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Results on distributed state estimation for LTI systems facing communication failures

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Abstract: We address distributed estimation of the state of a linear plant by a set of agents. The problem is cast in a setting where the communication capabilities of an agent might be deactivated from time to time, due to failures in the communication devices or malicious attacks. An observer architecture is proposed to achieve our estimation goal, based on a multi-hop subspace decomposition. Uniform exponential convergence to zero of the estimation errors is proven in the presence of communication failures, under a persistence of excitation assumption. Finally, the observer performance is evaluated in simulation, showing the merits of the proposed method and suggesting directions for future developments.

Keywords: Distributed estimation, Link failures, Multi-agent systems, Linear time-invariant systems, Switching topologies.

1. INTRODUCTION

Traditional control systems consist of centralized controllers with colocated actuators and sensors. Modern scalable solutions, however, often involve Networked Control Systems (NCS) architectures with distributed sensing and controlling nodes. Such NCS solutions pose new scientific challenges Zhang et al. (2015); Ge et al. (2017). Among these challenges, distributed monitoring and estimation is characterized by a network of agents sharing (partial) information to accomplish a collective plant state estimation goal (see the recent survey in Rego et al. (2019)).

When considering perturbed systems, there are mainly three families of distributed observers, namely, distributed Kalman, H_∞ and set-membership filters, each one valid and optimized for specific models of disturbances and noises. Distributed Kalman filters (DKF) (first presented in Olfati-Saber (2007)) provide an optimal state estimation when the system model and the measurements of the agents are affected by Gaussian noises. However, an accurate model of these noise distributions is needed and sometimes this is difficult to obtain. The H_∞ filtering theory is used to develop distributed estimators providing state estimates with guaranteed performance. This strategy have been successfully applied in Ugrinovskii (2011) and Shen et al. (2010). Usually, distributed H_∞ filters rely on costly LMI design methods. Finally, set-membership approaches aim at finding a compact set where the plant state is certainly confined, see Orihuela et al. (2018); Wang et al. (2017). They are conservative approaches that are adequate when the exogenous signals satisfy known bounds.

Regarding unperturbed and noiseless systems, different modifications of distributed Luenberger observers have been proposed. In pursuing a decentralized design with minimum information, the authors of Park and Martins (2017) use state augmentations. With similar objectives, but using subspace decompositions (observable and unobservable modes), recent results have been presented in del Nozal et al. (2019); Kim et al. (2016); Mitra and Sundaram (2018). These approaches conveniently provide necessary and sufficient design conditions, based on network-aware detectability properties, and exploit those properties in the observer.

The literature of distributed observers dealing with link failures and communication losses is less dense. The communication failures are modeled with different methods. For instance, in Ugrinovskii (2013), Markov processes are employed to model random communication topologies. However, the local mode information of the entire network topology is non-Markovian, which complicates the problem solvability. To overcome this difficulty, Ugrinovskii (2013) emploies a two-step design procedure. The corresponding solution requires solving linear matrix inequalities subject to rank constraints, which are generally difficult. A different approach to model the communication failures can be found in Liu et al. (2017) where Bernoulli variables are used. Liu et al. (2017) introduces a weighted matrix in the consensus steps in order to implement distributed filtering. In addition, boundedness properties are thoroughly investigated, using statistic information of the random link failures. Regarding the strategy used to deal with the distributed estimation problem two main approaches can be found in the literature. On the one hand, the use of $\mathcal{H}_\infty$ strategies as discussed in Yan et al. (2017) and Yu et al. (2013). In Yan et al. (2017), neural networks are used to estimate the system state using learning methods for the corresponding matrices. Instead, Yu et al. (2013) designs a filter on each node in the sensor network ensuring that the dynamics of the filtering error is mean-square stable and the prescribed average $\mathcal{H}_\infty$ performance constraint is met. On the other hand, the behaviour of the Kalman filter dealing with communication problems has been also studied. In Battilotti et al. (2018) a failure detection device is introduced in every agent to detect link failures in the network at the receiving side. In addition, by using the maximum a posteriori probability decision rule, the authors propose a method to identify online the generally correlated multiple-valued stochastic output delay which guarantees (with some approximation) the minimum probability of error, given the available observations. Finally, the algorithm presented in Alonso-Román and Beferull-Lozano (2016) provides unbiased estimations when the steady-state value of the average consensus process becomes a random variable.

Within this setting, we focus here on distributed estimation in the presence of communication losses, and provide:
• A distributed observer structure based on a multi-hop decomposition, which decomposes the state space in the observable subspace of each agent and the innovation introduced by the neighbors at each hop.
• A sufficient condition on the distributed observer gains ensuring uniform global exponential stability of the error dynamics over all possible communication losses satisfying a suitable persistence of excitation assumption.
• A set of assumptions (some necessary and some sufficient) under which it is always possible to design the gains of the distributed observers in order to meet the above mentioned sufficient conditions.

This paper is organized as follows. Section 2 states the main problem and presents the necessary assumptions. Section 4 presents the proposed observation structure and the main results of the paper concerning stability and feasibility. Section 5 shows the observer performance in simulations. Finally, conclusions are drawn in Section 6.

Notation. A graph is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ comprising a set $\mathcal{V} = \{1, 2, \ldots, p\}$ of *vertices* or *agents*, and a set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of *edges* or *links*. A *directed graph* is a graph in which edges have orientations, so that if $(j, i) \in \mathcal{E}$, then agent i obtains information from agent j. A directed path from node i_1 to node i_k is a sequence of edges such as (i_1, i_2), $(i_2, i_3), \ldots, (i_{k-1}, i_k)$ in a directed graph. The *neighborhood* of i, $\mathcal{N}_i \triangleq \{j : (j, i) \in \mathcal{E}\}$, is defined as the set of nodes with edges incoming to node i. Given $\rho \in \mathbb{Z}_+$, the *ρ-hop reachable set* of i, $\mathcal{N}_{i,\rho}$, is defined as the set of nodes with a direct path to i involving ρ edges. Note that the 1-hop reachable set of i corresponds to the neighborhood of i and the 0-hop reachable set of i matches with i.

Operators $\mathrm{col}(\cdot,\cdot), \mathrm{row}(\cdot,\cdot)$ stacks subsequent matrices into a column/row vector, e.g. for A and B of appropriate dimensions, $\mathrm{col}(A, B) = [A^\top \; B^\top]^\top$ and $\mathrm{row}(A, B) = [A \; B]$. $|x|$ is the Euclidean norm of vector x. $||A||$ stands for the induced matrix norm of matrix A.

2. PROBLEM STATEMENT

Consider a set of agents $\mathcal{V} = \{1, 2, \ldots, p\}$ intending to distributedly estimate the state of the discrete-time LTI system

$$x(k+1) = Ax(k), \tag{1}$$
$$y_i(k) = C_i x(k), \quad \forall i \in \mathcal{V}, \tag{2}$$

where x is the state vector, A is the system matrix, $y_i \in \mathbb{R}^{m_i}$ is the output locally measured by agent i and $C_i \in \mathbb{R}^{m_i \times n}$ is its output matrix.

Since the agents are not able to reconstruct the whole state x based only on the local measurement y_i (i.e. detectability of (C_i, A) is not assumed), a communication network among them is required.

Thus, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represents the directed graph modelling the communication network where no communication failures are allowed. For this directed graph, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represents every communication channel between pairs of agents.

2.1 Collective detectability assumption

We introduce here some key concepts, useful for the developments of the rest of the paper.

Definition 1. For the communication graph $\mathcal{G}$, the ρ-hop output matrix of agent i, $C_{i,\rho}$, is defined as:

$$C_{i,\rho} := \begin{bmatrix} C_{i,\rho-1} \\ \mathrm{col}(C_{j,\rho-1})_{j \in \mathcal{N}_i} \end{bmatrix}, \quad \forall \rho \geq 1, \tag{3}$$

where $C_{i,0} := C_i$.

Intuitively speaking, the ρ-hop output matrix $C_{i,\rho}$ of agent i, recursively defined in (3), comprises its output matrix C_i and the output matrices of all the agents j with a direct path to i involving ρ or less edges. Based on this concept, we can formulate our first assumption. A similar assumption was introduced in del Nozal et al. (2019).

Definition 2. System (1)-(2) is collectively detectable if, for any $i \in \mathcal{V}$, there exists a finite number of hops $\ell_i \in \mathbb{Z}_{>0}$ such that pair (C_{i,ℓ_i}, A) is detectable.

Assumption 1. System (1)-(2) is collectively detectable.

As shown in del Nozal et al. (2019), Assumption 1 is necessary for the existence of a converging distributed state estimator. According to Definition 2, system (1)-(2) is collectively detectable if, for each agent, the complete information provided by the network (that is, the ρ-hop output matrix with ρ sufficiently large) is sufficient to build a converging state observer. Recall that a strongly connected communication network is not required in contrast with other approaches as Wang et al. (2019)

2.2 Communication model and persistence assumption

In this paper, we consider that the topology of the network $\mathcal{G}$ can vary with time due to failures in the communication devices, jamming attacks (see Jin (2010)) or packet dropouts. To this end, a logic variable $k \mapsto \delta_i(k) \in \{0, 1\}$ is associated to each node i to represent communication failures for agent i at time k. Using δ_i, the set of links pointing to agent i, $(i, j) : \forall j \in \mathcal{N}_i$, are active at time k

if and only if $\delta_i(k) = 1$. Otherwise, when $\delta_i(k) = 0$, these links are inactive due to a communication failure. As a result, $\mathcal{E}(k)$ and $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$ represent, respectively, the set of active links at time k and characterize the time-varying graph at time k.

Whenever a loss occurs, agent i cannot receive information from its neighborhood. Consequently, it would be reasonable to operate based only on the system model and on the local measurement y_i. However, if this situation is extended in time, the plant state cannot be detected in the directions that are not observable from that output. To rule out this scenario, we assume the following persistence of excitation property.

Definition 3. Graph $k \mapsto \mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$ enjoys a uniform local persistence property if for each $i \in \mathcal{V}$ there exist a finite time horizon $\tau_i \in \mathbb{Z}_{>0}$ and an integer $n_{\tau i} \in \mathbb{Z}_{>0}$ such that

$$\sum_{h=0}^{\tau_i - 1} \delta_i(k+h) \geq n_{\tau i}, \quad \forall k \in \mathbb{Z}_{\geq 0}, \tag{4}$$

namely, for each time interval $\{k, \ldots, k+\tau_i - 1\}$, $k \in \mathbb{Z}_{>0}$, there exist at least $n_{\tau i}$ distinct values of $s \in \{k, \ldots, k + \tau_i - 1\}$ satisfying $\delta_i(s) = 1$.

Assumption 2. The graph $k \mapsto \mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$ enjoys a uniform local persistence property.

Assumption 2 requires that each agent i experiences no more than $\tau_i - n_{\tau i}$ communication losses in each time window spanning τ_i time instants. This implies that the agent receives enough information from the neighboring agents, which corresponds to some kind of persistence of excitation. While Assumption 2 is not necessary, in general, we emphasize that it does not hold true only in cases where the communication instants (when $\delta_i = 1$) become increasingly rare as time flows. Such a scenario is quite undesirable if one wants to achieve uniform convergence properties like those in our problem statement below.

2.3 Problem statement

Based on our standing Assumptions 1 and 2, we are ready to state our problem statement.

Problem 1. Consider system (1)-(2) and an interconnection graph $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$. Under Assumptions 1 and 2, design a distributed observer providing, at each node i, an estimate $\hat{x}_i$ of the state x of (1)-(2), such that these estimates converge uniformly and exponentially to x. In particular, for each δ_i, $i = 1, \ldots, p$, satisfying Assumption 2, there must exist scalars $M > 0$ and $\lambda \in (0, 1)$ such that, for any $x(0)$ and $\hat{x}_i(0)$, $i \in \mathcal{V}$,

$$|x(k) - \hat{x}_i(k)|^2 \leq M \lambda^k \sum_{j=1}^{p} |x(0) - \hat{x}_j(0)|^2, \quad \forall i \in \mathcal{V}. \tag{5}$$

The distributed observer that we design to solve the problem above generalizes the linear time-invariant solution in del Nozal et al. (2019). The novelty that we propose here is that we focus on linear dynamics subject to the "external" time-varying logical inputs $\delta_i(k)$. Due to these extra inputs, the linear cascaded arguments of del Nozal et al. (2019) cannot be adopted, but we may resort to nonlinear Input to State Stability (ISS) results for time-varying systems.

3. MULTI-HOP SUBSPACE DECOMPOSITION

Following the multi-hop subspace decomposition of del Nozal et al. (2019), there always exists a coordinate transformation matrix $\left[\bar{V}_{i,\rho}\ V_{i,\rho}\right] \in \mathbb{R}^{n \times n}$ associated to pair $(C_{i,\rho}, A)$, such that the change of variable $[\bar{V}_{i,\rho}\ V_{i,\rho}]^\top x \in \mathbb{R}^n$ transforms the original state-space representation into the observability staircase form Hespanha (2009). Note that $\bar{V}_{i,\rho} \in \mathbb{R}^{n \times n^{\bar{o}}_{i,\rho}}$ is composed by $n^{\bar{o}}_{i,\rho}$ column vectors in $\mathbb{R}^n$ that form an orthogonal basis of the unobservable subspace of pair $(C_{i,\rho}, A)$. Correspondingly, $V_{i,\rho} \in \mathbb{R}^{n \times n^o_{i,\rho}}$ is an orthogonal basis of its orthogonal complement.

Definition 4. The ρ-hop unobservable subspace from agent i, denoted $\bar{\mathcal{O}}_{i,\rho}$, is composed of all system modes that cannot be observed from the output locally measured by agent i and those measured by all the agents belonging to the s-hop reachable nodes from i, $\forall s \in \{0, \ldots, \rho\}$. Equivalently, the ρ-hop unobservable subspace from agent i is the unobservable subspace related to pair $(C_{i,\rho}, A)$ using the above coordinate transformation:

$$\bar{\mathcal{O}}_{i,\rho} := \mathrm{Im}(\bar{V}_{i,\rho}).$$

The orthogonal complement of $\bar{\mathcal{O}}_{i,\rho}$, with some abuse of notation, is denoted ρ-hop observable subspace from agent i, $\mathcal{O}_{i,\rho} := \mathrm{Im}(V_{i,\rho})$. We denote $n^o_{i,\rho} = \dim(\mathcal{O}_{i,\rho})$.

According to Definition 4, it is clear that:

$$\mathcal{O}_{i,\rho-1} \subseteq \mathcal{O}_{i,\rho}, \quad \forall i \in \mathcal{V}, \quad \rho \geq 0. \tag{6}$$

where we consider $\mathcal{O}_{i,-1} = \emptyset$. Then, the vectors of the "innovation" basis that generates $\mathcal{O}_{i,\rho} \cap (\mathcal{O}_{i,\rho-1})^\perp$ can be stacked into a matrix $W_{i,\rho} \in \mathbb{R}^{n \times n_{i,\rho}}$, where $n_{i,\rho} = n^o_{i,\rho} - n^o_{i,\rho-1}$, in such a way that:

$$\mathrm{Im}(W_{i,\rho}) := \mathcal{O}_{i,\rho} \cap (\mathcal{O}_{i,\rho-1})^\perp, \quad \rho \geq 0, \tag{7}$$

Let us define $\ell_i \in \mathbb{Z}_{>0}$, to be selected later, as an arbitrary number of hops. From these definitions it is clear that for all $\rho \in \{0, \ldots, \ell_i\}$ and all $i \in \mathcal{V}$, it holds that

$$\mathrm{Im}(V_{i,\rho}) = \mathrm{Im}\left([W_{i,\rho}\ V_{i,\rho-1}]\right), \tag{8}$$

$$\mathrm{Im}(\bar{V}_{i,\rho-1}) = \mathrm{Im}\left(\left[W_{i,\rho}\ \bar{V}_{i,\rho}\right]\right), \tag{9}$$

with $\bar{V}_{i,-1} := I_n$.

Thus, the transformation matrix $T_i \in \mathbb{R}^{n \times n}$, defined as $T_i = [\bar{V}_{i,\ell_i}\ V_{i,\ell_i}]$, can be partitioned as follows, using the innovations at each hop:

$$T_i := [\underbrace{\bar{V}_{i,\ell_i}\ W_{i,\ell_i}\ \cdots\ W_{i,\rho+1}}_{\bar{V}_{i,\rho}}\ \underbrace{W_{i,\rho}\ \cdots\ W_{i,0}}_{V_{i,\rho}}], \tag{10}$$

for all $\rho \in \{0, \ldots, \ell_i\}$, where it is easy to identify the observable and unobservable subspaces of the system by agent i at hop ρ. Note also that T_i is orthogonal by construction, namely $T_i^{-1} = T_i^\top$.

The following lemma, proven in (del Nozal et al., 2019, Lemma 3), introduces some important properties that are central for the derivations of this paper.

Lemma 1. (del Nozal et al., 2019, Lemma 3) For each agent $i \in \mathcal{V}$, and any $\ell_i > \mathbb{Z}_{>0}$, the next properties hold, $\forall \rho, \rho' \in \{1, \ldots, \ell_i\}$ such that $\rho \neq \rho'$:

(i) $W_{i,\rho}^\top W_{i,\rho'} = 0$,
(ii) $\mathrm{Im}(W_{j,\rho-1}) \subseteq \mathrm{Im}(V_{i,\rho})$, $\forall j \in \mathcal{N}_i$,
(iii) $\mathrm{Im}(W_{i,\rho}) \subseteq \bigoplus_{j \in \mathcal{N}_i} \mathrm{Im}(W_{j,\rho-1})$.

4. OBSERVER DESIGN FOR STABILITY DEALING WITH COMMUNICATION FAILURES

This section contains our main results. First we present the observer structure and then we derive the ensuing error dynamics. Finally, we provide design rules for the observer gains solving Problem 1 and we show that, under the prescribed assumptions, these design rules are feasible.

4.1 Observer structure and error dynamics

For each agent i, we propose the following observer structure:

$$\begin{aligned}\hat{x}_i(k+1) &= A\hat{x}_i(k) + W_{i,0}L_i(y_i(k) - \hat{y}_i(k)) \\ &+ \delta_i(k)\sum_{\rho=0}^{\ell_i}\sum_{j\in\mathcal{N}_i} W_{i,\rho}N_{i,j,\rho}W_{j,\rho-1}^\top(\hat{x}_j(k) - \hat{x}_i(k)),\end{aligned} \quad (11)$$

where L_i and $N_{i,j,\rho}$ are, respectively, a local gain and consensus gains to be selected later in such a way that Problem 1 is solved. The value of ℓ_i is chosen so that collective detectability is fulfilled as per Assumption 1. This structure was presented in del Nozal et al. (2019) for $\delta_i(k) = 1, \forall k$. For a more detailed explanation of the proposed observer structure, the reader is referred to that paper.

For each agent $i \in \mathcal{V}$, let us define the corresponding estimation error $e_i(k) := x(k) - \hat{x}_i(k)$. Similarly, it is possible to define the transformed estimation error as

$$\varepsilon_i := \mathrm{col}(\varepsilon_{i,\ell_i+1}, \ldots, \varepsilon_{i,0}) := T_i^\top e_i, \quad (12)$$

using the multi-hop subspace decomposition (10) introduced in Section 2. More specifically, the estimation error of agent $i \in \mathcal{V}$, at hop ρ, is defined as: $\varepsilon_{i,\rho}(k) := W_{i,\rho}^\top e_i(k), \quad \forall \rho = 0,\ldots,\ell_i+1$, where we denote $W_{i,\ell_i+1} = \bar{V}_{i,\ell_i}$ corresponding to the collectively unobservable but detectable system modes.

The following proposition clarifies the dynamics of these estimation errors. Its proof is a straightforward extension of the results in del Nozal et al. (2019) and is omitted.

Proposition 1. Consider the network of agents described by the graph $\mathcal{G}(k)$, where every agent i implements the observer structure (11) to estimate the state of system (1). Then the dynamics of the errors in (12) corresponds to

$$\varepsilon_{i,0}(k+1) = (W_{i,0}^\top AW_{i,0} - L_iC_iW_{i,0})\varepsilon_{i,0}(k), \quad (13)$$

$$\begin{aligned}\varepsilon_{i,\rho}(k+1) &= \sum_{r=0}^{\rho} D_{i,(\rho,r)}(\delta_i)\varepsilon_{i,r}(k) \\ &+\delta_i(k)\sum_{j\in\mathcal{N}_i} N_{i,j,\rho}\varepsilon_{j,\rho-1}(k),\ \rho\in\{1,\ldots,\ell_i\},\end{aligned} \quad (14)$$

with

$$D_{i,(\rho,r)}(\delta_i) = W_{i,\rho}^\top AW_{i,r} - \delta_i\sum_{j\in\mathcal{N}_i} N_{i,j,\rho}W_{j,\rho-1}^\top W_{i,r}.$$

The dynamics in Proposition 1 for $\rho \in \{1,\ldots,\ell_i\}$ can be compactly written as (we remove the dependence on k for simplicity):

$$\varepsilon_{i,0}^+ = (W_{i,0}^\top AW_{i,0} - L_iC_iW_{i,0})\varepsilon_{i,0}, \quad (15a)$$

$$\varepsilon_{i,\rho}^+ = D_{i,(\rho,\rho)}(\delta_i)\varepsilon_{i,\rho} + B_{i,\rho}(\delta_i)u_{i,\rho}, \quad \text{if } \rho \neq 0 \quad (15b)$$

where

$$B_{i,\rho}(\delta_i) = \Big[\mathrm{row}\left(D_{i,(\rho,r)}(\delta_i)\right)_{r\in\{0,\ldots,\rho-1\}} \,| \; | \,\mathrm{row}\left(\delta_i N_{i,j,\rho}\right)_{j\in\mathcal{N}_i}\Big],$$

$$u_{i,\rho}(k) = \begin{bmatrix}\mathrm{col}(\varepsilon_{i,r})_{r\in\{0,\ldots,\rho-1\}} \\ \mathrm{col}(\varepsilon_{j,\rho-1})_{j\in\mathcal{N}_i}\end{bmatrix},$$

which shows an interesting cascaded structure exploited in our main results of the next section.

4.2 Main result and tuning of the observer gains

By exploiting the cascaded dynamics (15), this section presents a design requirement that will be proven to be sufficient to guarantee the exponential estimation properties of Problem 1. Note that, since the evolution of the transformed estimation error at hop $\rho = 0$ does not depend on the agents connectivity, the local gain L_i can be easily tuned to ensure uniform exponential convergence to zero of the solutions to (15a). Instead, the connectivity properties in Assumption 2 are fundamental for the effectiveness of the design of the consensus gains $N_{i,j,\rho}$, for which the cascade structure revealed with the multi-hop decomposition becomes crucial.

Property 1. For each agent i, the local gain L_i and consensus gains $N_{i,j,\rho}$ for hops $\rho \in \{1,\ldots,\ell_i\}$ are designed in such a way that the matrix

$$(W_{i,0}^\top AW_{i,0} - L_iC_iW_{i,0}) \quad (16)$$

is Schur, and the following inequalities are met:

$$D_{i,(\rho,\rho)}(\delta_i)^\top P_{i,\rho}D_{i,(\rho,\rho)}(\delta_i) < \mu_{i,\rho}(\delta_i)P_{i,\rho},\ \delta_i\in\{0,1\} \quad (17)$$

$$\underline{\mu}_{i,\rho} := \mu_{i,\rho}(0)^{\tau_i - n_{\tau i}}\mu_{i,\rho}(1)^{n_{\tau i}} < 1, \quad (18)$$

where $\mu_{i,\rho}(\delta_i),\ \delta_i = 0,1$ are two a scalar parameters depending on the switching signal δ_i, satisfying $\mu_{i,\rho}(1) \leq \mu_{i,\rho}(0)$ and $P_{i,\rho}$ is a positive definite matrix with appropriate dimensions.

Now, we are in position to introduce the main result of the paper in Theorem 1, which establishes that observer (11) solves Problem 1 whenever Property 1 is satisfied. Its proof is postponed to Section 4.3 to avoid breaking the flow of the exposition.

Theorem 1. Consider plant (1) observed by a set of agents that can measure their local outputs (2), each of them implementing the observer structure (11). Under Assumptions 1 and 2, if the observer gains are designed according to Property 1, then Problem 1 is solved, namely the estimation errors satisfy (5).

The next theorem completes the statement of Theorem 1.

Theorem 2. It is always possible, under Assumptions 1 and 2, to design matrices $L_i, N_{i,j,\rho}, \forall i,\rho$ and $j\in\mathcal{N}_i$, that satisfy Property 1.

Proof. According to (del Nozal et al., 2019, Theorem 14), in the absence of communication failures, namely $\delta_i(k) = 1$ for all i and for all k, under Assumption 1 it is possible to design gain matrices L_i, and $N_{i,j,\rho}$ to fix the convergence rate of the estimator arbitrarily fast (a detailed design method is presented there). In other words the value $\mu_{i,\rho}(1)$ can be selected arbitrarily close to zero by appropriate choices of the gains. Due to the fact that the value of $\mu_{i,\rho}(0)$ is determined by the open-loop system dynamics, and therefore independent of the observer gains, it is then possible to choose $\mu_{i,\rho}(1)$ sufficiently small to ensure $\mu_{i,\rho}(0)^{\tau_i - n_{\tau i}}\mu_{i,\rho}(1)^{n_{\tau i}} < 1$ for any given pair τ_i, n_{τ_i} from Assumption 2. □

4.3 Sketch of the proof of Theorem 1

This section presents the main guidelines followed to prove Theorem 1 (for a complete proof of the theorem, reader is referred to Rodríguez del Nozal et al. (2020)). The proof is based on Input to State Stability (ISS) properties for systems of the form

$$\xi(k+1) = f(\xi(k), u(k), k), \tag{19}$$

which well represent dynamics (15b). In particular, we make use of the following property.

Definition 5. System (19) is uniformly globally exponentially finite-gain ISS with respect to u if there exist scalars $M > 0$, $\lambda \in (0,1)$ and $\gamma > 0$, such that for any initial time k_0, any initial condition $\xi(k_0)$ and any uniformly bounded input $k \mapsto u(k)$, the corresponding solution $k \mapsto \xi(k)$ satisfies [1]

$$|\xi(k)|^2 \leq M\lambda^{k-k_0}|\xi(k_0)|^2 + \gamma\|u\|_\infty^2, \quad \forall k \geq 0,$$

where $\|u\|_\infty := \sup_{k \geq k_0} |u(k)|$ denotes the l_∞ norm of the input u.

To prove the above ISS property for each one of the subsystems in (15b), for each $i \in \mathcal{V}$ and each $\rho \in \{1, \ldots, \ell_i\}$ we will use the following quadratic Lyapunov function

$$V_{i,\rho}(\varepsilon_{i,\rho}(k)) = \varepsilon_{i,\rho}(k)^\top P_{i,\rho}\varepsilon_{i,\rho}(k), \tag{20}$$

where $P_{i,\rho}$ is a symmetric positive definite matrix.

Based on the Lyapunov function (20), we can now prove an ISS property, in the sense of Definition 5, for dynamics (15b) for each $i \in \mathcal{V}$ and each $\rho \in \{1, \ldots, \ell_i\}$. This is established next.

Lemma 2. If Property 1 and Assumption 2 hold, then for each $i \in \mathcal{V}$ and each $\rho \in \{1, \ldots, \ell_i\}$ the error system with dynamics (15b) is uniformly globally exponentially finite-gain ISS with respect to $u_{i,\rho}$.

From Lemma 2, the proof of Theorem 1 can be presented.

Proof of Theorem 1. Since matrix (16) in Property 1 is Schur by assumption, the dynamics of the estimation error at hop $\rho = 0$ is exponentially stable for all agents, namely there exist $M_0 > 0$ and $\lambda_0 \in (0,1)$ such that $|\varepsilon_{i,0}(k)| \leq M_0\lambda_0^k|\varepsilon_{i,0}(0)|$ for all $k > 0$ and all $i \in \mathcal{V}$. Using this bound, and due to the cascaded-like expression of $u_{i,\rho}$ in (15b), we may concatenate the ISS bounds established in Lemma 2 to obtain that there exits $\lambda_\varepsilon \in (0,1)$ and $M_\varepsilon > 0$ such that vector $\varepsilon = \mathrm{col}(\varepsilon_1, \ldots, \varepsilon_p)$ satisfies the ISS bound

$$|\varepsilon(k)|^2 \leq M_\varepsilon\lambda_\varepsilon^k|\varepsilon(0)|^2. \tag{21}$$

Since ε is equivalent (through linear transformation) to $e = \mathrm{col}(e_1, \ldots, e_p)$ (where we recall that $e_i = x - \hat{x}_i$), then the previous bound implies bound (5) in Problem 1, thus completing the proof. □

[1] Note that the standard definition of ISS does not include square powers in signals norms. Nevertheless, if this expression is fulfilled it is trivial to go back to the standard definition.

Remark 1. We emphasize that the proof technique of this section, based on the time-varying dynamics (19), ensures that for each persistently exciting selection of δ_i, $i = 1, \ldots, p$, as per Definition 3, there exist M_ε and λ_ε satisfying (21) (equivalently (5) in Problem 1). However, we don't give here a guarantee that those scalars be uniform over the infinitely many persistently exciting selections of δ_i. Nevertheless, we conjecture that a different proof technique may be used to prove a uniform exponential bound, valid for all such selections. Proving this uniform exponential convergence property is left as future work.

5. SIMULATION RESULTS

This section presents some simulation results that demonstrate the effectiveness of the estimation algorithm. To this end, let us consider the following system:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^+ = \begin{bmatrix} 0.95 & 0 & 0 & 0 \\ 0 & 0.8606 & -1.3368 & 0 \\ 0 & 0.1485 & 0.9315 & 0 \\ 0 & 0 & 0 & 1.015 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

The system is observed by a set of three agents that communicate according to the interconnection graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, 2, 3\}$ and $\mathcal{E} = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$. In all of the examples discussed below, we consider that agents 1 and 3 have access to make measurements of states x_1 and x_4 and agent 2 measures state x_2.

Example 1. Let us assume a scenario where the agents experience communication failures. We assume that at every time interval $\{k, \ldots, k+\tau_i-1\}$, where $\tau_i = 100$, $\forall i \in \mathcal{V}$, there exists at least $n_{\tau_i} = 20$ times when every agents $i = 1, 2, 3$, can communicate with their neighborhood.

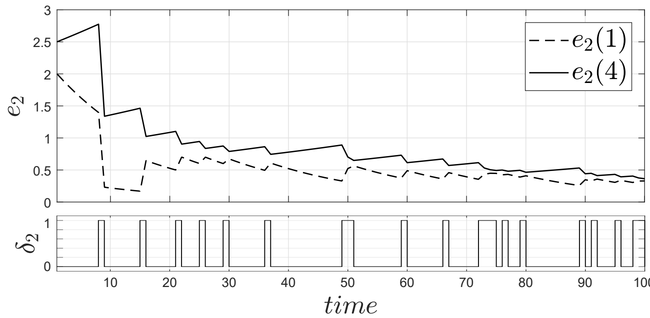

Fig. 1. Estimation error of agent 2 in Example 1.

Figure 1 shows the evolution of the estimation error of agent 2, estimating states x_1 and x_4. Recall that these states are measured by agents 1 and 3 and, according to the communication topology, these agents are one hop away from agent 2. When agent 2 can communicate, the estimation error decreases significantly. This is a consequence of the fast convergence rate fixed in the observer design. However, when agent 2 is not able to communicate with its neighbors the estimation error grows according to the unstable open-loop dynamics.

Example 2. This second example shows the unstable response of the state estimation error when Property 1 is not met. The consensus matrices designed in the previous example for $\tau_i = 100, n_{\tau_i} = 20$, do not satisfy Property 1 in the worsened scenario with $\tau_i = 100$, $n_{\tau_i} = 2$, for all $i \in \mathcal{V}$, namely with increased communication losses.

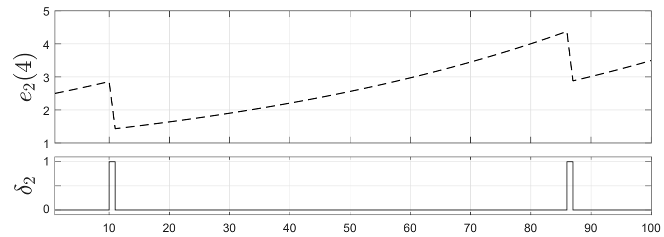

Fig. 2. Estimation error of agent 2 in Example 2.

Thus, by performing a parallel simulation to the one of the previous example, we now observe a diverging error response. In particular, Figure 2 shows the evolution of the estimation error of agent 2 estimating state x_4. Note that the estimation error decreases when the communications are active. However, this is not enough to stabilize the estimation error.

6. CONCLUSIONS

In this paper, the distributed state estimation problem of an autonomous LTI system by a lossy network of agents has been addressed. By using an observer structure based on a multi-hop subspace decomposition, each agent involved in the network can identify its observable subspace and the innovations introduced (whenever a communication loss does not occur) by its neighbors at each hop. Under some reasonable assumptions on the network connectivity, we have shown that it is always possible to find observer gains guaranteeing uniform exponential convergence to zero of all the estimation errors. Our architecture has been tested by means of simulations and the main results of the paper have been illustrated by a network of three agents. Future work includes proving a uniform version of our exponential bound.

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