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# Event-triggered Data-efficient Observers of Perturbed Systems ⋆

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**Abstract:** In this paper, an event-triggered, data-rate constrained observer for discrete-time linear systems with perturbations is presented. The system is connected to a remote location by a communication channel which can only transmit limited numbers of bits per time interval. The system is subject to perturbations in its state as well as errors in the output measurement. The objective is to reconstruct estimates of the state at the remote location, by sending messages over the communication channel. A new type of data-rate constrained observer which can be more efficient in terms of communication rate is presented. Relation between an admissible communication rate and the system parameters is evaluated. The observer's efficiency is illustrated by simulations.

*Keywords:* Observers, State estimation, Limited data rate, Discrete-time Systems, Perturbation

## 1. INTRODUCTION

Since wireless technologies were invented in 1880 by Alexander Graham and Charles Tainter, much progress has been done on increasing the maximum communication rate as well as reduce the required energy to transmit messages. In modern days, wireless technologies are omnipresent in industrial applications. Many of these applications also involve dynamical systems and as such, there is an entire subfield of the dynamics and control area that is dedicated to studying interactions between dynamical systems and communication channels. The problems studied in this subfield all include one or several dynamical systems, possibly with sensors and actuators at locations remote from one another and all connected together via one or several data-rate constrained communication channels with or without packet losses on the channels. A few examples of such systems are: cooperative driving of wirelessly connect cars, synchronization or formation control of drones, distributed sensing via networks of connected sensors, etc... The presence of limited data-rates and/or packet losses in conjunction with sources of uncertainty implies that it is necessary to develop specific solutions for those particular problems. The sources of uncertainty include: noise, perturbations, parametric uncertainty, and uncertainties in the initial conditions.

The earliest works involving dynamical systems and data-rate constraints can be found in Wong and Brockett (1997). After that, a lot of attention was given to linear systems (see e.g. Elia and Mitter (2001) and references therein). Overviews of the results for linear systems can be found in Nair et al. (2007), Baillieul and Antsaklis (2007) and Andrievsky et al. (2010).

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The nonlinear case has been studied extensively as well. The earliest results were obtained in De Persis (2003) and Baillieul (2004). More general results were then obtained in Nair et al. (2004) and Liberzon and Hespanha (2005) which generalized linear techniques to nonlinear systems. After that, different notions of entropy were used to provide bounds on the sufficient and/or necessary data-rates to observe/control nonlinear dynamical systems over data-rate constrained channels (see Kawan (2009), Matveev and Savkin (2009), Kawan (2017), Sibai and Mitra (2017), Liberzon and Mitra (2016), Matveev and Pogromsky (2016), Sibai and Mitra (2018), Voortman et al. (2019) and Matveev and Pogromsky (2019)).

The recent inclusion of event-triggered control in the dynamics and control field (see e.g. Heemels et al. (2012) or Hetel et al. (2017)) has also impacted the subfield of systems and control with communication channels. In Han et al. (2015), Shi et al. (2016), Huang et al. (2017), and Trimpe (2017) the authors consider different variants of the problem of state estimation for systems with Gaussian state perturbations and Gaussian output measurement errors. In Xia et al. (2017) and Muehlebach and Trimpe (2018) extensions for networked state estimation are presented.

We focus on developing an event-triggered observer for dynamical systems with bounded state perturbations and bounded output measurement errors. This observer transmits estimates of the state to a remote location by sending messages over a data-rate constrained communication channel. It is designed in such a way that the time between two consecutive messages can be chosen. The main contribution of the paper is providing, in function of the chosen time interval between two consecutive communications, an upper bound on the resulting observation error as well as the required minimum data-rate to implement the observer. Moreover, since the observer functions on an event-triggered basis, it can be much more efficient in terms of required rate than the theoretical worst-case required data-rate, as is proven through simulations.

This paper is structured as follows. In Section 2, we define the structure of the problem which we intend on solving. In Section 3, we provide a solution to the previously exposed problem in the form of a communication procedure. In Section 4, we prove two results. Firstly a proposition that gives an upper bound on the observation error, secondly a theorem which provides sufficient conditions on the minimum channel rate to implement the procedure of Section 3. Section 5 shows, through simulations, how this communication procedure can sometimes require bit-rates much lower than the theoretical maximum of the theorem in Section 4.

## 2. PROBLEM STATEMENT

We consider linear discrete-time systems of the following form

$$x[k+1] = Ax[k] + d[k],$$
$$y[k] = Cx[k] + w[k],$$

where $x[k] \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n\times n}$, $y[k] \in \mathbb{R}^m$ is the output, $C \in \mathbb{R}^{m\times n}$, $d[k] \in \mathbb{R}^n$ is an unknown state perturbation, and $w[k] \in \mathbb{R}^m$ is an unknown measurement error. The perturbations $d[k]$ and errors $w[k]$ verify

$$\|d[k]\|_2 \leq \delta,\ \|w[k]\|_2 \leq \omega,$$

$\forall k \geq 0$, where $\delta$ is the maximum state perturbation and $\omega$ is the maximum measurement error, both of which are known constants. In this paper, the notation $\|\cdot\|_2$ refers to the Euclidean norm in $\mathbb{R}^n$. We use the notation $\sigma_i(M)$ to refer to the $i$-th singular value of the matrix $M$, where the singular values are ranked in non-increasing order ($\sigma_1(M) \geq \cdots \geq \sigma_n(M)$).

The system is equipped with a smart sensor (a sensor admitting some computational capacities, which allows it to perform additional computations on the measured data) and it is connected to a remote location via a data-rate constrained communication channel which can only send messages that are of finite size. The objective is to provide estimates $\hat{x}[k]$ of $x[k]$ at the remote location by sending messages over this communication channel. The sensor and the remote location have an initial estimate $\hat{x}[0]$ which verifies

$$\|x[0] - \hat{x}[0]\|_2 \leq \varepsilon_0, \tag{1}$$

where $\varepsilon_0$ is a user-specified parameter corresponding to the error of initial conditions. The constraint on the channel data-rates is such that for any time interval between two consecutive transmission $\Delta k$, the channel can only transmit a maximum number of bits $b^+(\Delta k)$.

In order to generate the estimates, messages $m[k_j]$, where $k_j$ are the transmission times, are sent. Four ingredients interact with these messages: a sampler $\mathscr{S}$, a coder $\mathscr{C}$, an alphabet $\mathscr{A}$, and a decoder $\mathscr{D}$. The four devices together form a communication protocol. The following constants/parameters are known by all devices: the system matrices $A$ and $C$, the maximum state perturbation $\delta$, the maximum measurement error $\omega$, the discretization error $\varepsilon$ (which is induced by coding/decoding operation and for brevity of exposition in this note the constant $\varepsilon_0 = \varepsilon$), and the initial estimate $\hat{x}[0]$. At the system side, the sampler $\mathscr{S}$ generates the instants of transmission in the following way

$$k_{j+1} = \mathscr{S}(k_j, y[0], y[1], \ldots, y[k_j], m[k_1], \ldots, m[k_j]), \tag{2}$$

$k_0 = 0$, $m[0] = \emptyset$. The coder then generates the messages in the following way

$$m[k_j] = \mathscr{C}(\hat{x}[0], y[0], \ldots, y[k_j], m[k_1], \ldots, m[k_{j-1}]), \tag{3}$$

$\forall k_j : j > 0$. At each communication instant, the list of different possible messages is encoded into a finite-sized alphabet (the finite-sizedness being due to the data-rate constraints). The alphabet $\mathscr{A}$ determines the last index of the messages $l_j$ in the following way

$$l_j = \mathscr{A}(\hat{x}[0], m[k_1], \ldots, m[k_{j-1}]), \qquad \forall k_j : j > 0. \tag{4}$$

The restriction on the choice of messages is then

$$m[k_j] \subset \{1, \ldots, l_j\}, \qquad \forall k_j : j > 0.$$

After encoding the messages into sequences of bits, the number of transmitted bits should not exceed the maximum number of bits that can be sent during the communication interval. This implies the following constraint on the alphabet length.

$$\log_2 l_j \leq b^+(k_{j+1} - k_j) \qquad \forall k_j : j > 0. \tag{5}$$

At the remote location, the decoder $\mathscr{D}$ receives the messages and interprets them to generate an estimate of the state $\hat{x}[k]$ in the following way

$$\hat{x}[k] = \mathscr{D}(\hat{x}[0], m[k_1], \ldots, m[k_j]), \quad \forall k \in \{k_j, \ldots, k_{j+1} - 1\},$$

$\forall j \geq 0$. Because of the perturbation, measurement error and finite data-rate, it is impossible to provide exact estimates at the remote location. Instead, the design of the communication protocol should ensure that the estimation error $\|x[k] - \hat{x}[k]\|_2$ is bounded and proportional to the different sources of perturbations/noise that affect the system. The upper bound on the estimation error is decomposed as follows

$$\|x[k] - \hat{x}[k]\|_2 \leq \gamma_\varepsilon \varepsilon + \gamma_\delta \delta + \gamma_\omega \omega, \qquad \forall k \geq 0, \tag{6}$$

where $\gamma_\varepsilon, \gamma_\delta, \gamma_\omega \in \mathbb{R}^+$ are constants which indicate the contribution of the different errors/perturbation in the total error. The first objective of the paper is to design a data-rate constrained observer and communication scheme which on average performs better in terms of transmitted data if the perturbations are not the worst-case perturbations every time. The second objective of the present paper is to investigate the relationship between the time interval between subsequent communications $\Delta k_j := k_{j+1} - k_j$, the maximum number of bits per time interval $b^+(\cdot)$, and the error constants $\gamma_\varepsilon, \gamma_\delta, \gamma_\omega$ for the proposed communication scheme.

## 3. DESIGNING THE AGENTS

In this section, we introduce the different agents of the communication protocol. The main mechanism can be described as follows: at the sensor side, a local observer transforms the output into estimates of the state $\bar{x}[k]$. A copy of the decoder is also simulated by the computational capacity of the sensor such that it is known what estimate $\hat{x}[k]$ the decoder currently has. This copy of the remote estimate which is provided by the smart sensor will be denoted $\hat{x}_c[k]$. Starting at the initial estimate $\hat{x}[0]$ and in the absence of messages, the decoder simply updates the estimate by iterating the matrix $A$. When the distance between $\bar{x}[k]$ and $\hat{x}_c[k] = \hat{x}[k]$ becomes larger than the prescribed maximum error (including a margin for the observation error $\bar{e}[k] := x[k] - \bar{x}[k]$), the sampler decides to communicate and the coder sends a message to the decoder to provide a new estimate $\hat{x}[k]$. Figure 1 depicts how the different elements interact. Below, each of these algorithms are presented in details.

### 3.1 The Local Observer

The local observer equation is

$$\bar{x}[k+1] = A\bar{x}[k] - L(y[k] - C\bar{x}[k]), \tag{7}$$

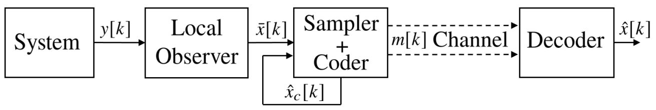


Fig. 1. Structure of setup.

where $L \in \mathbb{R}^{n\times m}$ is a gain matrix. Note that the dynamics of $\bar{e}[k]$ are thus

$$\bar{e}[k+1] = (A+LC)e[k] + d[k] + Lw[k]. \tag{8}$$

The observer uses $\bar{x}[0] = \hat{x}[0]$ as an initial point, which implies that $\|\bar{e}[0]\|_2 \le \varepsilon_0$. In order to minimize the worst case local observation error and to keep the local observation error bounded, $L$ is computed as the solution of

$$\begin{aligned} L = \arg\min_{L,\gamma_1} \; & \delta + \sigma_1(L)\omega - \gamma_1 \\ \text{s.t. } & \sigma_1(A+LC) < 1-\gamma_1, \\ & \gamma_1 \ge 0 \end{aligned} \tag{9}$$

This choice of nonlinear program implies that the worst case error is bounded, as proven by the following proposition.

*Proposition 1.* Assume that there exists an $L$ such that the inequality $\sigma_1(A+LC) < 1$ has a solution and $\varepsilon$ is chosen small enough, then choosing $L$ as in (9) leads to the following inequality

$$\|\bar{e}[k]\|_2 \le \frac{\delta + \sigma_1(L)\omega}{1-\sigma_1(A+LC)}, \qquad \forall k \ge 0.$$

*Proof:* We have that (8) implies that

$$\|\bar{e}[k+1]\|_2 \le \sigma_1(A+LC)\,\|\bar{e}[k]\|_2 + \delta + \sigma_1(L)\omega. \tag{10}$$

By choosing $\varepsilon_0 \le \delta + \sigma_1(L)\omega$, we have $\|\bar{e}[0]\|_2 \le \delta + \sigma_1(L)\omega$. Inequality (10) applied several times implies that

$$\|\bar{e}[k]\|_2 \le \sum_{i=0}^{k} \sigma_1(A+LC)(\delta + \sigma_1(L)\omega),$$

which, due to the sum of a geometric series and because $L$ is chosen such that $\sigma_1(A+LC) < 1$ implies that

$$\begin{aligned} \|\bar{e}[k]\|_2 &\le \frac{1-\sigma_1(A+LC)^k}{1-\sigma_1(A+LC)}(\delta + \sigma_1(L)\omega) \\ &\le \frac{\delta + \sigma_1(L)\omega}{1-\sigma_1(A+LC)}. \end{aligned}$$

■

For brevity, we define the notation $\eta := \frac{\delta+\sigma_1(L)\omega}{1-\sigma_1(A+LC)}$. The following proposition provides an alternative formulation of the previous nonlinear program in the form of an LMI program.

*Proposition 2.* If the following LMI, is feasible, then (9) has a solution:

$$\begin{aligned} L = \arg\min_{\gamma_i,} \; & -\gamma_1 + \delta^2\gamma_2 + \omega^2\gamma_3 - \gamma_4 \\ \text{s.t. } & \begin{bmatrix} -I_n+\gamma_4 & 0_{n\times n} & 0_{n\times m} & A^\top + C^\top L^\top \\ 0_{n\times n} & -\gamma_2 I_n & 0_{n\times m} & I_n \\ 0_{m\times n} & 0_{m\times n} & -\gamma_3 I_m & L^\top \\ A+LC & I_n & L & -I_n \end{bmatrix} \preceq 0, \\ & \begin{bmatrix} -I_n + \gamma_1 I_n & A^\top + C^\top L^\top \\ A+LC & I_n \end{bmatrix} \prec 0, \\ & \gamma_i \ge 0, \end{aligned} \tag{11}$$

where $I_n$ is the $n\times n$ identity matrix and $0_{n\times m}$ is the $n\times m$ zero matrix.

The proof of this proposition will be presented in the full version of this paper.

*Remark 1.* It is possible to use another Lyapunov function such as $\bar{V}(\bar{e}) = \bar{e}^\top P\bar{e}$ instead of $V(\bar{e}) = \bar{e}^\top\bar{e}$ in the above developments for more generality. In such a case in order to keep tightness in the inequalities used in the present work, it is necessary to define the singular values differently for the previous analysis. One way to achieve these tight bounds is to replace $\sigma_1(M)$, with the square root of the $P$-generalized eigenvalues of the matrix $M^\top PM$ (see e.g. Pogromsky and Matveev (2011) for more details about $P$-generalized eigenvalues). For brevity, the mathematical details of the usage of $\bar{V}(\bar{e})$ are not explored in this paper.

### *3.2 The Protocol Description*

We now present the communication procedure, which we will further reference as Procedure 1. It is composed of a sampler, alphabet, coder and decoder as described below. For this particular communication procedure, a minimum time interval between communications is going to be employed. This quantity, denoted as $\Delta k$ is known by all agents. It is a user-specified parameter which is to be chosen finite and it directly influences the upper bound on the estimation error. How exactly one might choose $\Delta k$ and how it influences the error will be discussed further in this paper.

To properly describe the communication instants, we will need several quantities. The indexes $j$ of the communication instants are inherently known by all agents. The quantity $\bar{j}$ refers to the index of the last instant of communication (initially, $\bar{j} = 0$). This quantity is always known by the sampler (because it knows how many communication instants it defined), the coder (because it knows how many messages it sent), as well as the decoder (because it knows how many messages it received). Finally, the sampler and coder interact to update the knowledge of the estimate $\hat{x}[k]$ at the decoder side.

*Procedure 1.*

*The Sampler:* At each time instant $k \ge k_{\bar{j}} + \Delta k$, the sampler verifies whether the following condition is satisfied

$$\|\bar{x}[k] - A\hat{x}_c[k-1]\|_2 \le \delta \sum_{f=1}^{\Delta k-1} \sigma_1(A)^{f-1} + \sigma_1(A)^{\Delta k-1}\varepsilon. \tag{12}$$

If the condition is verified, the sampler sets

$$\hat{x}_c[k] = A\hat{x}_c[k-1].$$

If the condition is not verified, a message must be sent to provide a new estimate. The sampler thus sets $\bar{j} = j = j+1$ and $k_{\bar{j}} = k$ ($j$ increases by one and $\bar{j} = j$).

*The Alphabet:* If $k = k_{\bar{j}}$, the alphabet agent builds a covering of the set $I_j$, where $I_j$ is defined as

$$\begin{aligned} I_j := \Big\{ x \in \mathbb{R}^n \,|\, \big\|x - A\hat{x}_c[k_j-1]\big\|_2 \le \delta \sum_{f=1}^{\Delta k} \sigma_1(A)^{f-1} \\ + \sigma_1(A)^{\Delta k}\varepsilon + (\sigma_1(A)^{\Delta k}+1)\eta \Big\}, \end{aligned} \tag{13}$$

with balls of size $\varepsilon$. The balls in the covering are numbered from 1 till $l_{\bar{j}}$, where $l_{\bar{j}}$ is the length of the alphabet.

*The Coder:* At the communication instants, the coder function finds the index of the ball in the covering made by the alphabet whose center is the closest to $\bar{x}[k_j]$ and sends this index over the communication channel. The coder also updates the local estimate $\hat{x}_c[k]$ by setting it to be equal to the center of the chosen ball.

*The Decoder:* If the decoder receives no message, it assigns

$$\hat{x}[k] = A\hat{x}[k-1],$$

otherwise it uses the center of the ball whose index it received as the new estimate.

The alphabet is based on the following idea. As was previously mentioned, in the absence of messages, new estimates are obtained at the decoder side by iterating $A$ (i.e. $\hat{x}[k+1] = A\hat{x}[k]$ and $\hat{x}_c[k+1] = A\hat{x}_c[k]$). After receiving a message, the state of the system $x[k]$ is contained in a ball of a certain radius whose center is the estimate $\hat{x}[k]$. In the absence of any messages, this "ball" of uncertainty is gradually deformed into a larger/small uncertain set. The uncertain set evolves due to two factors: first of all, the unknown state perturbation $d[k], d[k+1], \dots$ increases its radius by $\delta$ at each time step, secondly, the uncertainty set is stretched/compressed by action of the linear mapping $A$ (the deformations are proportional to the singular values of $A$). Given that the communication intervals are chosen to be finite, this uncertain set remains of finite size in between communications. It can thus be covered by a finite number of balls of size $\varepsilon > 0$. The balls in the covering can be indexed from 1 till $l_{\max} < \infty$. In order to produce such a covering, the only information needed are the initial ball and the different upper bounds on the uncertainties/errors, which implies that both the coder as well as the decoder can build the set. In order to transmit a new estimate, one can simply send the index of one of the balls whose center then serves as a new estimate with a precision that will depend on $\varepsilon$. The cost of communicating in that fashion is dependent on how many balls of size $\varepsilon$ are required to cover the uncertain set.

In practice, the previously described set is contained within the set $I_j$. The alphabet relies on the assumption that the estimate $\bar{x}[k]$ will lie within the set $I_j$ when the communications occur. This assumption guarantees that $\|\bar{x}[k] - \hat{x}[k]\|_2 \leq \varepsilon$ after each communication instant, which makes the procedure repeatable. The following lemma proves this assumption.

*Lemma 1.* For the set $I_j$ as defined in (13) and estimates $\bar{x}[k]$ as generated by the observer (7) the following holds: $\bar{x}[k_j] \in I_j$, $\forall j \geq 1$.

*Proof:* We first consider the case $k_j = k_{j-1} + \Delta k$. Starting from $k = k_{j-1} + 1$, the error $\hat{e}[k] := x[k] - A\hat{x}[k-1]$ follows particular dynamics. Indeed, we have

$$\begin{aligned} \hat{e}[k+1] &= x[k+1] - A\hat{x}[k] = Ax[k] + d[k] - A\hat{x}[k] \\ &= A\hat{e}[k] + d[k], \end{aligned} \tag{14}$$

$\forall k \in \{k_{j-1}, \dots, k_j - 1\}$. Because the estimates that was transmitted at $k_{j-1}$ is of precision $\varepsilon$, we have

$$\left\|\hat{e}[k_{j-1}]\right\|_2 \leq \left\|\bar{x}[k_{j-1}] - \hat{x}[k_{j-1}]\right\|_2 + \left\|\bar{e}[k_{j-1}]\right\|_2 \leq \varepsilon + \eta.$$

This implies that we have

$$\left\|\hat{e}[k_j]\right\|_2 \leq \sigma_1(A)^{\Delta k}(\varepsilon + \eta) + \sum_{f=1}^{\Delta k} \sigma_1(A)^{f-1}\delta.$$

By using Proposition 1, we can decompose the distance between $\bar{x}[k_j]$ and $A\hat{x}[k_j - 1]$

$$\begin{aligned} \left\|\bar{x}[k_j] - A\hat{x}[k_j-1]\right\|_2 &\leq \left\|\hat{e}[k_j]\right\|_2 + \left\|\bar{e}[k_j]\right\|_2 \\ &\leq \sigma_1(A)^{\Delta k}(\varepsilon + \eta) + \sum_{f=1}^{\Delta k} \sigma_1(A)^{f\text{-}1}\delta + \eta \end{aligned}$$

which implies that $\bar{x}[k_j] \in I_j$ for $k_j = k_{j-1} + \Delta k$. For $k_j > k_{j-1} + \Delta k$, (12) was necessarily verified at $k = k_j - 1$, which implies that

$$\left\|x[k_j - 1] - A\hat{x}[k_j - 2]\right\|_2 \leq \delta \sum_{j=1}^{\Delta k - 1} \sigma_1(A)^{j-1} + \sigma_1(A)^{\Delta k - 1}\varepsilon.$$

Because of the error dynamics of $\hat{e}[k]$, we have that

$$\begin{aligned} \left\|\bar{x}[k_j] - A\hat{x}[k_j - 1]\right\|_2 &\leq \left\|\hat{e}[k_j]\right\|_2 + \left\|\bar{e}[k_j]\right\|_2 \\ &\leq \delta \sum_{j=1}^{\Delta k} \sigma_1(A)^{j-1} + \sigma_1(A)^{\Delta k}\varepsilon + \eta, \end{aligned}$$

which implies that $\bar{x}[k_j] \in I_j$ for $k_j > k_{j-1} + \Delta k$. ■

We finish the current section with several remarks on the different features of the observer.

*Remark 2.*
- The estimates of the state at the sensor side $\hat{x}_c[k]$ are either updated by the sampler, if no message is sent, or by the coder, if a message is sent.
- The coordinates of the centers of the balls used in the covering are always relative to the previous estimates. By communicating in a relative fashion, it is possible to keep the size of the messages limited even if the system is unstable. Note that since $\hat{x}_c[k] = \hat{x}[k]$, both agents can build this set according to its definition (13).
- The alphabet procedure is easy from a computational point of view since it consists of covering a set which always has the same shape (the shape depends on which norm is used in the definition of $I_j$) except the whole set is shifted by a certain vector from the origin. Moreover, since this set is centered around the previous estimate, both the coder and decoder can build a covering for it and thus have access to the alphabet.

## 4. RATE AND ERRORS

With the observer and its agents fully introduced, it is time to determine what minimum number of bits per time interval is sufficient to implement the observer. This quantity is closely related to the observation error. The first result we present provides closed-form expressions for the constants that indicate the proportionality of the different errors/perturbations in the total error.

*Proposition 3.* The observer described in Procedure 1 ensures that (6) holds with

$$\gamma_\varepsilon = \max\{\sigma_1(A)^{\Delta k - 1}, 1\}, \tag{15}$$

$$\gamma_\delta = \frac{\max\{\sigma_1(A)^{\Delta k-1}, 1\}}{1 - \sigma_1(A + LC)} + \sum_{f=1}^{\Delta k - 1} \sigma_1(A)^{f-1}, \tag{16}$$

$$\gamma_\omega = \frac{\sigma_1(L)\max\{\sigma_1(A)^{\Delta k-1}, 1\}}{1 - \sigma_1(A + LC)}. \tag{17}$$

*Proof:* The proof is separated in three different cases, we first consider $k = k_j$, $j \in \{0, 1, \dots\}$. At the instants when the messages are sent, and because the estimates are the centers of ball of size $\varepsilon$, we have

$$\left\|x[k_j] - \hat{x}[k_j]\right\|_2 \leq \left\|\bar{x}[k_j] - \hat{x}[k_j]\right\|_2 + \left\|x[k_j] - \bar{x}[k_j]\right\|_2,$$

which, using Proposition 1, becomes

$$\left\|x[k_j] - \hat{x}[k_j]\right\|_2 \leq \varepsilon + \eta.$$

Evidently, since $\sigma_1(A) \geq 0$, the previous equation is upper bounded by

$$\begin{aligned} \left\|x[k_j] - \hat{x}[k_j]\right\|_2 \leq &\max\{\sigma_1(A)^{\Delta k-1}, 1\}\varepsilon \\ &+ \max\{\sigma_1(A)^{\Delta k-1}, 1\}\eta. \end{aligned}$$

which implies that (15)-(17) hold for $k = k_j$. For $k \in \{k_j+1, k_j+1, \dots, k_j+\Delta k-1\}$, we consider the dynamics of $e[k] = x[k]-\hat{x}[k]$. In between communication instants, we have

$$\begin{aligned} e[k+1] &= x[k+1]-\hat{x}[k+1] = Ax[k]+d[k]-A\hat{x}[k] \\ &= Ae[k]+d[k]. \end{aligned}$$

Since

$$\|e[k_j]\|_2 \le \varepsilon+\eta,$$

and $\|d[k]\|_2 \le \delta$, we have by induction that

$$\|e[k]\|_2 \le \sigma_1(A)^{\Delta k-1}(\varepsilon+\eta)+\delta\sum_{f=1}^{\Delta k-1}\sigma_1(A)^{f-1},$$

for $k \in \{k_j+1, k_j+1, \dots, k_j+\Delta k-1\}$. Evidently, the right-hand side of the previous equation is again upper bounded by

$$\begin{aligned} \|e[k]\|_2 \le &\max\{\sigma_1(A)^{\Delta k-1},1\}\varepsilon+\max\{\sigma_1(A)^{\Delta k-1},1\}\eta \\ &+\delta\sum_{f=1}^{\Delta k-1}\sigma_1(A)^{f-1}. \end{aligned}$$

which implies that (15)-(17) hold for $k \in \{k_j+1, k_j+1, \dots, k_j+\Delta k-1\}$. For $k \in \{k_j+\Delta k, \dots, k_{j+1}-1\}$, the sampler verifies whether (12) is satisfied. Since the instants $k$ we consider in this case are not instants of communication, the condition (12) is necessarily verified, which implies that

$$\begin{aligned} \|x[k]-\hat{x}[k]\|_2 &\le \|x[k]-\bar{x}[k]\|_2+\|\bar{x}[k]-\hat{x}[k]\|_2 \\ &\le \eta+\delta\sum_{j=1}^{\Delta k-1}\sigma_1(A)^{j-1}+\sigma_1(A)^{\Delta k-1}\varepsilon. \end{aligned}$$

The proof is completed by upper bounding the above expression as previously. ∎

The main result of this sections aims to provide a lower bound on $b^+(\cdot)$ for designed communication scheme. In the following theorem, the notation $\lceil\cdot\rceil$ refers to the ceiling function.

*Theorem 1.* The observer described in Procedure 1 with $\Delta k \ge 2$ is implementable on any channel with

$$b^+(\Delta k) \ge \log_2\left\lceil\sqrt{n}\frac{\sigma_1(A)^{\Delta k}\varepsilon+\delta\sum_{f=1}^{\Delta k}\sigma_1(A)^{f-1}+(\sigma_1(A)^{\Delta k}+1)\eta}{\varepsilon}\right\rceil^n \tag{18}$$

*Proof:* In order to implement Procedure 1, (5) should be verified for all $j$. The size of the alphabet is equal to the number of balls of radius $\varepsilon$ required to cover $I_j$. Since the radius of the set is

$$\sigma_1(A)^{\Delta k}\varepsilon+\delta\sum_{f=1}^{\Delta k}\sigma_1(A)^{f-1}+(\sigma_1(A)^{\Delta k}+1)\eta,$$

it can evidently be covered by no more than

$$\left\lceil\sigma_1(A)^{\Delta k}+\frac{\delta\sum_{f=1}^{\Delta k}\sigma_1(A)^{f-1}+(\sigma_1(A)^{\Delta k}+1)\eta}{\varepsilon}\right\rceil^n$$

hypercubes of radius $\frac{\varepsilon}{\sqrt{n}}$. These hypercubes are themselves contained in a sphere of radius $\varepsilon$. In order to verify (5), it thus suffices that

$$b^+(\Delta k) \ge n\log_2\left\lceil\sqrt{n}\frac{\sigma_1(A)^{\Delta k}\varepsilon+\delta\sum_{f=1}^{\Delta k}\sigma_1(A)^{f-1}+(\sigma_1(A)^{\Delta k}+1)\eta}{\varepsilon}\right\rceil$$

for the observer to be implementable. ∎

*Remark 3.* Without any state perturbations or measurement errors, the inequality of Theorem 1 becomes

$$b^+(\Delta k) \ge n\log_2\left\lceil\sqrt{n}\sigma_1(A)^{\Delta k}\right\rceil.$$

This right-hand side could be improved to

$$\sum_{i\in\{1,\dots,n\}}\max\{0,\log_2(\sqrt{n}\sigma_1(A)^{\Delta k})\}$$

by more accurately defining the sets $I_j$ in (13). Since the objective of this paper is to deal with the case with perturbations, such considerations are left for further research.

## 5. SIMULATIONS

In this section we provide simulations of the data-rate constrained observer. We will also demonstrate, through simulations, that the fact that we only communicate if (12) is verified, greatly reduces the average communication rate in some cases. For this purpose, we consider the following system

$$\begin{aligned} x[k+1] &= \underbrace{\begin{bmatrix} -0.3913 & 0.3149 \\ 0.6140 & 0.9115 \end{bmatrix}}_{A} x[k]+d[k], \\ y[k] &= \underbrace{\begin{bmatrix} -0.6071 & -0.8284 \end{bmatrix}}_{C} x[k]+w[k] \end{aligned} \tag{19}$$

with $\delta = 1$ and $\omega = 0.1$. We start by computing the gain of the local observer $L$. This gain is computed by solving the LMI program (11). For the matrix $A$ as in (19), solving (11) gives $L = [0.0191\ 0.82231]^\intercal$, for which $\sigma_1(L) = 0.8226$ and $\sigma_1(A+LC) = 0.5045$. With these values, we have $\eta = 2.184085$. We then used a Monte Carlo method with 100,000 different iterations of the communication scheme from $k=0$ till $k=100$ with initial conditions $(0,0)$ and random perturbations. For $\varepsilon = 1$, in Figure 2 an example of the errors between state, local estimates and remote estimates are plotted along with the maximum allowable error. The error resets twice due to communications. It can be seen that at the time of communication, there is still a margin between the actual error and the maximum allowable error $\gamma_\varepsilon\varepsilon+\gamma_\delta\delta+\gamma_\omega\omega$, which is due to too much conservatism in the definition of the set $I_j$.

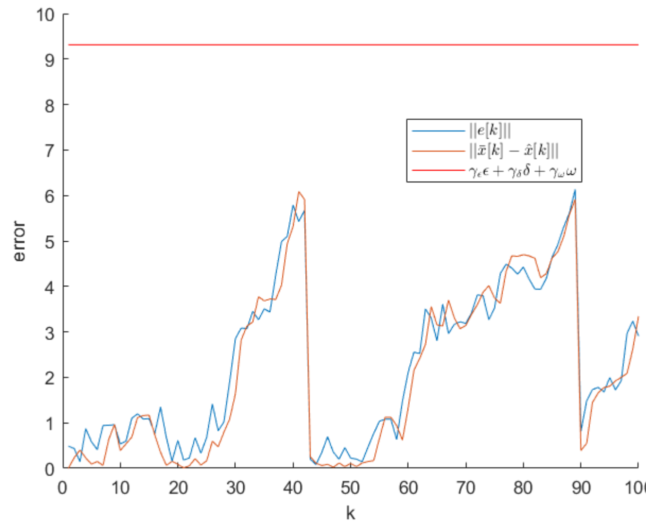


Fig. 2. Evolution of the different errors for one particular simulation.

The results in terms of average number of communications $\bar{J}_{\text{mean}}$ and number of transmitted bits $N_{\text{bits}}$ per communication can be seen in Table 1. As can be seen from the table, the communication scheme transmits messages much more rarely than every $\Delta k$ timesteps (e.g. with $\Delta k = 2$, the scheme could communicate up to 50 times every run, but only communicates 4.63678 times on average). Increasing the time interval between subsequent communications greatly reduces the total number of messages sent, which is due to the fact that a larger error between estimate and state is tolerated. In terms of rate rather than number of bits, for $\Delta k = 20$, the observers sends an average of $0.63829 \cdot 14/100 = 0.0893606$ bits/timestep. Overall, these simulations prove the effectiveness of the event-triggeredness of the communication scheme.

| $\Delta k$ | 2 | 3 | 5 | 10 | 20 |
|---|---|---|---|---|---|
| $\bar{j}_{\text{mean}}$ | 4.63678 | 2.78338 | 1.7587 | 1.00904 | 0.63829 |
| $N_{\text{bits}}$ | 8 | 8 | 9 | 11 | 14 |

Table 1. Results for various $\Delta k$

## 6. CONCLUSION

In this paper we presented an event-triggered, data-rate constrained observer for discrete-time linear systems with perturbations. After posing the problem statement, the design of the agents that form the communication scheme was explained. We then provided a theorem that upper bounded the minimum bit-rate required to implement this communication protocol. The protocol was tested via simulations, which confirmed that the time interval between subsequent communications is much longer than the minimum allowable time interval. Such an observation implies that the communication scheme is very efficient at reducing the average number of communications. This work will be adapted for the observation of mobile robots as well as for more general nonlinear systems.

## REFERENCES

Andrievsky, B., Matveev, A., and Fradkov, A. (2010). Control and Estimation under Information Constraints: Toward a Unified Theory of Control, Computation and Communications. *Automation and Remote Control*, 71, 572–633.

Baillieul, J. (2004). Data-rate requirements for nonlinear feedback control. *Proc. 6th IFAC Symp. Nonlinear Control Syst.*, 1277–1282.

Baillieul, J. and Antsaklis, P.J. (2007). Control and Communication Challenges in Networked Real-Time Systems. *Proceedings of the IEEE*, 95(1), 9–28.

De Persis, C. (2003). A Note on Stabilization via Communication Channel in the presence of Input Constraints. *Proceedings of the 42nd IEEE Conference on Decision and Control*, 187–192.

Elia, N. and Mitter, S. (2001). Stabilization of linear systems with limited information. *IEEE Transactions on Automatic Control*, 46(9), 1384–1400.

Han, D., Mo, Y., Wu, J., Weerakkody, S., Sinopoli, B., and Shi, L. (2015). Stochastic event-triggered sensor schedule for remote state estimation. *IEEE Transactions on Automatic Control*, 60(10), 2661–2675.

Heemels, M., Johansson, K.H., and Tabuada, P. (2012). An Introduction to Event-triggered and Self-triggered Control. *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, 3270–3285.

Hetel, L., Fiter, C., Omran, H., Seuret, A., Fridman, E., Richard, J.P., and Niculescu, S.I. (2017). Recent developments on the stability of systems with aperiodic sampling: An overview. *Automatica*, 76, 309–335.

Huang, J., Shi, D., and Chen, T. (2017). Event-triggered state estimation with an energy harvesting sensor. *IEEE Transactions on Automatic Control*, 62(9), 4768–4775.

Kawan, C. (2009). *Invariance Entropy for Deterministic Control Systems An Introduction*. Springer Cham Heidelberg New York Dordrecht London.

Kawan, C. (2017). State estimation , entropy and Lyapunov exponents. *Systems & Control Letters*.

Liberzon, D. and Hespanha, J. (2005). Stabilization of nonlinear systems with limited information feedback. *IEEE Transactions on Automatic Control*, 50(6), 910–915.

Liberzon, D. and Mitra, S. (2016). Entropy and minimal data rates for state estimation and model detection. *Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control*, 247–256.

Matveev, A. and Pogromsky, A. (2016). Observation of nonlinear systems via finite capacity channels: Constructive data rate limits. *Automatica*, 70, 217–229.

Matveev, A.S. and Pogromsky, A.Y. (2019). Observation of nonlinear systems via finite capacity channels, Part II: Restoration entropy and its estimates. *Automatica*, 103, 189–199.

Matveev, A.S. and Savkin, A.V. (2009). *Estimation and Control over Communication Networks*. Birkhauser Basel.

Muehlebach, M. and Trimpe, S. (2018). Distributed event-based state estimation for networked systems: An LMI approach. *IEEE Transactions on Automatic Control*, 63(1), 269–276.

Nair, G., Fagnani, F., Zampieri, S., and Evans, R. (2007). Feedback Control Under Data Rate Constraints: An Overview. *Proceedings of the IEEE*, 95(1), 108–137.

Nair, G.N., Evans, R.J., Mareels, I.M., and Moran, W. (2004). Topological feedback entropy and nonlinear stabilization. *IEEE Transactions on Automatic Control*, 49(9), 1585–1597.

Pogromsky, A. and Matveev, A. (2011). Estimation of the topological entropy via the direct Lyapunov method. *Nonlinearity*, 24, 1937–1959.

Shi, D., Chen, T., and Darouach, M. (2016). Event-based state estimation of linear dynamic systems with unknown exogenous inputs. *Automatica*, 69, 275–288.

Sibai, H. and Mitra, S. (2017). Optimal Data Rate for State Estimation of Switched Nonlinear Systems. *Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control*, 71–80.

Sibai, H. and Mitra, S. (2018). State Estimation of Dynamical Systems with Unknown Inputs: Entropy and Bit Rates. In *proceedings of the 2018 HSCC*, 217–226. Porto, Portugal.

Trimpe, S. (2017). Event-based state estimation: An emulationbased approach. *IET Control Theory and Applications*, 11(11), 1684–1693.

Voortman, Q., Pogromsky, A.Y., Matveev, A.S., and Nijmeijer, H. (2019). Data-Rate Constrained Observers of Nonlinear Systems. *Entropy*, 21(282), 1–29.

Wong, W.S. and Brockett, R.W. (1997). Systems with finite communication bandwidth constraints - Part I: State estimation problems. *IEEE Transactions on Automatic Control*, 42(9), 1294–1299.

Xia, M., Gupta, V., and Antsaklis, P.J. (2017). Networked state estimation over a shared communication medium. *IEEE Transactions on Automatic Control*, 62(4), 1729–1741.