Finite-time Distributed Convex Optimization with Zero-Gradient-Sum Algorithms

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Abstract: This article considers the distributed finite-time optimization problem of multi-agent systems within the Zero-Gradient-Sum (ZGS) framework. We employ a distributed algorithm to drive the estimate of each agent to converge to the optimal solution of the global objective function, the sum of the local objectives. In a general case with non-quadratic local functions, we can obtain a finite-time convergence. Furthermore, when all the local cost functions are quadratic, the proposed algorithm can achieve a fixed-time result such that the upper bound of settling time can be estimated regardless of the initial conditions. Considering that the communication network may be affected by some external disturbances, we also extend to consider the case with switching topologies. Finally, the algorithms are demonstrated via an example simulation.

Keywords: Multi-agent system, convex optimization, finite-time stability, distributed control.

1. INTRODUCTION

With the development of Cyber-Physical Systems (CPS), distributed control strategies have been widely studied in various areas Knorn et al. (2015), such as formation control Cheng et al. (2019), energy dispatch Kong et al. (2019), distributed estimation Wang and Ren (2018), machine learning Boyd et al. (2010), and so on. In these applications, a multi-agent system may need to achieve some global common objectives during the tasks' progress, which can be described as an optimization problem. The traditional optimization methodology usually requires a central unit to process the global information and assign instructions to local units, which is not suitable for large-scale networked systems. In order to improve the robustness of the networked system and reduce the excessive computing and communication, it is imperative to study how to solve these problems in a distributed manner. In other words, the distributed optimization problem is an essential topic in networked systems.

In recent years, various distributed optimization protocols have been presented in the literature Nedic and Liu (2018); Yang et al. (2019); Xie et al. (2018); Li et al. (2019). Most of them can be divided into two categories, the discrete-time and continuous-time algorithms. In the discrete-time case, the major difference between the existing distributed algorithms is whether the step-sizes are diminishing or fixed. In general, the later ones can achieve a faster convergence rate than the former ones. For rigorous analysis with the well-developed continuous-time stability theory, the continuous-time distributed algorithms have been de-

signed for two types of optimization problems where the first-order gradient and second-order Hessian information are used, respectively. Based on the feedback control perspective, the authors in Wang and Elia (2011) propose a unified framework, which facilitates the analysis of given convex optimization problems. Following this direction, a series of distributed optimization algorithms have been developed Gharesifard and Cortes (2014); Kia et al. (2015); Li et al. (2018); Qiu et al. (2019). In addition to these first-order gradient-based algorithms, a few second-order Newton algorithms are proposed to establish the exponential stability in Varagnolo et al. (2016). When it comes to using the second-order information, the so-called Zero-Gradient-Sum (ZGS) algorithms provide many insights and new viewpoints of research Lu and Tang (2012). Compared to the first-order distributed algorithms, these second-order protocols result in a faster convergence rate by employing the Hessian information. However, most of the aforementioned distributed optimization algorithms can only steer to the optimal solutions either asymptotically or exponentially, which means that the optimization is solved over an infinite time horizon. Hence, it is critical to consider the research about the finite-time optimization strategy.

Except for achieving a fast convergence, the superfluous burden can be hugely reduced by using some finite-time protocols. Therefore, many techniques have been employed to address the finite-time, even fixed-time, optimization problems in discrete-time or continuous-time setting Yao et al. (2018); Mai and Abed (2018); Song and Chen (2016); Lin et al. (2017); Li et al. (2017); Chen and Li (2018); Ning et al. (2019); Feng et al. (2019). Inspired by the discontinuous finite-time consensus protocols, a large part of existing algorithms adopt the non-smooth

signum function which will result in undesired chattering behaviors. To overcome this drawback, the authors in Feng et al. (2019) utilize a smoothing factor to remove the chattering. On the other hand, the basically fixed-time consensus protocols proposed in Parsegov et al. (2013); Zuo and Tie (2016), which have been employed to deal with the fixed-time optimization problem Ning et al. (2019). However, the algorithms in Ning et al. (2019) may still suffer from the undesired chattering behaviors since a non-smooth signum function is contained. So far, the fixed-timed optimization schemes have not been fully explored in distributed systems.

This paper aims to design a finite-time distributed solution strategy for non-quadratic convex optimization problems. By combining the ZGS method and fixed-time consensus protocols, a set of distributed optimization algorithms are proposed over fixed and switching topologies. Compared with the existing literature, the main contributions include that the proposed algorithms can tackle a more generalized class of objective functions, and for the quadratic optimization problem, we can obtain a fixed-time convergence without depending on initial states of the system. Besides, if there exist uncertainties in a communication network, some of them can be modeled as the problem with switching topologies. Here, the finite-time algorithms can be extended to the case with switching topology straightly.

The rest of the paper is organized as following. Section 2 presents the notations, convex analysis, and some necessary theories used in this paper. The main results including the design of algorithm and convergence analysis are given in Section 3. The simulation examples are illustrated in Section 4. Finally, we offer the conclusion in Section 5.

2. PRELIMINARIES

2.1 Notations and Convex analysis

Let $\mathbf{R}^{m \times n}$ be the set of $m \times n$ dimensional real matrices. For a vector $x = [x_1, x_2, \ldots, x_m]^T \in \mathbf{R}^m$, we define the ρ-norm as $\|x\|_\rho = (|x_1|^\rho + |x_2|^\rho + \cdots + |x_m|^\rho)^{\frac{1}{\rho}}$ with $\rho > 0$. Given a symmetric matrix $A \in \mathbf{R}^{n \times n}$, A^T denotes its transpose, and $A > 0$ (or $A \geq 0$) means that A is a positive (or non-negative) definite matrix. $I_n \in \mathbf{R}^{n \times n}$ denotes the identity matrix of dimension $n \times n$. $\nabla f(\cdot)$ and $\nabla^2 f(\cdot)$, respectively, denote the gradient and Hessian matrices of the function $f(\cdot)$.

Next, we provide some properties of convex functions. A twice continuously differentiable function $f(\cdot) : \mathbf{R}^n \to \mathbf{R}$ is locally θ-strongly convex, if for any convex and compact set $\Omega \subset \mathbf{R}^n$, there exists a constant $\theta > 0$ such that the following equivalent conditions hold (Lu and Tang (2012)):

$$f(y) - f(z) - \nabla f(z)^T(y - z) \geq \frac{\theta}{2}\|y - z\|_2^2, \quad (1)$$
$$\forall y, z \in \Omega$$

$$(\nabla f(y) - \nabla f(z))^T(y - z) \geq \theta\|y - z\|_2^2, \ \forall y, z \in \Omega \quad (2)$$

$$\nabla^2 f(z) \geq \theta I_n, \ \forall z \in \Omega \quad (3)$$

where θ is called the convexity parameter, and $\nabla^2 f(z) \geq \theta I_n$ means that $\nabla^2 f(z) - \theta I_n \geq 0$. Finally, for any twice continuously differentiable function $f(\cdot) : \mathbf{R}^n \to \mathbf{R}$, any

convex set $\Omega \subset \mathbf{R}^n$ and any constant $\Theta > 0$, the following inequalities are equivalent:

$$f(y) - f(z) - \nabla f(z)^T(y - z) \leq \frac{\Theta}{2}\|y - z\|_2^2, \quad (4)$$
$$\forall y, z \in \Omega$$

$$(\nabla f(y) - \nabla f(z))^T(y - z) \leq \Theta\|y - z\|_2^2, \ \forall y, z \in \Omega \quad (5)$$

$$\nabla^2 f(z) \leq \Theta I_n. \ \forall z \in \Omega \quad (6)$$

2.2 Graph Theory and Stability Theory

In this paper, the communication network of a multi-agent system containing N nodes is encoded as a graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ with the node set $\mathcal{V} \triangleq \{v_1, \ldots, v_N\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. A directed edge from node v_i to node v_j is denoted by $(v_i, v_j) \in \mathcal{E}$. If all the channels in the network are bidirectional, then the graph is undirected, which means that $(v_i, v_j) \in \mathcal{E} \Leftrightarrow (v_j, v_i) \in \mathcal{E}$. A sequence of ordered edges $\{(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_s}, v_j)\}$ represent a directed path of $\mathcal{G}$ form v_i to v_j. An undirected graph $\mathcal{G}$ is connected if there exists a path between any two nodes. The graph $\mathcal{G}$ can be denoted by an adjacency matrix $\mathcal{A} \triangleq [a_{ij}] \in \mathbf{R}^{N \times N}$ with $a_{ii} = 0$, $a_{ij} = a_{ji} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Define the Laplacian matrix of graph $\mathcal{G}$ as $\mathcal{L} \triangleq [\mathcal{L}_{ij}] \in \mathbf{R}^{N \times N}$, where $\mathcal{L}_{ii} = \sum_{j=1}^N a_{ij}$ and $\mathcal{L}_{ij} = -a_{ij}, i \neq j$. The communication graph $\mathcal{G}$ is assumed to be undirected and connected throughout this paper.

Lemma 1. (Li and Duan (2014)). For an undirected connected graph, zero is a simple eigenvalue of $\mathcal{L}$. The corresponding eigenvector of the simple eigenvalue is the N dimension column vector $\mathbf{1}_N$ with all entries equal to 1. The smallest nonzero eigenvalue $\lambda_2(\mathcal{L})$ satisfies $\lambda_2(\mathcal{L}) = \min_{x \neq 0, \mathbf{1}_N^T x = 0} \frac{x^T \mathcal{L} x}{x^T x}$.

Next, we will introduce the definitions of finite-time and fixed-time stability and related theories.

Definition 1. (Zuo et al. (2018)). A closed-loop system is globally finite-time stable, if and only if the equilibrium $\bar{x}$ of the system is Lyapunov stable and there exists a positive definite function $T(x_0)$ called the settling time function such that, for all initial state $x_0 \in \mathbf{R} \setminus \{\bar{x}\}$, $\lim_{t \to T(x_0)} x(t, x_0) = \bar{x}$ and $x(t, x_0) = \bar{x}, \forall t > T(x_0)$. Furthermore, if the settling time function $T(x_0)$ is bounded by a real number $T_{\max} > 0, \forall x_0 \in \mathbf{R}$, then the closed-loop system is fixed-time stable.

Lemma 2. (Parsegov et al. (2013)). Given a differential equation

$$\dot{v} = -\alpha v^{1-\frac{1}{\gamma}} - \beta v^{1+\frac{1}{\gamma}}, \ v(0) = v_0 \quad (7)$$

where $v \in \mathbf{R}$ and v_0 denotes the initial state, $\alpha, \beta > 0$, and $\gamma > 1$. Then the equilibrium is fixed-time stable for (7) and the following upper bound of the settling time T holds,

$$T(v_0) \leq T_{\max} = \frac{\pi\gamma}{2\sqrt{\alpha\beta}}. \quad (8)$$

Lemma 3. (Chen and Li (2018)). Let $z_1, z_2, \ldots z_N \geq 0$. Then

$$\sum_{i=1}^N z_i^\varepsilon \geq (\sum_{i=1}^N z_i)^\varepsilon, \text{ if } 0 < \varepsilon \leq 1,$$

$$\sum_{i=1}^N z_i^\varepsilon \geq N^{1-\varepsilon}(\sum_{i=1}^N z_i)^\varepsilon, \text{ if } 1 < \varepsilon < \infty.$$

3. MAIN RESULTS

Consider a multi-agent system where each agent evolves according to the dynamics of

$$\dot{x}_i = u_i, \ v_i \in \mathcal{V}, \tag{9}$$

where $x_i \in \mathbf{R}^n$ is agent v_i's estimate of the unique global minimizer x^* and $u_i \in \mathbf{R}^n$ is the control input to be designed. Specifically, the objective of this dynamical system is to solve the convex optimization problem

$$\text{minimize } f(x) = \sum_{i=1}^{N} f_i(x), \tag{10}$$

only with the local interaction and information. The local cost function of agent v_i is $f_i(x) : \mathbf{R}^n \to \mathbf{R}$, which only can be accessed by agent v_i itself. For the cost function, we have the following assumption:

Assumption 1. For each $v_i \in \mathcal{V}$, the local cost function $f_i(\cdot)$ is twice continuously differentiable, strongly convex with convexity parameter $\theta_i > 0$, and has a locally Lipschitz Hessian $\nabla^2 f_i$.

Hence, for each $v_i \in \mathcal{V}$, there exists a unique $x_i^* \in \mathbf{R}^n$ such that $f_i(x_i^*) \leq f_i(x), \ \forall x \in \mathbf{R}^n$ and $\nabla f_i(x_i^*) = 0$. And then, we have that f has a unique minimizer $x^* \in \mathbf{R}^n$, so that problem (10) is well-posed:

Proposition 1. (Lu and Tang (2012)). Under Assumption 1, $\forall x \in \mathbf{R}^n$, there exists a unique x^* such that $f(x^*) \leq f(x)$, and $\nabla f(x^*) = 0$.

We now ready to present a distributed optimization algorithm as

$$u_i = -\alpha_1 (\nabla^2 f_i(x_i))^{-1} \sum_{j=1}^{N} a_{ij}(x_i - x_j)^{1 - \frac{q}{p}} \tag{11a}$$
$$\quad -\alpha_2 (\nabla^2 f_i(x_i))^{-1} \sum_{j=1}^{N} a_{ij}(x_i - x_j)^{1 + \frac{q}{p}},$$
$$x_i(0) = x_i^*, \tag{11b}$$

where α_1 and α_2 are constant gains and the odd integer p, the even integer q satisfy $p > q > 0$. Since x_i and x_j are vectors in $\mathbf{R}^n$, the operations $(x_i - x_j)^{1 - \frac{q}{p}}$ and $(x_i - x_j)^{1 + \frac{q}{p}}$ are element-wise. In light of Assumption 1 and (11b), we have

$$\sum_{i=1}^{N} \nabla f_i(x_i^*) = 0. \tag{12}$$

By taking the time derivative of $\sum_{i=1}^{N} \nabla f_i(x_i(t))$ and using the facts that $a_{ij}(x_i - x_j)^{1 - \frac{q}{p}} = -a_{ji}(x_j - x_i)^{1 - \frac{q}{p}}$ and $a_{ij}(x_i - x_j)^{1 + \frac{q}{p}} = -a_{ji}(x_j - x_i)^{1 + \frac{q}{p}}$, we can achieve that

$$\sum_{i=1}^{N} \nabla^2 f_i(x_i(t))\dot{x}_i(t) = 0, \tag{13}$$

which means that $\sum_{i=1}^{N} \nabla f_i(x_i(t))$ is constant for $t \geq 0$. Together with $\sum_{i=1}^{N} \nabla f_i(x_i^*) = 0$, we can obtain $\sum_{i=1}^{N} \nabla f_i(x_i(t)) = 0$ for $t \geq 0$. Therefore, the first and the third conditions proposed in (Lu and Tang (2012)) are satisfied. The second condition will be discussed in the proof of the following theorem.

Theorem 1. Suppose that Assumption 1 holds. The proposed distributed algorithm in (11) can enable the agents

to converge to the optimal solution of Problem (10) in a finite time, i.e., $\lim_{t \to T_1} x_i = x^*, \ \forall v_i \in \mathcal{V}$, where

$$T_1 \leq T_{\max} = \frac{2\pi p N^{\frac{q}{4p}}}{q\sqrt{\alpha_1\alpha_2}(\frac{4\lambda_2(\mathcal{L}_1)}{\Theta_{\max}})^{\frac{1}{2} - \frac{q}{4p}}(\frac{4\lambda_2(\mathcal{L}_2)}{\Theta_{\max}})^{\frac{1}{2} + \frac{q}{4p}}},$$

and the constant $\Theta_{\max}$ will be determined in the proof.
Proof 1. Construct a positive-definite function as

$$V_1(x(t)) = \sum_{i=1}^{N} (f_i(x^*) - f_i(x_i) - \nabla f_i(x_i)^T(x^* - x_i)), \tag{14}$$

Since Assumption 1 is satisfied, it is not difficult to obtain that

$$V_1(x(t)) \geq \sum_{i=1}^{N} \frac{\theta_i}{2}\|x^* - x_i\|_2^2, \tag{15}$$

The derivative of V_1 along (11) is

$$\dot{V}_1(x(t))$$
$$= \sum_{i=1}^{N} (x_i - x^*)^T \nabla^2 f_i(x_i)\dot{x}_i$$
$$= -\alpha_1 \sum_{i=1}^{N} (x_i - x^*)^T \sum_{j=1}^{N} a_{ij}(x_i - x_j)^{1 - \frac{q}{p}}$$
$$\quad -\alpha_2 \sum_{i=1}^{N} (x_i - x^*)^T \sum_{j=1}^{N} a_{ij}(x_i - x_j)^{1 + \frac{q}{p}} \tag{16}$$
$$= -\alpha_1 \sum_{i=1}^{N} \tilde{x}_i^T \sum_{j=1}^{N} a_{ij}(\tilde{x}_i - \tilde{x}_j)^{1 - \frac{q}{p}}$$
$$\quad -\alpha_2 \sum_{i=1}^{N} \tilde{x}_i \sum_{j=1}^{N} a_{ij}(\tilde{x}_i - \tilde{x}_j)^{1 + \frac{q}{p}}$$

where we replace $x_i - x^*$ with $\tilde{x}_i$. Then, in light of the fact that $\tilde{x}_i - \tilde{x}_j = x_i - x_j$, we can obtain the last equation in (16).

Due to the symmetry of the graph, the first term on the right hand side of (16) can be written as

$$\alpha_1 \sum_{i=1}^{N} \tilde{x}_i^T \sum_{j=1}^{N} a_{ij}(\tilde{x}_i - \tilde{x}_j)^{1 - \frac{q}{p}}$$
$$= \frac{\alpha_1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{x}_i^T a_{ij}(\tilde{x}_i - \tilde{x}_j)^{1 - \frac{q}{p}}$$
$$\quad -\frac{\alpha_1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{x}_j^T a_{ij}(\tilde{x}_i - \tilde{x}_j)^{1 - \frac{q}{p}} \tag{17}$$
$$= \frac{\alpha_1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(\tilde{x}_i - \tilde{x}_j)^{2 - \frac{q}{p}}.$$

Following similar lines, we have

$$\alpha_2 \sum_{i=1}^{N} \tilde{x}_i^T \sum_{j=1}^{N} a_{ij}(\tilde{x}_i - \tilde{x}_j)^{1 + \frac{q}{p}}$$
$$= \frac{\alpha_2}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(\tilde{x}_i - \tilde{x}_j)^{2 + \frac{q}{p}}. \tag{18}$$

Substituting (17) and (18) into (16) yields

$$\dot{V}_1(x(t)) \tag{19}$$

$$= -\frac{\alpha_1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (\tilde{x}_i - \tilde{x}_j)^{2-\frac{q}{p}}$$

$$-\frac{\alpha_2}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (\tilde{x}_i - \tilde{x}_j)^{2+\frac{q}{p}}$$

$$= -\frac{\alpha_1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (a_{ij}^{\frac{2p}{2p-q}} (\tilde{x}_i - \tilde{x}_j)^2)^{1-\frac{q}{2p}}$$

$$-\frac{\alpha_2}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (a_{ij}^{\frac{2p}{2p+q}} (\tilde{x}_i - \tilde{x}_j)^2)^{1+\frac{q}{2p}}$$

$$\leq -\frac{\alpha_1}{2} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^{\frac{2p}{2p-q}} (\tilde{x}_i - \tilde{x}_j)^2 \right)^{1-\frac{q}{2p}}$$

$$-\frac{\alpha_2}{2} N^{-\frac{q}{2p}} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^{\frac{2p}{2p+q}} (\tilde{x}_i - \tilde{x}_j)^2 \right)^{1+\frac{q}{2p}}.$$

Here, we use Lemma 3 to get the inequality. Letting the consensus error be $\xi_i(t) = x_i(t) - \frac{1}{N} \sum_{j=1}^{N} x_j(t) = x_i(t) - \zeta(t)$, then we can obtain that $x_i(t) - x_j(t) = \xi_i(t) - \xi_j(t)$. Hence, we have

$$L_1 = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^{\frac{2p}{2p-q}} (x_i - x_j)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^{\frac{2p}{2p-q}} (\xi_i - \xi_j)^2,$$

and

$$L_2 = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^{\frac{2p}{2p+q}} (x_i - x_j)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^{\frac{2p}{2p+q}} (\xi_i - \xi_j)^2.$$

By defining $\xi = [\xi_1^T, \xi_2^T, \ldots, \xi_N^T]^T$, we can express $L_1 = 2\xi^T(\mathcal{L}_1 \otimes I_n)\xi$ and $L_2 = 2\xi^T(\mathcal{L}_2 \otimes I_n)\xi$ in a compact form. According to Lemma 1, we have $L_1 \geq 2\lambda_2(\mathcal{L}_1)\xi^T\xi$ and $L_2 \geq 2\lambda_2(\mathcal{L}_2)\xi^T\xi$. Thus,

$$\dot{V}_1(x(t)) \leq -\frac{\alpha_1}{2} (2\lambda_2(\mathcal{L}_1)\xi^T\xi)^{1-\frac{q}{2p}}$$
$$-\frac{\alpha_2}{2} N^{-\frac{q}{2p}} (2\lambda_2(\mathcal{L}_2)\xi^T\xi)^{1+\frac{q}{2p}}. \tag{20}$$

Now, it follows from (14) and (20) that the set $\mathcal{C}_i = \{x_i \in \mathbf{R}^n | f_i(x^*) - f_i(x_i) - \nabla f_i(x_i)^T(x^* - x_i) \leq V_1(x(0))\}$ is nonempty and invariant. Besides, based on Assumption 1, we know that $\mathcal{C}_i$ is compact. Next define a convex hull $\mathcal{C} = \text{conv} \cup_{v_i \in \mathcal{V}} \mathcal{C}_i$. Since the set $\mathcal{C}$ is compact and $x_i(t) \in \mathcal{C}$, $\forall v_i \in \mathcal{V}$. Then, again from Assumption 1, we know that there exists a constant $\Theta_i \geq \theta_i$ such that

$$\nabla^2 f_i(x_i) \leq \Theta_i I_n, \ \forall x_i(t) \in \mathcal{C}. \tag{21}$$

Note that $\zeta(t) \in \mathcal{C}$ for $\mathcal{C}$ is convex. This, together with the fact that x^* is the unique solution to the optimization problem (10), implies that $\sum_{i=1}^{N} f_i(x^*) \leq \sum_{i=1}^{N} f_i(\zeta(t))$. Thus, it follows from (13) and (14) that

$$V_1(x(t)) \leq \sum_{i=1}^{N} f_i(\zeta(t)) - f_i(x_i(t))$$
$$- \nabla f_i(x_i(t))^T(\zeta(t) - x_i(t)).$$

This together with (4), (6) and (21) implies that for all $t \geq 0$,

$$V_1(x(t)) \leq \sum_{i=1}^{N} \frac{\Theta_i}{2} \|\zeta(t) - x_i(t)\|_2^2 \leq \frac{\Theta_{\max}}{2} \xi^T \xi. \tag{22}$$

where $\Theta_{\max} = \max\{\Theta_i, \ v_i \in \mathcal{V}\}$. Therefore, we have

$$\dot{V}_1(x(t))$$

$$\leq -\frac{\alpha_1}{2} \left(\frac{2\lambda_2(\mathcal{L}_1)\xi^T\xi}{(\Theta_{\max}/2)\xi^T\xi} V_1 \right)^{1-\frac{q}{2p}}$$

$$-\frac{\alpha_2}{2} N^{-\frac{q}{2p}} \left(\frac{2\lambda_2(\mathcal{L}_2)\xi^T\xi}{(\Theta_{\max}/2)\xi^T\xi} V_1 \right)^{1+\frac{q}{2p}}$$

$$= -\frac{\alpha_1}{2} \left(\frac{4\lambda_2(\mathcal{L}_1)}{\Theta_{\max}} \right)^{1-\frac{q}{2p}} V_1^{1-\frac{q}{2p}}$$

$$-\frac{\alpha_2}{2} N^{-\frac{q}{2p}} \left(\frac{4\lambda_2(\mathcal{L}_2)}{\Theta_{\max}} \right)^{1+\frac{q}{2p}} V_1^{1+\frac{q}{2p}}. \tag{23}$$

In light of Lemma 2 and the comparison principle, it is straightforward to claim that $V_1 = 0$ with a settling time T_1. Observe that the constants $\Theta_{\max}$ in $T_{\max}$ indeed depends on the initial state $x(0)$ via the sets $\mathcal{C}$ and $\mathcal{C}_i$. Although, we use a fixed-time stability theory here. Yet, strictly speaking, the proposed algorithm achieves a finite-time convergence. In particular, the real convergence time T_1 will not exceed to $T_{\max}$, i.e., $T_1 \leq T_{\max}$. Additionally, since $V(t) = 0$ for $t \geq T_1$, together with $V(x(t)) \geq \sum_{i=1}^{N} \frac{\theta_i}{2} \|x^* - x_i\|^2$, we can obtain that $\lim_{t \to T_1} x_i = x^*$, $\forall v_i \in \mathcal{V}$. This is consistent with Theorem 1 in (Lu and Tang (2012)).

Remark 1. Here we propose a finite-time algorithm for the distributed convex optimization problem (10). Based on the framework of the ZGS algorithm, we rigorously prove the triple conditions provided in (Lu and Tang (2012)). We must admit that the limitations of the family of ZGS algorithms still exist, such as the initial condition requires to solve local problems to obtain x_i^*. The computing cost of inverting the Hessian in the algorithm is high. These problems are worthy of further study.

Remark 2. In theory, one can reduce the upper bound of settling time to be arbitrarily small by increasing α_1 and α_2, which requires a huge control input energy. In practice, the upper limit of the controller's capability should be considered. Besides, although we insert a classic fixed-time consensus protocol into an optimization algorithm, the initial states of the system still affect the convergence time. Hence, in a general case, we only can achieve a finite-time result. However, when all the local cost functions are quadratic functions, the algorithm can be slightly modified to achieve a fixed-time convergence.

For a special case where all the local cost functions $f_i(x_i) = x_i^T A_i x_i + B_i^T x_i + C_i$ are quadratic functions with $A_i = A_i^T \in \mathbf{R}^{n \times n} (A_i > 0)$, $B_i \in \mathbf{R}^n$, and $C_i \in \mathbf{R}$, the controller can be reduced to

$$u_i = -\alpha_1 A_i^{-1} \sum_{j=1}^{N} a_{ij} (x_i - x_j)^{1-\frac{q}{p}}$$
$$-\alpha_2 A_i^{-1} \sum_{j=1}^{N} a_{ij} (x_i - x_j)^{1+\frac{q}{p}}, \tag{24a}$$

$$x_i(0) = x_i^*. \tag{24b}$$

Corollary 1. Suppose that Assumption 1 holds and all the local cost functions are quadratic functions. The proposed

distributed algorithm in (24) can enable the agents to converge to the optimal solution of Problem (10) in a fixed time, i.e., $\lim_{t \to T_2} x_i = x^*$, $\forall v_i \in \mathcal{V}$, where

$$T_2 \leq T_{\max} = \frac{2\pi p N^{\frac{q}{4p}}}{q\sqrt{\alpha_1 \alpha_2}\left(\frac{4\lambda_2(\mathcal{L}_1)}{\Lambda_{\max}}\right)^{\frac{1}{2}-\frac{q}{4p}}\left(\frac{4\lambda_2(\mathcal{L}_2)}{\Lambda_{\max}}\right)^{\frac{1}{2}+\frac{q}{4p}}},$$

where $\Lambda_{\max} = \max\{\Lambda_i, v_i \in \mathcal{V}\}$ and Λ_i can be taken as the largest eigenvalues of A_i.

Remark 3. In this case, we utilize the property of each local quadratic function to characterize the upper bound of the settling time. Hence, the dependency on the initial states of the closed-loop system is removed, which can straightly lead to a fixed-time result. In addition, the proposed algorithm in this paper can avoid the undesired chattering behaviors caused by the discontinuous controller Ning et al. (2019).

Consider a multi-agent system that communicates with each other over a wireless network. Due to the existence of some interferences and obstructions in a real environment, it is inevitable that some of the existing connected links may fail aperiodically or permanently. The graph property at any particular time, $\mathcal{G}(t) = \mathcal{G}_{\delta(t)}$, is denoted by a switching signal $\delta(t) : \mathbf{R}_+^m \to \Delta$, where a finite set Δ contains the indexes associated to specific connected graph. We assume that the communication graph changes from one to another indexed in Δ over time, and the graph is fixed between any two sequential switches. Hence, the Laplacian matrix of graph $\mathcal{G}([a_{ij}^{\frac{2p}{2p-q}}(t)])$ and $\mathcal{G}([a_{ij}^{\frac{2p}{2p+q}}(t)])$ can be denoted by $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ at time t. Furthermore, we can define $\underline{\lambda}_2(\mathcal{L}_1) = \min\{\lambda_2(\mathcal{L}_1(t))\}$ and $\underline{\lambda}_2(\mathcal{L}_2) = \min\{\lambda_2(\mathcal{L}_2(t))\}$. Since the expression of V is independent from the topology, so it is ready to extend the fixed topology results to the switching case.

Theorem 2. For a switching networked system, the communication graph is switching among Δ, i.e., $\mathcal{G}(t) = \mathcal{G}_{\delta(t)}$, and Assumption 1 holds. The proposed distributed algorithm in (11) can enable the agents to converge to the optimal solution of Problem (10) in a finite time, i.e., $\lim_{t \to T_3} x_i = x^*$, $\forall v_i \in \mathcal{V}$, where

$$T_3 = \frac{2\pi p N^{\frac{q}{4p}}}{q\sqrt{\alpha_1 \alpha_2}\left(\frac{4\underline{\lambda}_2(\mathcal{L}_1)}{\Theta_{\max}}\right)^{\frac{1}{2}-\frac{q}{4p}}\left(\frac{4\underline{\lambda}_2(\mathcal{L}_2)}{\Theta_{\max}}\right)^{\frac{1}{2}+\frac{q}{4p}}}. \tag{25}$$

Proof 2. At time t, $\delta(t) \in \Delta$, by calculating the derivative of V, we have

$$\dot{V}(x(t))$$
$$\leq -\frac{\alpha_1}{2}\left(\frac{4\underline{\lambda}_2(\mathcal{L}_1)}{\Theta_{\max}}\right)^{1-\frac{q}{2p}} V^{1-\frac{q}{2p}} \tag{26}$$
$$-\frac{\alpha_2}{2} N^{-\frac{q}{2p}}\left(\frac{4\underline{\lambda}_2(\mathcal{L}_2)}{\Theta_{\max}}\right)^{1+\frac{q}{2p}} V^{1+\frac{q}{2p}}.$$

Obviously, $\forall \delta(t) \in \Delta$ (26) holds. In fact, V serves as a common Lyapunov function for any $\delta(t) \in \Delta$. Therefore, the stability of (9) is guaranteed under the switching topology when using protocol (11). Since the rest analysis are the same as the proof in Theorem 1, we omit it here.

4. SIMULATION

This example presents a multi-agent system where 6 agents cooperate to solve a finite-time distributed optimization problem with non-quadratic objective functions: $f_i = \frac{i}{2}(x - \frac{1}{6-i})^2$ for $i = 1, 2$, $f_i = \frac{2}{3}(x - \frac{i}{3})^2 + \frac{3}{4}(x - \frac{i}{3})^4$ for $i = 3, 4$, $f_i = \frac{4}{5}(x - \frac{i}{4})^2 + \frac{5}{6}(x - \frac{i}{4})^6$ for $i = 5, 6$. where $x \in \mathbf{R}$ denotes the global variable. Hence, it can be derived that the optimal value of cost function $\sum_{i=1}^N f_i(x)$ is given by 0.967 via a centralized calculation method.

In this simulation, each agent estimates this global optimal solution based on the local information they can accessed. The communication graph is depicted in Fig. 1.
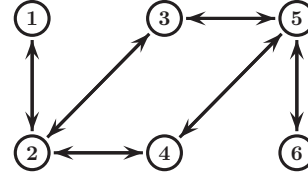

Fig. 1. The interaction graph between six agents.

The parameters of the proposed algorithm (11) are selected as $\alpha_1 = \alpha_2 = 2$, $q = 2$ and $p = 3$. As a comparison, we also process a standard algorithm

$$u_i = -\alpha_3(\nabla^2 f_i(x_i))^{-1} \sum_{j=1}^N a_{ij}(x_i - x_j),$$

where $\alpha_3 = 2$. The simulation result Fig. 2 demonstrates that the estimate of each agent converges to the optimal solution (dotted line) under both the finite-time algorithm (solid line) and standard algorithm (dotted line). The proposed algorithm (11) can achieve the optimal solution in a finite time while the trajectories with the standard algorithm are asymptotically stable.
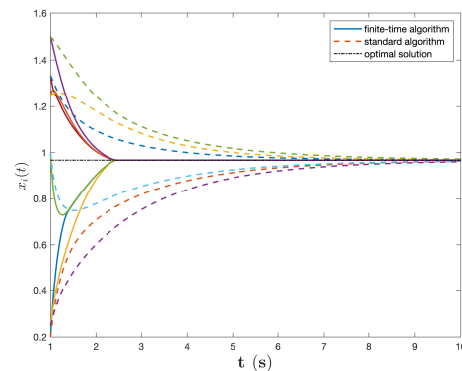

Fig. 2. State trajectories of six agents.

5. CONCLUSION

This paper systematically studies a distributed algorithm for a set of convex optimization problems. Combining with the ZGS strategy, a continuous controller is designed to seek the optimal solution of the global objective in finite-time. Furthermore, when all the local cost functions are quadratic functions, the proposed algorithm can achieve a fixed-time convergence such that the upper bound of settling time can be estimated regardless of the initial conditions. Besides, this algorithm is extended to the multi-agent systems with switching topology. Future work will explore the distributed optimization problem with time-varying cost functions and external interferences.

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