i = u_i$, $i = 1, ..., n$, are to converge to a predefined formation shape defined by a set of desired distances d_{ij}, using only local information from neighboring agents. However, this assumes that the graph $\mathcal{G}$ defining these neighbor sets is undirected and rigid, c.f., Sun et al. (2017).

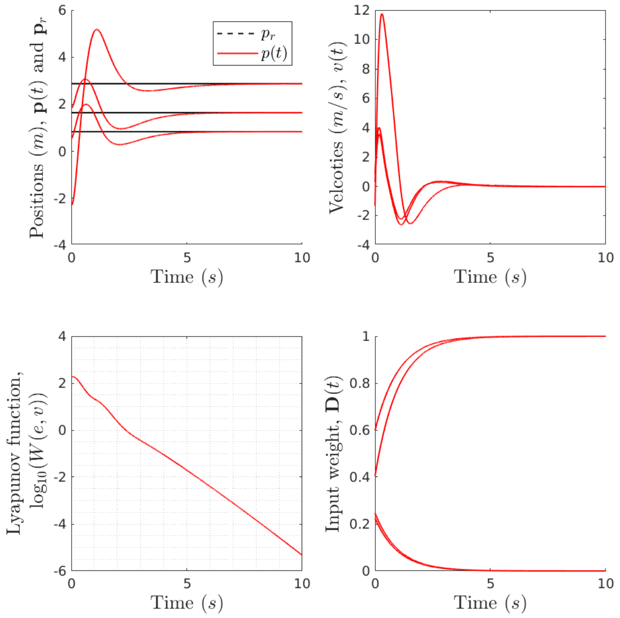

Fig. 2. *Top left*: Positional response *Top right*: Velocity response. *Bottom left*: Lyapunov function in the 10-logarithm. *Bottom right*: matrix elements of $D(t)$.

To see this, consider a set of m squared distance errors, $e_k = \|p_i - p_j\|^2 - d_{ij}^2$, for each edge $(i,j) \in \mathcal{E}$, and define $e = [e_1^\top, \cdots, e_m^\top]^\top \in \mathbb{R}^{dm}$. The potential function for formation shape stabilization is given by

$$V(e) = \frac{1}{4}\sum_{(i,j)\in\mathcal{E}} (\|p_i - p_j\|^2 - d_{ij}^2)^2 = \frac{1}{4}e^\top e. \quad (18)$$

In addition, define the concatenated vectors and matrix

$$p = [p_1^\top, \cdots, p_n^\top]^\top \in \mathbb{R}^{dn}, \quad (19a)$$
$$v = [v_1^\top, \cdots, v_n^\top]^\top \in \mathbb{R}^{dn}, \quad (19b)$$
$$u = [u_1^\top, \cdots, u_n^\top]^\top \in \mathbb{R}^{dn}, \quad (19c)$$
$$D = \text{blockdiag}(D_1, \cdots, D_n) \in \mathbb{R}^{dn\times dn}. \quad (19d)$$

Note that if $D(t) = Q\Lambda(t)Q^\top = \text{blockdiag}(D_i, \cdots, D_n)$, and each $D_i(t) = Q_i\Lambda_i(t)Q_i^\top$, then

$$Q = \text{blockdiag}(Q_1, \cdots, Q_n),$$
$$\Lambda = \text{blockdiag}(\Lambda_1, \cdots, \Lambda_n).$$

Now, with a feedback law $u(t) = -D(t)(\nabla_p V(e) + kv)$, the closed-loop double-integrator formation system is

$$\dot{p}(t) = v(t) \quad (20a)$$
$$\dot{v}(t) = -D(t)(\nabla_p V(e) + kv) \quad (20b)$$

It is clear that the time-derivative of the gradient of the potential function $d\nabla_p V(e)/dt = (1/2)e^\top \dot{e}$ is bounded, as e and v are bounded, hence the gradient is uniformly continuous in time, and (A2) in Proposition 1 holds.

Next, consider the matrix inequality condition in (A3) of Proposition 1. Since D admits a spectral decomposition,

$$0 \prec 2kDD + \dot{D} \Leftrightarrow 0 \prec 2k\Lambda\Lambda + \dot{\Lambda} \quad (21a)$$
$$\Leftrightarrow 0 \prec 2k\Lambda_i\Lambda_i + \dot{\Lambda}_i \;\forall i \in 1, \cdots, n \quad (21b)$$
$$\Leftrightarrow 0 \prec 2kD_iD_i + \dot{D}_i \;\forall i \in 1, \cdots, n \quad (21c)$$

and if each $D_i(t) = Q_i\Lambda_i(t)Q_i^\top$ for system i satisfies the matrix inequality in (A3), then D necessarily satisfies the same inequality. Furthermore, it is clear that $\rho(D) = \max_{i\in[1,n]}(\rho(D_i)) \leq C$ and $\dot{D}$ is uniformly continuous if $\dot{D}_i$ is uniformly continuous for all $i = 1, ..., n$. Consequently, if the conditions (A3) in Theorem 1 hold for each individual D_i, then they will necessarily hold for D.

As discussed in Deghat et al. (2015); Sun et al. (2017), the nominal double-integrator formation system (with $D_i = I_n$, $\forall i$) is locally asymptotically stable in the sense that $v \to 0$ and $e \to 0$ and all agents locally converge to a desired formation shape. However, we have shown that the conditions (A1)-(A2) in Theorem 1 are satisfied, and (A3) holds if satisfied for each individual $D_i(t)$. This leads to a more general statement summarized below.

Proposition 4. The desired equilibrium point $(e, v) = (0, 0)$ is asymptotically stable for the formation control system in (20) with time-varying inputs, provided each individual $D_i(t)$ satisfies assumptions (A3) in Theorem 1.

Remark 5. Note that the update law for system i becomes

$$u_i(t) = -D_i(t)(\nabla_{p_i} V(e(t)) + kv_i(t)), \quad (22)$$

and can be implemented in a decentralized fashion, permitting local updates of $D_i(t)$ using Proposition 3.

We now present a numerical example to illustrate the above result. Consider a 4-agent formation system with six edges, while the desired distances for the six edges are $d = [3, 4, 3, 4, 5, 5]^\top$. The control action is implemented in a distributed manner by (22). For simulation purposes we choose random Q_i, and the time-varying input gain matrix $D_i(t) = Q_i\text{diag}(\lambda_{i1}, \cdots, \lambda_{id})Q_i^\top$ is updated by $\lambda_{ij}(t) = 1 - e^{-t} + \lambda_{ij}(0)e^{-t}$ with random positive $0 < \lambda_{ij}(0) < 1$, which then satisfies the condition in (A3) of Theorem 1. As shown in Fig. 3, under the double-integrator formation systems with time-varying input, the target formation shape is asymptotically achieved and all distance errors converge to zero. Furthermore, in the second simulation depicted in Fig. 4, the weight associated with the control signal of system $i = 1$ is instead modified so as to approach a small positive value using the feedback in Proposition 3. This drastically changes the convergence behaviour, and the systems instead converge to the desired shape in a location closer to agent $i = 1$. Such a weight updating strategy can be used to achieve additional control objectives such as collision avoidance (for agents 1 and 2 in the example shown in Fig. 4) or adjusting the traveling distances between different agents.

4.2 Synchronization of power generators with unknown time-varying inertia tensors

Yet another interesting application example pertains to the synchronization of power generators with an unknown time-varying inertia $J_i(t) = J_i(t)^\top \succ 0$, and dynamics

$$\dot{p}_i(t) = v_i(t), \quad (23a)$$
$$J_i(t)\dot{v}_i(t) = u_i(t). \quad (23b)$$

In this example we will assume no knowledge of $J_i(t)$, apart from the existence of a constants $k, C_i > 0$, s.t.,

$$-2kI \prec \dot{J}_i(t), \qquad \sup_{t_0\leq t} \rho(J_i(t)) \leq C_i, \quad (24)$$

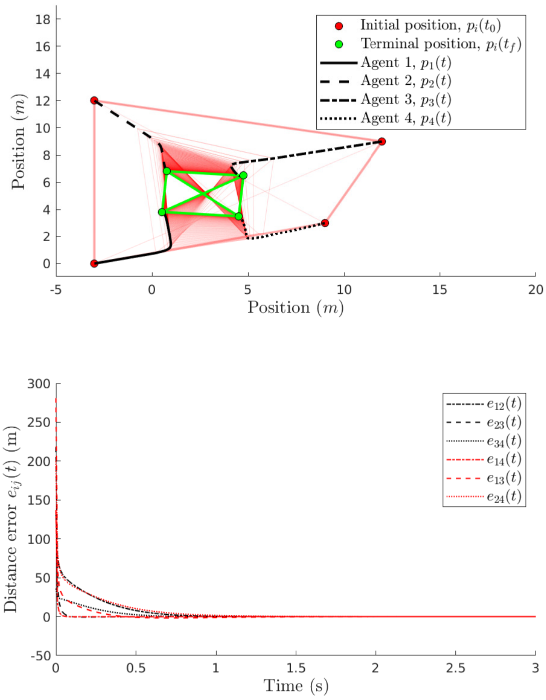

Fig. 3. Convergence of a 4-agent formation system modelled by double-integrator equations with sinusoidal time-varying weights acting on the input. *Top*: Positional trajectories. *Bottom*: Distance errors.

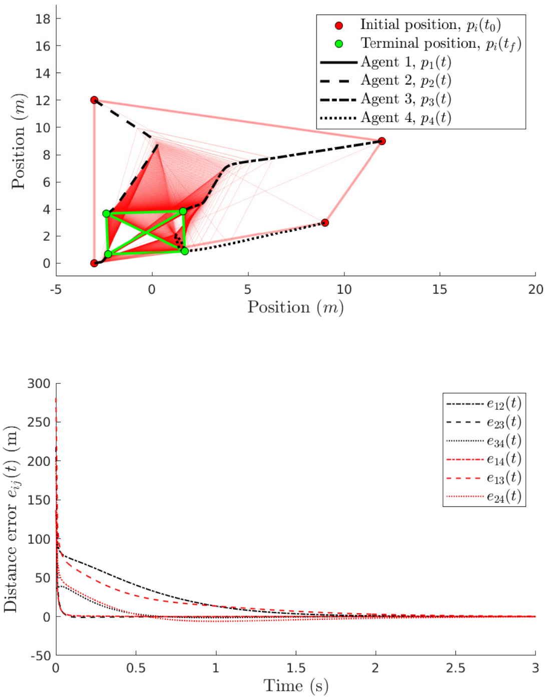

Fig. 4. Convergence of a 4-agent formation system where the weight D_1 associated with system $i = 1$ is adapted locally to become small in the spectral norm. *Top*: Positional trajectories. *Bottom*: Distance errors.

and that $\dot{J}_i(t)$ is uniformly continuous for all $i = 1, \cdots n$. Just as in the previous example, we let the systems be configured in $p_i \in \mathbb{R}^d$. The can be viewed as a power network system with linearized swing equations Machowski et al. (2011); Goldin (2013), in which the gradient vector term $\nabla_p V(e)$ is described by $\nabla_p V(e) = \sum_{j \in \mathcal{N}_i}(p_i - p_j)$. Given that the connectivity is described by a connected graph $\mathcal{G}$, our goal is to design a control law to reach state synchronization without knowing $J_i(t)$.

To make use of Theorem 1, we first concatenate the system state vectors and control signals as in (19a)-(19c), we let $e_k = p_i - p_j$, for each edge $(i, j) \in \mathcal{E}$, and define $e = [e_1^\top, \cdots, e_m^\top]^\top \in \mathbb{R}^{dm}$ with a configuration error defined as in (18). Similarly to (19c), we define matrix

$$J = \text{blockdiag}(J_1, \cdots, J_n) \in \mathbb{R}^{dn \times dn}. \tag{25}$$

Consider a feedback similar to that in the previous section,

$$u = -\nabla_p V(e) - kv. \tag{26}$$

The closed-loop system becomes

$$\dot{p}(t) = v(t),$$
$$\dot{v}(t) = J(t)^{-1}u(t) = -J(t)^{-1}(\nabla_p V(e) + kv),$$

which is recognized as the form in (5) but with $D(t) = J(t)^{-1}$. Furthermore, we see that if $-2kI \prec \dot{J}_i(t)$, then

$$-2kI \prec \dot{J}_i(t) \Leftrightarrow 0 \prec 2kD(t)D(t) + \dot{D}(t)$$

Uniform continuity of D follows from the uniform continuity of J, and since J is positive definite, D is necessarily bounded in the spectral norm. Therefore, the non-restrictive assumptions posed on J in (23a) are equivalent to D satisfying the conditions in (A3) of Theorem 1. As in the previous example, conditions (A1) and (A2) are met.

Proposition 6. The feedback law in (26) yields an asymptotically stable equilibrium point (state synchronization) in $(e, v) = (0, 0)$ for the double-integrator system (23a), without inertia needing to be known, provided (24) holds.

To illustrate this, consider a cyclic graph of $m = n = 10$ double integrator systems in $d = 3$ dimensional space (see Fig. 5), where each double integrator has an associated time-varying inertia tensor $J_i(t)$. The initial states of each integrator system are sampled from a unit Gaussian distribution, and the eigenvalues of the matrices $J_i(t)$ are randomized on the interval $[1/2, 1]$ and formed with a random time-invariant bases Q_i, also chosen at random. The gain is chosen as $k = 2$, and new reference eigenvalues λ_{fj} are chosen for the inertia tensor inverses $D(t) = J_i(t)^{-1}$ at the times $t_s = \{4, 8, 12\}$, and the inertia tensors are changed by application of the update law in Proposition 3 to $D(t) = J(t)^{-1}$, with smallest possible gain $\tau = (2k)^{-1}$, such that the condition (A3) is satisfied at all times. We stress that the inertia tensors are completely unknown to the systems, and that the double integrator feedback law is implemented in a distributed manner.

The system response is illustrated in Fig. 6. It is clear that the overall system reaches a stable synchronization equilibrium where $p_i = p_j$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, n$, where then $e = 0$ and $v = 0$, implying state synchronization is achieved. The Lyapunov function $W(e, v)$ is monotonically decreasing in time, despite the rapidly changing time-varying inertia tensor inverses.

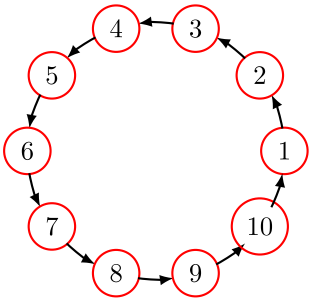

Fig. 5. Connectivity in the considered generator network.

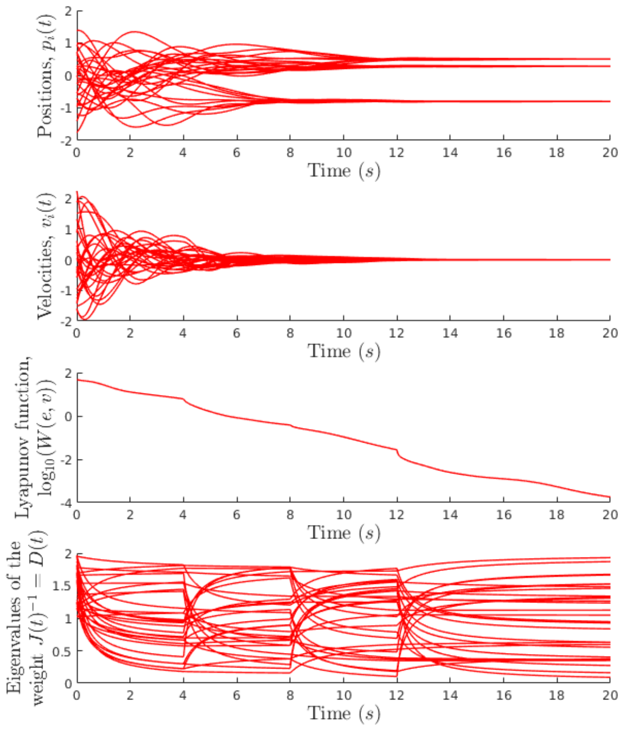

Fig. 6. *Top:* Positional and velocity response. *Bottom center:* The Lyapunov function $W(e, v)$ in the 10-logarithm, the system converges to a consensus where $e \equiv 0$. *Bottom:* The eigenvalues of each $D(t) = J(t)^{-1}$, here updated in time using Proposition 3 to satisfy assumption (A3) in Proposition 1 at all times.

5. CONCLUSION

In this paper we consider coordination control of double-integrator systems while the acceleration-driving input is weighted by time-varying matrices. The introduction of the time-varying input weight allows more freedoms in the control design, and can be used to locally and dynamically regulate transient behaviours by a time-varying gain matrix. We provide a sufficient condition on the time-varying matrix that ensures the asymptotic convergence of double-integrator systems under time-varying input. Typical examples that demonstrate the applications of the double-integrator systems in distributed control and networked systems are shown to support the results.

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