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Investigation of Conditions for Non-degeneracy and Normality in Control Problems with Equality and Inequality State Constraints

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Abstract: This work aims to investigate conditions for normality and non-degeneracy of the maximum principle for general state-constrained optimal control problems in which the state constraints are given by equalities and inequalities. Non-degeneracy is proved under a regularity condition formulated in terms of the limiting normal cone to the feasible control set. The same regularity condition implies normality of the maximum principle if one of the end-points is free.

Keywords: Optimal control, maximum principle, state constraints.

1. INTRODUCTION

In this article, a general state-constrained optimal control problem is investigated in which the state constraints are defined by equalities and inequalities. Normality and non-degeneracy conditions for the informative maximum principle are presented. Such conditions are based on the notion of inward/outward regularity of a control process which, in turn, essentially relies on the notions of the Mordukhovich limiting normal cone and the Mordukhovich constraint qualification (regularity condition) as the case of closed feasible control set is examined.

Non-degeneracy conditions in state-constrained optimal control problems have been studied in Arutyunov and Tynyanskiy (1985); Dubovitskij and Dubovitskij (1985); Arutyunov (1991); Ferreira and Vinter (1994); Ferreira et al. (1999); Arutyunov and Aseev (1997); Arutyunov (2000); Vinter (2000); Arutyunov et al. (2011); Fontes and Frankowska (2015); Arutyunov et al. (2017); Arutyunov and Karamzin (2020). Normality conditions in state-constrained optimal control problems have been the subject of investigation in, for example, Rampazzo and Vinter (1999); Arutyunov et al. (2003); Frankowska (2009); Fontes and Frankowska (2015); Bettiol et al. (2016). These lists of contribution are certainly far from exhaustive. Our work is different in that the equality state constraints are involved in the problem formulation. The equality state constraints in optimal control problems have been studied in, for example, Pontryagin et al. (1962); Russak (1970); Arutyunov and Karamzin (2015, 2016).

This article is organized as follows. In Section 2, the problem formulation and the main definitions are presented, including the notion of inward/outward regularity of a control process. In Section 3, the degenerate maximum principle is presented, including several substantial remarks. In Section 4, the main results are formulated and proved. These include the non-degenerate maximum principle and normal maximum principle. Finally, Section 5 contains the conclusion.

2. PROBLEM FORMULATION AND MAIN DEFINITIONS

Consider the following control problem

$$\begin{aligned} &\varphi(p) \to \min, \\ &\dot{x} = f(x,u), \\ &p = (x(0), x(1)) \in C, \\ &u(t) \in U \ \text{ for a.a. } t \in [0,1], \\ &g_1(x(t)) \le 0, \ \ g_2(x(t)) = 0 \ \ \forall\, t \in [0,1]. \end{aligned} \tag{1}$$

Here, x is the state variable in $\mathbb{R}^n$, and u is the control variable in $\mathbb{R}^m$. Assume that C is closed and U is compact. The feasible trajectory $x(\cdot)$ is an absolutely continuous function such that it satisfies the equation: $\dot{x}(t) = f(x(t), u(t))$ for a.a. $t \in [0,1]$, the boundary conditions $p \in C$, and also the state constraints $g_1(x(t)) \le 0$, $g_2(x(t)) = 0 \ \forall\, t \in [0,1]$. The mappings $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$, and $g_i : \mathbb{R}^n \to \mathbb{R}^{k_i}$, $i = 1,2$ are supposed sufficiently smooth.

Let us introduce the notion of regularity. Consider a feasible process (x^*, u^*). Denote

$$\Gamma_i(x,u,t) = \frac{\partial g_i}{\partial x}(x,t) f(x,u), \ \ i = 1,2,$$
$$U(x) := \{u \in U : \Gamma_2(x,u) = 0\}.$$

Definition 1. Trajectory $x^*(\cdot)$ is said to be regular with respect to the equality state constraints provided that for all $t \in [0,1]$ and $u \in U(x^*(t))$ the following full rank condition is valid:

$$\operatorname{rank} \frac{\partial \Gamma_2}{\partial u}(x^*(t), u) = k_2,$$

$$\operatorname{im}\frac{\partial\Gamma_2^*}{\partial u}(x^*(t),u)\cap N_U(u)=\{0\}.$$

Here, $N_U(u)$ signifies the limiting normal cone to set U at point u in the sense proposed by B. Mordukhovich, see in Mordukhovich (1976). For $u \in U$, this cone is defined as

$$N_U(u) := \operatorname*{Ls}_{y\to u}\operatorname{cone}(y-\Pi_U(y)),$$

where $\Pi_U(y)$ is the Euclidean projection of y to U,

$$\Pi_U(y)=\{u\in U:\ |y-u|=\operatorname{dist}(y,U)\},$$

where $\operatorname{dist}(y,U)=\inf_{u\in U}|y-u|$ is the distance to the set; cone is the conic hull, and Ls is the upper topological limit (it contains all possible limit points of sequences of vectors from $\operatorname{cone}(y_k-\Pi_U(y_k)$ as $y_k\to u$).

Denote by $\mathcal{U}(t)$ the closure with respect to the measure of control function $u^*(t)$. Recall that set-valued map $\mathcal{U}(t)$ is defined for $t\in(0,1)$ as the set of vectors $u\in\mathbb{R}^m$ such that

$$\ell\Big(\Big\{s\in[t-\varepsilon,t+\varepsilon]:\ u^*(s)\in B_\varepsilon(u)\Big\}\Big)>0\ \ \forall\,\varepsilon>0.$$

For $t=0,1$, it is $\mathcal{U}(t)=\operatorname{Ls}_{s\to t}\mathcal{U}(s)$. Here, $B_\varepsilon(u)=\{v\in\mathbb{R}^m:\ |v-u|\le\varepsilon\}$, and ℓ is the Lebesgue measure on $\mathbb{R}$.

Consider set $J(x):=\{j: g_1^j(x)=0\}$. It is the so-called set of active indexes.

Definition 2. A control process $(x^*(t),u^*(t))$ is said to be inward regular with respect to the inequality state constraints provided that there exists a number $\varepsilon_0>0$ such that for all $t\in[0,1]$ there exists $\delta=\delta(t)>0$ such that for all $s\in(t-\delta,t+\delta)\cap[0,1]$ and for all $u\in\mathcal{U}(s)$ one can find a unit vector

$$d_1=d_1(t,s,u)\in\ker\frac{\partial\Gamma_2}{\partial u}(x^*(s),u)\cap N_U^\circ(u)$$

such that

$$\Big\langle\frac{\partial\Gamma_1^j}{\partial u}(x^*(s),u),d_1\Big\rangle<-\varepsilon_0\ \ \forall\,j\in J(x^*(t)).\tag{2}$$

The process is said to be outward regular with respect to the inequality state constraints provided that for all $t\in[0,1]$ there exists $\delta=\delta(t)>0$ such that for all $s\in(t-\delta,t+\delta)\cap[0,1]$, $u\in\mathcal{U}(s)$ one can find a unit vector

$$d_2=d_2(t,s,u)\in\ker\frac{\partial\Gamma_2}{\partial u}(x^*(s),u)\cap N_U^\circ(u)$$

such that

$$\Big\langle\frac{\partial\Gamma_1^j}{\partial u}(x^*(s),u),d_2\Big\rangle>\varepsilon_0\ \ \forall\,j\in J(x^*(t)).\tag{3}$$

The inward and outward regular process will be called regular with respect to the inequality state constraints.

Here, N_U° stands for the polar cone to N_U.

Definition 3. A control process $(x^*(t),u^*(t))$ is said to be inward/outward regular with respect to the state constraints provided that it is inward/outward regular with respect to the inequality state constraints while the trajectory $x^*(t)$ is regular with respect to the equality state constraints. The inward and outward regular process will be called regular with respect to the state constraints.

Definition 4. The controllability conditions are said to be satisfied with respect to the state constraints at the endpoint $p^*=(x^*(0),x^*(1))$, provided that for $s=1,2$,

$$\exists\, f_s\in\operatorname{conv}f(x_s^*,U(x_s^*)):$$
$$(-1)^s\Big\langle\frac{\partial g_1^j}{\partial x}(x_s^*),f_s\Big\rangle>0\ \ \forall\,j\in J(x_s^*).$$

Here, conv signifies the convex hull of the set.

This definition is also referred to in literature as "inward pointing condition" and "outward pointing condition" (IPC/OPC), when $s=1$ and $s=2$, respectively. The IPC/OPC can also be considered along the reference trajectory, should x_s^* be replaced with $x^*(t)$ in the above inequalities.

There is an important relation between these concepts.

Lemma 1. The inward regular with respect to the state constraints control process implies the inward pointing condition while outward regular with respect to the state constraints control process implies the outward pointing condition along the reference trajectory. Thus, the regular with respect to the state constraints control process implies the controllability conditions with respect to the state constraints at the endpoint.

The state constraints are said to be compatible with the endpoint constraints provided that

$$C\subseteq G\times G,$$

where

$$G:=\{x\in\mathbb{R}^n: g_1(x)\le0, g_2(x)=0\}.$$

The compatibility of constraints is not burdensome. Obviously, it can always be achieved by replacing the set C with the set $C\cap(G\times G)$. Therefore, in what follows, it is assumed that C is embedded in the state-constrained set $G\times G$.

3. MAXIMUM PRINCIPLE

Consider the extended Hamilton-Pontryagin function

$$\bar H(x,u,\psi,\mu):=\big\langle\psi,f(x,u)\big\rangle-\big\langle\mu,\Gamma(x,u)\big\rangle,$$

where $\psi\in\mathbb{R}^n$, $\mu=(\mu_1,\mu_2)$, $\mu_i\in\mathbb{R}^{k_i}$, and $\Gamma=(\Gamma_1,\Gamma_2)$.

Definition 5. Control process (x^*,u^*) is said to satisfy the maximum principle provided that there exist Lagrange multipliers: a number $\lambda\ge0$, an absolutely continuous vector-valued function $\psi\in\mathbb{W}_{1,\infty}([0,1];\mathbb{R}^n)$, a vector-valued component-wise decreasing function of bounded variation $\mu_1\in\mathbb{BV}([0,1];\mathbb{R}^{k_1})$, and a measurable vector-valued function $\mu_2\in\mathbb{L}_\infty([0,1];\mathbb{R}^{k_2})$ such that the following conditions hold:

- Non-triviality Condition
$$\lambda+\sum_{j=1}^{k_1}\operatorname{Var}\mu_1^j(t)|_0^1+$$
$$\operatorname{dist}\Big(\psi(t)-\mu_1(t)\frac{\partial g_1}{\partial x}(x^*(t)),\operatorname{im}\frac{\partial g_2^*}{\partial x}(x^*(t))\Big)>0\quad\forall\,t\in[0,1];$$
- Adjoint Equation
$$\dot\psi(t)=-\frac{\partial\bar H}{\partial x}(x^*(t),u^*(t),\psi(t),\mu(t))\ \text{ for a.a. } t\in[0,1];$$
- Transversality Condition
$$(\psi(0)-\mu_1(0)\frac{\partial g_1}{\partial x}(x_0^*),-\psi(1)+\mu_1(1)\frac{\partial g_1}{\partial x}(x_1^*))\in\lambda\frac{\partial\varphi}{\partial p}(p^*)+N_C(p^*);$$
- Maximum Condition
$$\max_{u\in U(x^*(t))}\bar H(x^*(t),u,\psi(t),\mu(t))=\bar H(x^*(t),u^*(t),\psi(t),\mu(t))\ \text{for a.a. } t\in[0,1];$$
- Conservation Law
$$\exists\,c\in\mathbb{R}:\ \mathcal{M}(t)=c\ \ \forall\,t\in[0,1],$$
where $\mathcal{M}(t):=\max_{u\in U(x^*(t))}\bar H(x^*(t),u,\psi(t),\mu(t))$;

- Euler-Lagrange Inclusion
$$\frac{\partial \bar{H}}{\partial u}(x^*(t), u^*(t), \psi(t), \mu(t)) \in \operatorname{conv} N_U(u^*(t));$$
- Complementary Slackness Condition
$$\int_0^1 \left\langle g_1(x^*(t)), d\mu_1(t) \right\rangle = 0.$$

Moreover, the following estimate on the multipliers is valid. There exists a number $\kappa > 0$ such that

$$|\mu_2(t) + \xi| \leq \kappa \cdot \left| \psi(t) - \mu_1(t) \frac{\partial g_1}{\partial x}(x^*(t)) + \xi \frac{\partial g_2}{\partial x}(x^*(t)) \right| \tag{4}$$
$$\text{for a.a. } t \in [0, 1] \text{ and for all } \xi \in \mathbb{R}^{k_2}.$$

Let us provide a few comments to this definition. While Non-triviality Condition, Adjoint Equation, Transversality Condition, Maximum Condition, Conservation Law, Euler-Lagrange Inclusion, and Complementary Slackness Condition are fairly known, Estimate (4) is new, and moreover, is an essential part of the maximum principle. This estimate on the multipliers is strongly related to the property of regularity with respect to the equality constraints and the fact that the case of a merely closed set U is examined. Note that it cannot be deduced from Euler-Lagrange Condition by virtue of regularity condition stated in Definition 1 due to the convexification of the right hand-side and thus should be proved separately. When U is convex, Estimate (4) is redundant.

Consider several important remarks.

Remark 1. Along with the set of Lagrange multipliers (λ, ψ, μ), the conditions of the maximum principle are also satisfied by the following set of the Lagrange multipliers

$$\left(\lambda, \psi(t) + a\frac{\partial g}{\partial x}(t), \mu(t) + a\right),$$

where $g = (g_1, g_2)$ and a is an arbitrary vector from $\mathbb{R}^k$, $k = k_1 + k_2$. See Pontryagin et al. (1962); Arutyunov et al. (2011).

Remark 2. Function $\mu_1^j(t)$ is constant on any time interval $[a, b]$, on which the optimal trajectory lies in the interior of the j-th state constraint set, that is, when $g_1^j(x^*(t)) < 0$ $\forall\, t \in [a, b]$.

Remark 3. Conservation Law is not a totally independent condition as its following part is simple to derive from the rest of the maximum principle, notably, from Maximum Condition, Adjoint Equation and from the monotonicity of the measure multiplier μ_1:

$$\exists\, c \in \mathbb{R}: \; \mathcal{M}(t) = c \;\; \forall\, t \in (0, 1),$$
$$\mathcal{M}(0) \geq c, \; \mathcal{M}(1) \geq c.$$

The rest of Conservation Law represents true (independent) optimality conditions:

$$\mathcal{M}(0) \leq c, \; \mathcal{M}(1) \leq c,$$

which may be regarded as transversality conditions with respect to time constraints: $t \geq 0$ and $t \leq 1$, being the time treated as state variable while Hamiltonian, or Energy, treated as adjoint function to time.

The assertion of the maximum principle is as follows.

Theorem 1. Suppose that control process (x^*, u^*) is optimal in Problem (1). If trajectory $x^*(t)$ is regular with respect to the equality state constraints then control process (x^*, u^*) satisfies the maximum principle.

The proof of this theorem can be given by virtue of the same arguments as in Arutyunov and Karamzin (2016, 2015) and essentially relying on the results obtained in Arutyunov et al. (2016), see Estimate (10) therein, which, at the end, implies Estimate (4). The proof is rather lengthy, and therefore, hereby is omitted.

Note that the statement of Theorem 1 represents the degenerate maximum principle in which degeneration may occur with respect to the inequality state constraints for the same reason as in the standard setting, Arutyunov (2000). So, it will always be the case should one of the endpoints lie at the boundary of the state-constrained set. Then, the conditions of the maximum principle can be satisfied by a trivial set of Lagrange multipliers, for a detailed explanation see Arutyunov (2000); Arutyunov et al. (2011).

Next section deals with condition ensuring the absence of such trivial sets of multipliers and also the existence of such sets of multipliers in which $\lambda > 0$.

4. NONDEGENERACY AND NORMALITY

The main results are as follows.

Theorem 2. Suppose that control process (x^*, u^*) is optimal in Problem (1). If (x^*, u^*) is regular with respect to the state constraints then it satisfies the maximum principle, in which the stronger non-triviality condition is valid:

$$\lambda + \operatorname{dist}\Big(\psi(t) - \mu_1(t)\frac{\partial g_1}{\partial x}(x^*(t)), \operatorname{im}\frac{\partial g_2^*}{\partial x}(x^*(t))\Big) > 0 \quad \forall\, t \in [0, 1]; \tag{5}$$

Non-triviality condition (5) forbids the degenerate multipliers under which the maximum principle holds trivially. Moreover, if $\lambda = 0$ then the maximum condition is informative due to (5). Next result concerns normality, that is, the conditions ensuring that $\lambda > 0$.

Theorem 3. Suppose that control process (x^*, u^*) is optimal in Problem (1). Suppose that (x^*, u^*) is inward regular with respect to the state constraints while the right endpoint is free, which means that $C = C_0 \times G$ for some closed $C_0 \subseteq G$. Then, control process (x^*, u^*) satisfies the maximum principle, albeit with $\lambda > 0$. Similarly, if (x^*, u^*) is outward regular with respect to the state constraints and the left endpoint is free, which means that $C = G \times C_1$ for some closed $C_1 \subseteq G$, then (x^*, u^*) satisfies the maximum principle with $\lambda > 0$.

Before proceeding to the proofs of these theorems, first we prove Lemma 1. Consider the system of constraints

$$F(x) \in S,$$

where F is a smooth map and S is a closed set. Consider a point x_*: $F(x_*) \in S$. The following regularity concept was first proposed in Mordukhovich (1988)

$$\ker[F'(x_*)]^* \cap N_S(F(x_*)) = \{0\}. \tag{6}$$

This regularity allows for metrical regularity of $F(x)$ with respect to S, that is,

$$\operatorname{dist}(x, F^{-1}(y + S)) \leq \text{const} \cdot \operatorname{dist}(F(x), y + S)$$

in the proximity of $(x_*, 0)$, Mordukhovich (2006). In particular, putting $y = 0$, one has,

$$\operatorname{dist}(x, F^{-1}(S)) \leq \text{const} \cdot \operatorname{dist}(F(x), S).$$

As a simple consequence, the Generalized Lyusternik theorem on the tangent cone is obtained.

Lemma 2. Assume that regularity condition (6) is satisfied at x_*. Then,

$$T_{F^{-1}(S)}(x_*) = \Big\{ d \in \mathbb{R}^n : F'(x_*)d \in T_S(F(x_*)) \Big\}.$$

Indeed, consider a vector $d \in \mathbb{R}^n$: $F'(x_*)d \in T_S(F(x_*))$. Then, there exist a sequence of numbers $t_i \to 0+$, and a sequence of vectors $\xi_i \to 0$ such that

$$F(x_*) + t_i F'(x_*)d + t_i\xi_i \in S \ \Rightarrow$$
$$\operatorname{dist}(F(x_* + t_i d), S)/t_i \to 0.$$

Then, by virtue of metric regularity,

$$\operatorname{dist}(x_* + t_i d, F^{-1}(S))/t_i \to 0,$$

and thereby, d is in the tangent cone to the solution set $F^{-1}(S)$.

Vice-versa, let d be from the tangent cone. Then, there exist a sequence of numbers $t_i \to 0+$, and a sequence of vectors $\xi_i \to 0$ such that

$$F(x_* + t_i d + t_i\xi_i) \in S.$$

Then, the Taylor's expansion completes the proof.

Proof of Lemma 1. Consider a point $t_0 \in [0,1)$. The regularity of the equality state constraints implies that, for the data

$$F(u) = (\Gamma_2(x^*(t_0), u), u), \ \ S = \{0\} \times U,$$

Regularity condition (6) is satisfied which is simple to verify. Thus, Lemma 2 can be invoked.

Consider function

$$h(x) := \big\langle \alpha, g_1(x) \big\rangle,$$

where $\alpha \neq 0$ is some vector from $\mathbb{R}^{k_1}$ such that $\alpha^j \geq 0$, and $\alpha^j = 0$ for $j \notin J(x^*(t_0))$. It is clear that $h(x^*(t)) \leq 0$ $\forall\, t \in [0,1]$ while $h(x^*(t_0)) = 0$. Thus, this convolution h can be regarded as a scalar state constraint for which the inward/outward regularity of the reference control process is obviously satisfied.

Note that there exists a vector $u \in U(x^*(t_0))$ such that $\big\langle \alpha, \Gamma(x^*(t_0), u) \big\rangle < 0$. Indeed, otherwise $\big\langle \alpha, \Gamma(x^*(t_0), u) \big\rangle = 0\ \forall\, u \in \mathcal{U}(t_0)$. Take any $u_0 \in \mathcal{U}(t_0)$ and consider the inward pointing vector $d_1 \in \ker \frac{\partial \Gamma_2}{\partial u}(x^*(t), u_0) \cap N_U^\circ(u_0)$ figuring in Definition 2. Since $N_U^\circ(u_0) \subseteq T_U(u_0)$, by Lemma 2, we derive that $d_1 \in T_{U(x^*(t_0))}(u_0)$. Then, it is clear that the inward regularity condition (2) implies the existence of the above vector u by virtue of definition of α.

Consider the convex cone

$$K := \operatorname{conv} \operatorname{cone}\Big\{ \frac{\partial g_1^j}{\partial x}(x^*(t_0)),\ j \in J(x^*(t_0)) \Big\}.$$

Cone K is pointed as the inward regularity condition implies (2) positive linear independence of the gradients which generate K. Therefore, $\operatorname{int} K^\circ \neq \varnothing$.

Consider the set

$$F = \operatorname{conv} f(x^*(t_0), U(x^*(t_0))).$$

It is simple to demonstrate that

$$K^\circ = \Big\{ y \in \mathbb{R}^n : \Big\langle y, \frac{\partial g_1^j}{\partial x}(x^*(t_0)) \Big\rangle \leq 0\ \forall\, j \in J(x^*(t_0)) \Big\}.$$

Therefore, in order to ensure the IPC, it is sufficient to establish that

$$F \cap \operatorname{int} K^\circ \neq \varnothing. \tag{7}$$

Assume the contrary. Then, the separability theorem applied to the disjoint convex sets $\operatorname{conv} F$ and $\operatorname{int} K^\circ$ yields the existence of a nonzero vector $\xi \in \mathbb{R}^n$ such that $\big\langle \xi, y \big\rangle \geq 0\ \forall\, y \in F$ and $\xi \in K^{\circ\circ}$. At the same time, K is closed and convex, and thus, $K^{\circ\circ} = K$ and $\xi \in K$.

Then, by virtue of definition of K, there exists a vector $\alpha \in \mathbb{R}^{k_1}$ such that

$$\xi = \alpha \frac{\partial g_1}{\partial x}(x^*(t_0))$$

and $\alpha^j \geq 0\ \forall\, j \in J(x^*(t_0))$, $\alpha^j = 0$, $\forall\, j \notin J(x^*(t_0))$. At the same time, it is proved above for the given α that there exists $y \in F$ such that $\big\langle \xi, y \big\rangle < 0$. This contradiction proves (7).

By carrying out similar argument at point $t_0 \in (0,1]$, but using the outward regular condition (3), we obtain the OPC. The proof is complete.

Proof of Theorem 2. Application of Theorem 1 yields the existence of a set of Lagrange multipliers (λ, ψ, μ), satisfying the maximum principle.

Assume that (5) is violated. Then, $\lambda = 0$, and

$$\psi(t_0) - \mu_1(t_0)\frac{\partial g_1}{\partial x}(x^*(t_0)) = \xi \frac{\partial g_2}{\partial x}(x^*(t_0))$$

for some $t_0 \in [0,1]$ and $\xi \in \mathbb{R}^{k_2}$.

Using Remark 1, it is not restrictive to consider the new set of multipliers

$$\lambda,\ \psi(t) - a\frac{\partial g}{\partial x}(x^*(t)),\ \mu - a,$$

where $a = (\mu_1(t_0), \xi)$, which still satisfies the maximum principle. Relabeling, we keep for the new multipliers the same notation. Then, obviously,

$$\psi(t_0) = 0,\ \mu_1(t_0) = 0.$$

Consider the cover of $[0,1]$ by open intervals

$$O_t := (t - \delta, t + \delta),$$

where $\delta = \delta(t)$ is taken from Definition 2. Using the upper-semicontinuity of the mapping $J(x)$, by decreasing if necessary positive δ, one may consider that the neighbourhood O_t is such that $J(x^*(s)) \subseteq J(x^*(t))$ for all $s \in O_t$. Using the compactness, consider the finite subcover O_{t_i}, where $t_i \in [0,1]$, $i = 1, ..., N$.

Consider the Euler-Lagrange inclusion

$$\frac{\partial \bar{H}}{\partial u}(x^*(t), u^*(t), \psi(t), \mu(t)) \in \operatorname{conv} N_U(u^*(t)),$$

which holds for a.a. $t \in [0,1]$. By the Denjoy theorem, we have $u^*(t) \in \mathcal{U}(t)$ for a.a. $t \in [0,1]$. Let $t_0 \in O_{t_M}$ for some $M \in \{1, .., N\}$, and $t_0 \leq t_M$. Then, considering the outward pointing vector $d_2 = d_2(t_M, t, u^*(t))$ from Definition 2 and multiplying the above inclusion by d_2, we have

$$O(|\psi(t)|) - \mu_1(t)\frac{\partial \Gamma_1}{\partial u}(x^*(t), u^*(t))d_2 \leq 0$$

on the interval $(t_0, t_M + \delta(t_M))$. Using (3), and Remark 2, since $\mu_1^j(t) \leq 0\ \forall\, t \geq t_0$, while $\mu_1^j(t) = 0$ for all $t \in O_{t_M}$ for $j \notin J(x^*(t_M))$, we derive the estimate

$$|\mu_1(t)| \leq \text{const}\, |\psi(t)|,\ \ t \geq t_0,\ \ t \in O_{t_M},$$

where constant const does not depend on t, t_0.

Then, in view of (4),

$$|\mu_2(t)| \leq \text{const}\, |\psi(t)|,\ \ t \geq t_0,\ \ t \in O_{t_M}.$$

Therefore, applying the Gronwall inequality to Adjoint equation on O_{t_M}, by virtue of the above estimates on the multipliers, we derive that $\psi(t) = 0\ \forall\, t \in [t_0, t_M + \delta(t_M)]$. Then $\mu(t) \equiv 0$ on $[t_0, t_M + \delta(t_M))$. By continuing the same arguments in a finite number of steps, we prove that $\psi(t) = 0$ on $\forall t \in [t_0, 1]$, and $\mu(t) = 0 \; \forall t \in [t_0, 1)$.

By repeating the same arguments, albeit on the left from point t_0 and using for this the inward pointing vector d_1 and Condition (2) from Definition 2, we obtain that $\psi(t) = 0$ on $\forall t \in [0, t_0]$, and $\mu(t) = 0 \; \forall t \in (0, t_0]$. Thus, $\psi(\cdot) = 0$, while $\mu(t) = 0 \; \forall t \in (0, 1)$.

By virtue of Lemma 1, Controllability conditions with respect to the state constraints are fulfilled at the endpoints. This, as is simple to see, along with Conservation Law leads to a contradiction with Non-triviality Condition. Indeed, consider the point $t = 0$. From Conservation Law, since $c = 0$, we have

$$\max_{u \in U(x_0^*)} \left\langle -\mu_1(0), \Gamma_1(x_0^*, u) \right\rangle = 0 \quad \Rightarrow$$

$$\min_{u \in U(x_0^*)} \left\langle \mu_1(0), \Gamma_1(x_0^*, u) \right\rangle = 0.$$

However, the IPC asserts that the above minimum is strictly negative if $\mu_1(0) \neq 0$. Thus, $\mu_1(0) = 0$. The point $t = 1$ is considered similarly, but by invoking the OPC. Thus, all the multipliers vanish contradicting Non-triviality Condition.

The proof is complete.

Proof of Theorem 3. Consider the first case of the theorem, that is, the case of inward regularity and when the terminal point x_1^* is free. Theorem 1 yields the existence of a set of Lagrange multipliers (λ, ψ, μ), satisfying the conditions of the maximum principle. Assume that $\lambda = 0$. In view of Remark 1, it is not restrictive to consider that $\mu_1(1) = 0$. Then, from Transversality Condition, it follows the existence of vectors $\xi_i \in \mathbb{R}^{k_i}$, $i = 1, 2$, such that

$$\psi(1) = \xi_1 \frac{\partial g_1}{\partial x}(x_1^*) + \xi_2 \frac{\partial g_2}{\partial x}(x_1^*),$$

and $\xi_1^j \leq 0$, $j = 1, ..., k_1$. Using Remark 1, consider the new set of Lagrange multipliers

$$\lambda, \;\; \tilde{\psi}(t) := \psi(t) - \xi_1 \frac{\partial g_1}{\partial x}(x^*(t)) - \xi_2 \frac{\partial g_2}{\partial x}(x^*(t)),$$

and

$$\tilde{\mu}(t) := \mu(t) - \xi,$$

where $\xi = (\xi_1, \xi_2)$. This set of multipliers again satisfies all the conditions of the maximum principle. Keeping the same notation, we relabel by $\psi := \tilde{\psi}$ and $\mu := \tilde{\mu}$. Note that $\psi(1) = 0$, and $\mu(t) \geq 0 \; \forall t \in [0, 1]$ by construction.

Following the proof to Theorem 2, consider the cover of $[0, 1]$ by open intervals

$$O_t := (t - \delta, t + \delta),$$

where $\delta = \delta(t)$ is taken from Definition 2. Using the upper-semicontinuity of the mapping $J(x)$, by decreasing if necessary positive δ, one may consider that the neighbourhood O_t is such that $J(x^*(s)) \subseteq J(x^*(t))$ for all $s \in O_t$. Using the compactness, consider the finite subcover O_{t_i}, where $t_i \in [0, 1]$, $i = 1, ..., N$, $t_1 < t_2 < ... < t_N$.

Consider Euler-Lagrange Inclusion on $[0, 1]$. Then, considering neighbourhood $O(t_N)$ which contains point $t = 1$ and inward pointing vector $d_1 = d_1(t_N, t, u^*(t))$ from Definition 2 and multiplying this inclusion by d_1, since $u^*(t) \in \mathcal{U}(t)$ for a.a. $t \in [0, 1]$, we have

$$O(|\psi(t)|) - \mu_1(t) \frac{\partial \Gamma_1}{\partial u}(x^*(t), u^*(t)) d_1 \leq 0$$

on the interval $(t_N - \delta(t_N), 1)$. Using (2), and Remark 2, since $\mu_1^j(t) \geq 0 \; \forall t$, while $\mu_1^j(t) = 0$ for all $t \in O_{t_N}$ for $j \notin J(x^*(t_N))$, we derive the estimate

$$|\mu_1(t)| \leq \text{const} \, |\psi(t)|, \;\; t \in O_{t_M} \cap [0, 1],$$

where constant const does not depend on t.

Then, in view of (4),

$$|\mu_2(t)| \leq \text{const} \, |\psi(t)|, \;\; t \in O_{t_M} \cap [0, 1].$$

Therefore, applying the Gronwall inequality to Adjoint equation on O_{t_N}, by virtue of the above estimates on the multipliers, we derive that $\psi(t) = 0 \; \forall t \in [t_N - \delta(t_N), 1]$. Then, by virtue of the above estimates on μ_1 and monotonicity of μ_1, it follows that $\mu(t) \equiv 0$ on $[t_N - \delta(t_N), 1]$. Repeating the same arguments, in a finite number of steps, we prove that $\psi(t) = 0$ on $\forall t \in [0, 1]$, and $\mu(t) = 0 \; \forall t \in (0, 1]$.

By virtue of Lemma 1, Controllability conditions with respect to the state constraints are fulfilled at point x_0^*. This, as is shown in the proof to Theorem 2, along with Conservation Law implies that $\mu_1(0) = 0$ and thus, leads to a contradiction with Non-triviality Condition.

The case of outward regularity and free initial point x_0^* is considered similarly, albeit using point $t = 0$ as a starting point instead of point $t = 1$.

The proof is complete.

Remark 4. In Theorems 1, 2, 3, set U is compact, thus, the case of bounded feasible set is examined. The case of *unbounded* but still closed feasible set can be treated by virtue of the technique proposed in Arutyunov et al. (2016). In particular, due to the results obtained therein, the statements of these theorems will hold true under U convex or semi-algebraic. There is a number of other types of the sets suitable for such an extension, see Arutyunov et al. (2016). In particular, the case $U = \mathbb{R}^m$, which is important in view of Calculus of Variations, is certainly covered.

5. CONCLUSION

In this work, the conditions for nondegeneracy and normality of the maximum principle for optimal control problems with equality and inequality state constraints have been presented and proved. These conditions essentially rely on the notion of inward and outward regularity of the reference control process. It has been shown that the inward regularity implies the inward pointing condition while the outward regularity implies the outward pointing condition with respect to the state constraints. The case of closed feasible set U has been considered that which invoked such concepts of non-smooth analysis as the Mordukhovich limiting normal cone and the Mordukhovich regularity condition.

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