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Delayed Newton-Based Multivariable Extremum Seeking with Sequential Predictors ⋆

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Abstract: We provide a new method for Newton-based multivariable extremum seeking which allows different delays in each of the input channels. We allow arbitrarily long input delays. Our sequential predictor delay compensation method eliminates the need for the distributed terms that were required in earlier methods. We illustrate our method in a source seeking example.

Keywords: Systems with time delays, extremum seeking, averaging

1. INTRODUCTION

Extremum seeking is a useful approach for the optimization of unknown nonlinear maps in real time, because it is a model independent adaptive control method that applies through the tuning of certain parameters; see Guay and Dochain (2015) and Guay et al. (2004). It uses a sinusoidal excitation in order to disturb the parameters that we wish to tune. By measuring the effects of parameters on an output, extremum seeking methods search for optimal values; see Ariyur and Krstic (2003); Krstic (2014).

Although the well known extremum seeking algorithms do not address delays, it is valuable to consider delayed extremum seeking, because of the ubiquity of delays in engineering processes. For delayed extremum seeking, the objective is to estimate the extremum from available delayed output measurements, because current measurements are not available; see Oliveira and Krstic (2015a) and Oliveira et al. (2017). One standard approach to delay compensation is called emulation, and involves designing the control without using information about the input or output delays, and then studying the effects of replacing the state or output values in the feedback by the delayed values that are available for measurement; see Mazenc et al. (2008). On the other hand, emulation generally leads to bounds on the permissible delays that are sometimes too conservative to use in some engineering settings.

This motivated the development of other methods that can compensate for arbitrarily long input or output delays, by exploiting known information about the delays. The reduction model approach is one such method, and was introduced by Artstein in the early 1980s for linear systems; see Artstein (1982). A second method combines backstepping and predictive delay compensation, and uses Lyapunov functionals to study the robustness of the stability of the closed loop system. A third method is based on sequential predictors (as first discussed in Besançon et al. (2007)), in which the distributed terms found in standard predictive control methods are replaced by dynamic extensions that are made up of copies of the original system evolving on different time scales; see Cacace and Germani (2017) and Mazenc and Malisoff (2017). Sequential predictors can be a useful alternative to the computational challenges that can arise from using standard predictive control approaches.

The first mathematical approach to address delayed extremum seeking appears to be the one proposed by Oliveira and Krstic (2015a) and Tsubakino et al. (2015); see Section 3 below. The motivation for Oliveira and Krstic (2015a) and Tsubakino et al. (2015) came from the processing of output measurements in engineering applications that is often required prior to generating control values; see Aarsnes et al. (2019); Butler (2013); Lin et al. (2000), and Swinnen et al. (2004). The delays in these applications are usually constant, known, and large relative to the time scale of the system dynamics. Using distributed terms in the feedbacks, these earlier results on delayed extremum seeking produce controls that are expressed in terms of integral equations that cannot be solved to obtain an explicit control formula, which could complicate applications that typically call for explicit control formulas.

This motivates this note, which is unlike earlier works because we will use a sequential predictor with a vector valued delay to compensate for arbitrarily long sensor and actuator delays in Newton-based extremum seeking, to allow situations with different delays in different arguments of the output, which can arise from input and measurement delays. This work is a one stage sequential prediction because it only requires one sequential predictor. This note contrasts with our gradient-based extremum seeking result in Malisoff and Krstic (2020, to appear), because while the estimation error in Malisoff and Krstic (2020, to appear) depends on the inverse of the unknown

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Hessian, the approach here provides an estimation error that is independent of this inverse.

We use standard notation, where the dimensions of the Euclidean spaces are arbitrary unless otherwise noted and $|\cdot|$ is the standard Euclidean norm and the corresponding matrix norm. Given a function $Z : \mathbb{R} \to \mathbb{R}^n$ and $J = [J_1 \dots J_n]^\top \in \mathbb{R}^n$, we set $Z(t+J) = [Z_1(t+J_1) \dots Z_n(t+J_n)]^\top$. Also, $K_{i,j}$ denotes the entry in the ith row and jth column of any matrix valued function K for all i and j, and $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. We use $K < 0$ to indicate that K is a negative definite constant matrix, and $x_t(s) = x(t+s)$ for all functions x and values of $s \le 0$ and $t \ge 0$ for which $t+s$ is in the domain of x. We choose the initial times of our solutions to be $t_0 = 0$, and constant initial functions, and $\mathcal{O}(x)$ denotes any function $\mathcal{O} : (0,\infty) \to (0,\infty)$ such that there is a constant $c > 0$ such that $\mathcal{O}(x)/x$ is bounded above on the interval $(0, c)$.

2. STATEMENT OF PROBLEM

This section provides the necessary notation and definitions from Oliveira et al. (2017) for the multivariable Newton-based extremum seeking problem that we will study. As in Oliveira et al. (2017), our objective is the maximization of a real valued output function

$$y(t) = Q(\theta(t-D)) \tag{1}$$

by considering the effects of different choices of an input function $\theta = [\theta_1\ \theta_2\ \dots\ \theta_n]^\top$, where $D = [D_1\ \dots\ D_n]^\top \in \mathbb{R}^n$ is a vector of nonnegative known constants that represent delays. Focusing on the case of maximum seeking, we assume that there is a $\theta^* \in \mathbb{R}^n$ such that

$$\frac{\partial Q(\theta^*)}{\partial\theta} = 0, \quad \frac{\partial^2 Q(\theta^*)}{\partial\theta^2} = H < 0, \text{ and } H = H^T, \tag{2}$$

where $H \in \mathbb{R}^{n\times n}$ and θ^* are unknown and constant. Further assuming that y is at least locally quadratic, we have the representation

$$y(t) = y^* + \frac{1}{2}\big(\theta(t-D) - \theta^*\big)^T H \big(\theta(t-D) - \theta^*\big), \tag{3}$$

in which $y^* = Q(\theta^*)$ is the extremum. We measure $y(t)$ by using the $\mathbb{R}^n$-valued dither signals

$$S(t) = [a_1 \sin(\omega_1(t+D_1))\ \dots\ a_n \sin(\omega_n(t+D_n))]^\top$$
$$\text{and } M(t) = \left[\frac{2}{a_1}\sin(\omega_1 t)\ \dots\ \frac{2}{a_n}\sin(\omega_n t)\right]^\top, \tag{4}$$

whose probing frequencies ω_i and amplitudes a_i are positive constants that will be specified below. The paper (Oliveira et al., 2017, Section II.E) uses

$$G(t) = M(t)y(t) \tag{5}$$

to estimate the unknown gradient $\partial Q(\theta)/\partial\theta$ of $Q(\theta)$ in an averaging sense. Then the calculated input

$$\theta(t) = \theta_e(t) + S(t) \tag{6}$$

is a real-time estimate $\theta_e(t)$ of θ^* that we specify later, which is perturbed by $S(t)$.

The elements of the $n \times n$ demodulating matrix $N(t)$ that are used to build the estimator

$$\hat{H}(t) = N(t)y(t) \tag{7}$$

of the Hessian are defined by

$$N_{i,i}(t) = \frac{16}{a_i^2}\left(\sin^2(\omega_i t) - \frac{1}{2}\right) \text{ for } i = 1, 2, \dots, n \text{ and} \tag{8}$$

$$N_{i,j}(t) = \frac{4}{a_i a_j}\sin(\omega_i t)\sin(\omega_j t) \text{ for } i \neq j\,. \tag{9}$$

We also choose the probing frequencies $\omega_i > 0$ such that

$$\omega_i = \omega_i' \omega = \mathcal{O}(\omega)\,, \quad i = 1, 2, \dots, n, \tag{10}$$

where $\omega \in \mathbb{R}$ is positive and $\omega_i' \in \mathbb{R}$ is rational. Similar to Oliveira et al. (2017), we require these conditions for all distinct i, j, k, and l values:

$$\omega_i' \notin \left\{\omega_j', \frac{1}{3}\omega_j', \frac{1}{2}(\omega_j' + \omega_k'), \omega_j' + 2\omega_k', \omega_j' + \omega_k' \pm \omega_l'\right\}, \tag{11}$$

with ω_i/ω_j being a ratio of odd integers for all i and j, where $\omega_j' + \omega_k' \pm \omega_l'$ is not present if $n \le 3$, and where the third and fourth elements in the curly braces are not present when $n = 2$.

Our goal is to find a formula for an output feedback controller u for the dynamics

$$\theta_e'(t) = u \tag{12}$$

that ensures that the estimation error

$$\tilde{\theta}(t) = \theta_e(t) - \theta^* \tag{13}$$

converges to a suitably small neighborhood of zero, where the controller u must be computed from delayed values of y; see below. Therefore, we have a delay compensation problem that can be solved by a variant of backstepping based predictive methods, and also by the sequential predictor approaches from Mazenc and Malisoff (2017). In the next section, we summarize the backstepping based approach from Oliveira et al. (2017), and Section 4 provides our sequential predictors approach.

3. BACKSTEPPING BASED PREDICTION

To compare our extremum seeking approach with earlier predictive methods for extremum seeking with delays, this section summarizes the delay compensation method from (Oliveira et al., 2017, Section VI), which covers the special case of Newton-based extremum seeking in which all of the delays D_i are equal to common value D. This special case was first presented in Oliveira and Krstic (2015b), and utilizes averaging to produce the controller

$$U(t) = \frac{c}{s+c}\left\{-K\left[z(t) + \int_{t-D}^{t} U(\tau)d\tau\right]\right\}, \tag{14}$$

where K is a diagonal matrix whose diagonal entries are all positive, $z(t) = \Gamma(t)G(t)$, Γ is a solution of the Riccati differential equation

$$\Gamma'(t) = \omega_r\Gamma(t) - \omega_r\Gamma(t)\hat{H}(t)\Gamma(t) \tag{15}$$

where Γ is used to estimate the inverse of H^{-1}, G and $\hat{H}$ are from (5) and (7), and $\frac{c}{s+c}\{\cdot\}$ is the usual notation for a low pass filter, for a design constant $\omega_r > 0$. The controller (14) was designed to ensure local stability of the extremum seeking, insofar that for each constant $\delta > 0$, there exists a positive constant $\underline{\omega}(\delta) > 0$ so that this condition holds for each constant $\omega > \underline{\omega}(\delta)$: there exist a constant $\rho > 0$ and a function $\mathcal{O}$ such that the calculated input (6) satisfies

$$\limsup_{t\to+\infty} |\theta(t) - \theta^*| \le \mathcal{O}(|a| + 1/\omega) \tag{16}$$

for all initial states $\theta_e(0)$ for the estimator such that $|(\theta_e(0), \Gamma(0)) - (\theta^*, H^{-1})| \le \rho$. Oliveira et al. (2017) is based on backstepping, and proposes a sequential backstepping analog that extends the preceding result to allow different delays in each of the input channels. Localness in the preceding result is a necessary consequence of using averaging, whose local conclusions are an analog of the local conclusions that occur from linearization, so only local results can be expected. The control (14) is distributed, because it depends on an integral containing past control values, and because this integral equation does not have an explicit solution.

4. MAIN RESULT FOR NEWTON-BASED CASE

This section provides an alternative delay compensating Newton-based extremum seeking method using a dynamic extension that yields a controller that is free of distributed terms. This provides a final bound (19) on the estimation error that does not depend on the inverse of the Hessian. In order to obtain an estimation error that does not depend on H^{-1}, we use a higher order dynamical controller that also contains an estimator for H^{-1}. We can prove:

Theorem 1: Let $a_i > 0$ for $i = 1, 2, \ldots, n$, $\theta^* \in \mathbb{R}^n$, $H \in \mathbb{R}^{n\times n}$, and ω_i for $i = 1, 2, \ldots, n$ satisfy the requirements from Section 2. Given constants $D_i \geq 0$ for $i = 1, 2, \ldots, n$, choose a constant $\bar{w} > 0$ such that

$$3\sqrt{n}\bar{w}\max_i D_i < 1. \tag{17}$$

Then for the choices of M, N, and S from Section 2 and all constants $\omega_r > 0$ and $\bar{k} > 0$ and each constant $\delta > 0$, there exists a constant $\underline{\omega}(\delta) > 0$ such that the following holds for each choice of the constant $\omega > \underline{\omega}(\delta)$: There is a constant $\rho > 0$ such that for each constant initial function $\phi : [-\max_i D_i, 0] \to \mathbb{R}^{n(n+2)}$ that satisfies $|\phi(0) - (\theta^*, 0, H^{-1})| \leq \rho$, the solution of the system

$$\begin{cases} \theta_e'(t) = -\bar{k}z(t) \\ z'(t) = -\bar{k}z(t) + \bar{w}\big(\Gamma(t)M(t)y(t) - z(t-D)\big) \\ \Gamma'(t) = \omega_r\big(-I_n + N(t)\Gamma(t)y(t)\big) \\ y(t) = y^* + \dfrac{1}{2}S_D^\top(\theta_{e,t})HS_D(\theta_{e,t}), \text{ where} \\ S_D(\theta_{e,t}) = \theta_e(t-D) + S(t-D) - \theta^* \end{cases} \tag{18}$$

with outputs is such that

$$\limsup_{t\to\infty} |\theta_e(t) - \theta^*| \leq \delta \tag{19}$$

and

$$\limsup_{t\to\infty} |y(t) - y^*| \leq \frac{1}{2}|H|\big(|a|+\delta\big)^2 \tag{20}$$

are satisfied. □

5. SKETCH OF PROOF OF THEOREM 1

The proof of Theorem 1 can be outlined as follows. Although (18) is nonlinear, we can show that the corresponding averaged system is a linear time-invariant system having a similar structure to the sequential predictor systems in Mazenc and Malisoff (2017), except that the delay in the averaged system is vector valued. We can then use averaging to conclude that (18) satisfies the stability properties that are asserted by Theorem 1. The complete proof is in Malisoff and Krstic (2019, in preparation).

To sketch the proof of Theorem 1, we first state two lemmas. The first one is an averaging result from Oliveira et al. (2017) that is a consequence of results from Hale and Lunel (1990). It involves a system of functional differential equations of the form

$$x'(t) = f(t/\varepsilon, x_t) \tag{21}$$

with constants $\epsilon > 0$ and the related averaged system

$$\alpha'(t) = F_{\rm av}(\alpha_t), \tag{22}$$

where

$$F_{\rm av}(\psi) = \lim_{T\to\infty} \tfrac{1}{T}\int_0^T f(s,\psi)\mathrm{d}s \tag{23}$$

for all choices of ψ.

Lemma 1: Consider the system (21) where $f : \mathbb{R}\times\Omega \to \mathbb{R}^n$ is a continuous functional for a neighborhood Ω of 0 of the supremum-normed Banach space $C([-r, 0]; \mathbb{R}^n)$ of all continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ and $r > 0$ is a constant. Assume that $f(t, \varphi)$ is periodic in t uniformly with respect to φ and that f has a continuous Fréchet derivative $\partial f(t, \varphi)/\partial\varphi$ in φ on $\mathbb{R}\times\Omega$. If $y = y_0 \in \Omega$ is an exponentially stable equilibrium for the averaged system (22), then, for some $\epsilon_0 > 0$ and each $\epsilon \in (0, \epsilon_0]$, there is a periodic solution $t \mapsto x^*(t, \epsilon)$ of (21) that is continuous in t and ϵ, satisfying

$$|x^*(t,\epsilon) - y_0| \leq \mathcal{O}(\epsilon) \tag{24}$$

for all $t \in \mathbb{R}$, and such that there is a constant $\rho > 0$ so that if $x(t)$ is a solution of (21) with $x(0) = \varphi$ and $|\varphi - y_0| < \rho$, then for some constants $C > 0$ and $\gamma > 0$, we have

$$|x(t) - x^*(t,\epsilon)| \leq Ce^{-\gamma t} \tag{25}$$

for all $t \geq 0$. □

Our next lemma is an analog of the scalar valued delay compensating sequential predictors from Mazenc and Malisoff (2017), using vector valued delays. Our motivation for using sequential predictors for the delay compensation is that we can view the system (26) as a dynamics that is driven by single sequential predictor and a vector valued delay, and that (26) has the same structure as an averaged version of the closed loop system (18) that we derive in the proof of Theorem 1. The proof of the lemma uses a Lyapunov Krasovskii argument that we also sketch.

Lemma 2: Let $\mathcal{N} \in \mathbb{R}^{n\times n}$ be negative definite, $D = [D_1 \ldots D_n]^\top \in \mathbb{R}^n$ be a vector of nonnegative constants, and $\bar{w} > 0$ be a constant such that (17) is satisfied. Then

$$\begin{cases} q'(t) = \mathcal{N}z(t-D) \\ z'(t) = \mathcal{N}z(t) - \bar{w}(z(t-D) - q(t)) \end{cases} \tag{26}$$

is uniformly globally exponentially stable to 0 on $\mathbb{R}^{2n}$. □

Proof of Lemma 2 (Sketch). Along all solutions of the dynamics

$$\mathcal{E}'(t) = -\bar{w}\mathcal{E}(t-D) \tag{27}$$

for the error variable $\mathcal{E}(t) = z(t) - q(t+D)$, we can check that the time derivative of $V(\mathcal{E}) = \frac{1}{2}|\mathcal{E}|^2$ is such that

$$\begin{aligned} \dot{V} \leq & -\bar{w}V(\mathcal{E}(t)) \\ & +2n\bar{w}^3\max_i D_i\int_{t-2\max_i D_i}^t V(\mathcal{E}(\ell))\mathrm{d}\ell \end{aligned} \tag{28}$$

for all $t \geq 0$, by applying the Fundamental Theorem of Calculus to each component $\mathcal{E}_i$ of $\mathcal{E}$, and also using Jensen's inequality and the triangle inequality. This is done by rewriting (27) as $\mathcal{E}'(t) = -\bar{w}\mathcal{E}(t) + \bar{w}[\mathcal{E}(t) - \mathcal{E}(t-D)]$ and applying the triangle inequality to the terms

$$\Big\{\sqrt{\bar{w}}|\mathcal{E}(t)|\Big\}\Big\{\bar{w}^{3/2}\sqrt{n}\int_{t-2\max_i D_i}^t |\mathcal{E}(\ell)|\mathrm{d}\ell\Big\} \tag{29}$$

in curly braces that come from computing $\dot{V}$, and then applying Jensen's inequality to the squared integral term that results from squaring the second term in curly braces.

This provides a constant $v_* > 0$ such that the time derivative of

$$V^\sharp(\mathcal{E}_t) = V(\mathcal{E}(t)) + v_* \int_{t-2\max_i D_i}^{t} \int_s^t V(\mathcal{E}(\ell))\mathrm{d}\ell \mathrm{d}s \quad (30)$$

satisfies

$$\dot{V}^\sharp(\mathcal{E}_t) \le -\bar{\epsilon}_0 V^\sharp(\mathcal{E}_t) \quad (31)$$

along all solutions of the $\mathcal{E}$ dynamics for all $t \ge 2\max_i D_i$, where

$$\bar{\epsilon}_0 = \bar{w}\min\left\{1 - 9n\bar{w}^2 \max_i D_i^2, 5/(18\bar{w}(\max_i D_i + 1))\right\} \quad (32)$$

is positive because of (17). Hence, the $\mathcal{E}$ dynamics are uniformly globally exponentially stable to 0. The lemma follows by rewriting (26) as a linear system that is input-to-state stable with respect to the variable $\mathcal{E}(t-D)$. □

Returning to the sketch of the proof of Theorem 1, we show that the average of a delayed error dynamics associated with (18) is globally exponentially stable to 0, and then we apply Lemma 1. Consider the system $X'(t) = f(t/\varepsilon, X_t)$ with $\varepsilon = 1/\omega$ and the state

$$X = (\Delta, z, \tilde{\Gamma}e_1, \dots, \tilde{\Gamma}e_n), \text{ where } \Delta(t) = \tilde{\theta}(t-D), \quad (33)$$

and where

$$\begin{aligned} &\tilde{\Gamma} = \Gamma - H^{-1},\ \tilde{\theta} = \theta_e - \theta^*, \\ &f = (f_1, f_2, f_3),\ f_3 = (f_{31}, \dots, f_{3n}), \\ &X_1 = \Delta,\ X_2 = z,\ X_3 = (\tilde{\Gamma}e_1, \dots, \tilde{\Gamma}e_n), \end{aligned} \quad (34)$$

$$\begin{aligned} &f_1(t, X_t) = -\bar{k}X_2(t-D), \\ &f_2(t, X_t) = -\bar{k}X_2(t) \\ &+\bar{w}\big((\tilde{\Gamma}(t) + H^{-1})M(t/\omega)Y(t, X_t) - X_2(t-D)\big) \\ &\text{and for all } i,\ f_{3i}(t, X_t) = \\ &\omega_r\big(-I_n + N(t/\omega)(\tilde{\Gamma}(t) + H^{-1})Y(t, X_t)\big)e_i, \end{aligned} \quad (35)$$

where e_i is the ith standard basis vector,

$$\begin{aligned} &Y(t, X_t) = y^* + \tfrac{1}{2}S_*^\top(t, X_t)HS_*(t, X_t) \\ &S_*(t, X_t) = S\left(\tfrac{t}{\omega} - D\right) + X_1(t). \end{aligned} \quad (36)$$

Using the averaging formulas in (Oliveira et al., 2017, Section II.E), the averaged X system is

$$\begin{cases} \Delta'(t) = -\bar{k}z(t-D) \\ z'(t) = -\bar{k}z(t) - \bar{w}\big(z(t-D) - \Gamma(t)H\Delta(t)\big) \\ \tilde{\Gamma}'_{i*}(t) = \omega_r H\tilde{\Gamma}_{i*}(t) \text{ for } i = 1, 2, \dots, n, \end{cases} \quad (37)$$

where $\tilde{\Gamma}_{i*} = \tilde{\Gamma}e_i$

Also, the system (37) is globally exponentially stable to 0 on its state space $\mathbb{R}^{n(n+2)}$. This exponential stability result can be obtained in four steps, using the dynamics

$$\begin{cases} \Delta'(t) = -\bar{k}\Delta(t) - \bar{k}\mathcal{E}(t-D) \\ z'(t) = -\bar{k}z(t) - \bar{w}\mathcal{E}(t-D) + \bar{w}\tilde{\Gamma}(t)H\Delta(t) \\ \mathcal{E}'(t) = -\bar{w}\mathcal{E}(t-D) + \bar{w}\tilde{\Gamma}(t)H\Delta(t) \end{cases} \quad (38)$$

where $\mathcal{E}(t) = z(t) - \Delta(t+D)$. First, we obtain positive constants v_* and v_{**} such that with the choice (30), the time derivative of

$$V^{\sharp\sharp}(\mathcal{E}_t, \Delta, z) = v_* V^\sharp(\mathcal{E}_t) + \frac{1}{2}\left(|\Delta|^2 + |z|^2\right) \quad (39)$$

along all solutions of (38) that satisfy $\tilde{\Gamma} = 0$ also satisfies

$$\dot{V}^{\sharp\sharp}(t) \le -v_{**}V^{\sharp\sharp}(\mathcal{E}_t, \Delta(t), z(t)). \quad (40)$$

In the second step, we use the triangle inequality and the exponential stability of the $\tilde{\Gamma}$ dynamics to find a constant $v_a \in (0, v_{**})$ and a $\Gamma_0 \in \mathcal{K}_\infty$ so that $\Gamma_0(s) \ge s$ for all $s \ge 0$ and such that (40) holds with v_{**} replaced by v_a along all solutions of (38) for all $t \ge \Gamma_0(|\tilde{\Gamma}(0)|)$. In the third step, we can use the second step to obtain an exponential decay condition for (37) for all $t \ge \Gamma_0(|\tilde{\Gamma}(0)|)$. Finally, using linear growth of the right side of (37) and Gronwall's inequality, we can derive a constant $\bar{B}$ such that

$$|(\Delta(t), z(t), \tilde{\Gamma}(t))| \le \bar{B}|(\Delta(0), z(0), \tilde{\Gamma}(0))|e^{\Gamma_0(|\tilde{\Gamma}(0)|)-t} \quad (41)$$

holds for all $t \in [0, \Gamma_0(|\tilde{\Gamma}(0)|)]$ and all solutions of (37), which we combine with the third step to get a global exponential stability estimate for (37) for all $t \ge 0$.

Next, note that (18) can be expressed as

$$X'(t) = f(t/\varepsilon, X_t) \quad (42)$$

with the choice (35) of f and $\varepsilon = 1/\omega$. Hence, Lemma 1 applies to the 0 equilibrium for (37). We conclude that for each constant $\delta > 0$, there exists a constant $\underline{\omega}(\delta) > 0$ such that for suitable constant initial states ϕ and for all $\omega \ge \underline{\omega}(\delta)$, we have $\limsup_{t\to\infty}|X(t)| \le \delta$. Then (6) gives

$$\limsup_{t\to\infty}|\theta_e(t) - \theta^*| = \limsup_{t\to\infty}|\tilde{\theta}(t)| \le \delta \quad (43)$$

for appropriate constant initial states for (18), which gives the first conclusion of the theorem. The second conclusion can be obtained by recalling the structure of the output y.

6. ILLUSTRATION

We apply Theorem 1 to a source seeking example whose output function y represents an unknown concentration field. The goal is to find a source of a signal, which could be acoustic, chemical, or electromagnetic; see Oliveira et al. (2017). Following Oliveira et al. (2017), we assume that

$$H = \begin{bmatrix} -2 & -2 \\ -2 & -4 \end{bmatrix}. \quad (44)$$

We also choose $a_1 = a_2 = 1$, the delays $D_1 = 10$ and $D_2 = 20$, the frequencies $\omega_1 = 7\omega$ and $\omega_2 = 5\omega$ where $\omega = 150$, $\omega_r = 10$, the constants $n = 2$ and $\bar{k} = 1$ and

$$\bar{w} = 0.99/(3\sqrt{n}\max\{D_1, D_2\}) = 0.116673, \quad (45)$$

the optimizer $\theta^* = (0, 1)$, and the maximum value $y^* = 1$. However, unlike Oliveira et al. (2017), or our work Malisoff and Krstic (2020, to appear) which applied a sequential predictor gradient-based extremum seeking approach to this example, here we apply our novel Newton-based sequential prediction method from Theorem 1 above.

In Fig. 1 below, we use Mathematica to plot the estimator values obtained from (18) in simulations with the initial states for the z_i's being zero and the choices in the preceding paragraph, and with the initial states $\theta_e(0) = (0.9.0.1)$, $\theta_e(0) = (1.2, -0.4)$, and $\theta_e(0) = (0.7, -0.25)$ for the estimator. In each case, we took the initial state

$$\Gamma(0) = \begin{bmatrix} -2.1 & -1.9 \\ -2.1 & -3.8 \end{bmatrix} \quad (46)$$

for the estimator for the inverse H^{-1} of the Hessian. Then Fig. 2 plots the estimator values from a second set of Mathematica simulations that used the same choices as in the first simulations from Fig. 1, except with $\omega = 150$ replaced by $\omega = 550$. Enlarging ω made it possible to

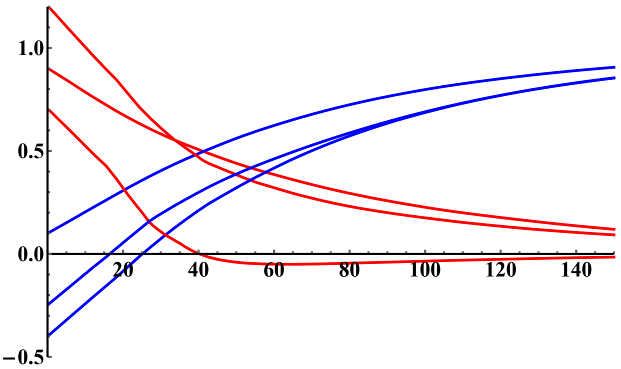

Fig. 1. First (Red) and Second (Blue) Components of θ_e with Initial States 0 for z_i's and Initial States (0.9.0.1), (1.2, −0.4), and (0.7, −0.25) for θ_e and $\omega = 150$ Showing Convergence toward $\theta^* = (0, 1)$.

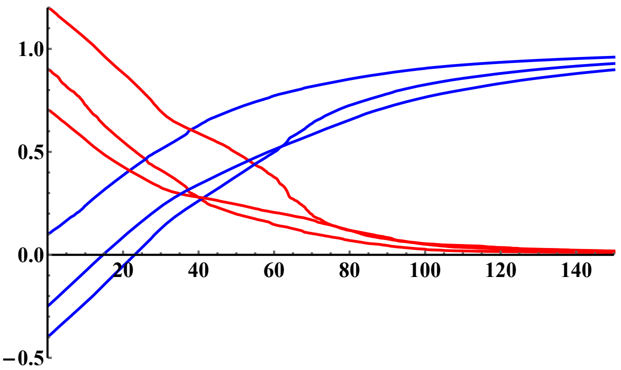

Fig. 2. First (Red) and Second (Blue) Components of θ_e with Initial States 0 for z_i's and Initial States (0.9.0.1), (1.2, −0.4), and (0.7, −0.25) for θ_e and $\omega = 550$ Showing Convergence toward $\theta^* = (0, 1)$.

allow a smaller estimation error δ for our theorem, which can explain why the convergence of θ_e is faster in Fig. 2 than in Fig. 1. Since the figures show convergence of the estimate to $\theta^* = (0, 1)$, they help verify the theorem.

7. CONCLUSIONS

We contributed to the literature on extremum seeking, which is a useful non-model based adaptive control method to find extrema of unknown functions. We solved multivariable Newton-based extremum seeking problems where the delays in each component of the estimator can differ, to address the need to consider latencies in source seeking problems. We used a new one-stage sequential predictor for a vector valued delay, whose potential advantage is that it has no distributed terms, while compensating for arbitrarily long delays. We plan to develop analogs for uncertain delays and applications to oil drilling models from Aarsnes et al. (2019), and PDE versions for diffusion dynamics that arise from the control of traffic flows.

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