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# Reduction-based stabilization of nonlinear discrete-time systems through delayed state measurements ⋆

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**Abstract:** In this paper, the problem of stabilizing nonlinear discrete-time systems affected by delayed state measures is addressed under average passivity-based control. The contribution stands in the introduction of a new delay-free dynamics which is used for the design of the stabilizing feedback. Simulations over an academic example illustrate the performances in a comparative sense.

*Keywords:* Asymptotic stabilization; Delay systems; Lyapunov methods.

## 1. INTRODUCTION

Throughout the last decades, time-delay systems have been widely investigated due to their involvement in practical scenarios (see Fridman (2014); Valmorbida et al. (2019) for an overview). Among the several strategies, the prediction-based control method represents one of the most appealing approaches from its early introduction by Smith (1957) for systems affected by input or measurement delays (e.g., Krstic (2008); Karafyllis and Krstic (2013a,b); Gonzalez et al. (2012); Zhou (2014); Zhou et al. (2017)).
A different and more general approach for input-delayed systems has been initiated by Artstein (1982). It relies upon the idea of transporting the control problem over a new delay-free dynamics (the so-called *reduced dynamics*) which is equivalent to the retarded one in terms of stabilizability. Such an approach has been recently extended to the nonlinear context by Mattioni et al. (2018a,b) for both continuous and discrete-time dynamics affected by input delays.
Inspired by input-reduction, the contribution of this paper is twofold: the novel concept of *output-reduction* is introduced for stabilizing nonlinear discrete-time dynamics when the delay occurs on the measurements of the states; passivity-based control methodologies, developed in Monaco and Normand-Cyrot (2011) for nonlinear discrete-time dynamics in terms of average passivity, are generalized to the presence of state measurement delays. The reduction approach stands in the possibility of leading the control design over a suitably transformed delay-free dynamics. In addition, it is shown that the construction of the output-reduction variable (and thus of the stabilizing feedback) requires the knowledge of the instantaneous delayed measurement of the state plus a finite number of past delayed signals over the time-delayed window only. In addition, contrarily to classical prediction, the free evolution of the retarded dynamics is preserved by reduction with a new forced component which accounts for the delay. Moreover, as the delay approaches to zero, the retarded output mapping and damping feedback naturally recover the delay-free ones. The problem is then specified to the case of a linear time-invariant system and the solution validated on an academic example through simulations enlightening the improvements, also in terms of robustness, with respect to classical prediction-based control.

The paper is organized as follows. In Section 2 recalls are provided and the problem is formulated. In Section 3 the main result is enhanced and specified to the case of linear dynamics. In Section 4 an academic example is carried out and simulated whereas Section 5 concludes the paper.

*Notations.* $\mathbb{R}$ and $\mathbb{N}$ denote, respectively, the set of real and natural numbers including 0. For any vector $v \in \mathbb{R}^n$, $|v|$ and $v^\top$ define the norm and transpose of $v$ respectively. Given a square matrix $R \geq 0$ and $v \in \mathbb{R}^n$, the weighted square seminorm is defined as $\|v\|^2_R := v^\top Rv$. $I_d$ denotes the identity function or identity matrix while $I$ denotes the identity operator. Given a real-valued function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ assumed differentiable, $\nabla V$ represents its jacobian vector when $\nabla$ denotes the $\mathbb{R}^n$ vector of partial derivatives. Given a smooth vector field over $\mathbb{R}^n$, the Lie operator is defined as $\mathrm{L}_f = \sum_{i=1}^n f_i(x)\frac{\partial}{\partial x_i}$. The arguments of the functions are dropped when clear from the context. A function $R(x, \delta) = O(\delta^p)$ is said of order $\delta^p$, $p \geq 1$ if it can be written as $R(x, \delta) = \delta^{p-1}\tilde{R}(x, \delta)$

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and there exist a function $\theta \in \mathcal{K}_\infty$ and $\delta^* > 0$ such that $\forall \delta \le \delta^*$, $|\tilde{R}(x,\delta)| \le \theta(\delta)$. To simplify the notations, when no confusion arises the time arguments are dropped out.

## 2. RECALLS AND PROBLEM STATEMENT

### *2.1 State space representations for discrete-time dynamics*

As discussed in Monaco and Normand-Cyrot (1998), a general discrete-time system

$$\Sigma: \quad x(k+1) = F(x(k), u(k)) \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ can be equivalently described through a couple of differential-difference equations

$$x^+ = F_0(x) \tag{2a}$$

$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u), u) \quad \text{with} \quad x^+(0) = x^+ \tag{2b}$$

where: $F_0(\cdot) = F(\cdot, 0)$ is a $\mathbb{R}^n$-valued smooth map and $G(\cdot, u) = G_1(\cdot) + \sum_{i>0} \frac{u^i}{i!} G_{i+1}(\cdot)$ is a vector field on $\mathbb{R}^n$ assumed complete verifying $\nabla_u F(x,u) = G(F(x,u),u)$ for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}$. In such a formalism, $x^+(u)$ represents a generic curve in $\mathbb{R}^n$ parameterized by $u \in \mathbb{R}$. For all $(k, x(k), u(k)) \in \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}$, integrating (2b) over the interval $[0, u(k))$ with initial condition fixed through (2a) (i.e. $x^+(0) = F_0(x(k))$), one recovers the representation (1) in the form of a map

$$F(x(k), u(k)) = F_0(x(k)) + \int_0^{u(k)} G(x^+(v), v)\mathrm{d}v. \tag{3}$$

Hence, $x^+(u(k)) := x(k+1)$ represents the one-step ahead evolution while $x^+ = x^+(0) = F_0(x(k)) = F(x(k), 0)$ defines the one-step free evolution. In this respect, one gets that (1) rewrites for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}$ as

$$x^+(u) = F(x,u) = e^{u\mathcal{G}(\cdot,u)} I_d\big|_{F_0(x)} \tag{4}$$

where $e^{u\mathcal{G}(\cdot,u)}$ is the flow associated with $G(\cdot,u)$ as the solution to (2b) and characterized by the exponent series $u\mathcal{G}(\cdot,u)$ defined as (see Monaco et al. (2007))

$$u\mathcal{G}(\cdot,u) = uG_1 + \frac{u^2}{2}G_2 + \frac{u^3}{3!}(G_3 + \frac{1}{2}[G_1, G_2]) + O(u^4). \tag{5}$$

From (4) and (5), one computes the first terms in the expansion of $F(\cdot,u)$ in $u$ as follows

$$F(x,u) = e^{uG_1 + \frac{u^2}{2}G_2 + \frac{u^3}{3!}(G_3 + \frac{1}{2}[G_1,G_2]) + O(u^4)} I_d\big|_{F_0(x)} =$$

$$F_0(x) + u\mathrm{L}_{G_1} I_d\big|_{F_0(x)} + \frac{u^2}{2}(\mathrm{L}^2_{G_1} + \mathrm{L}_{G_2}) I_d\big|_{F_0(x)} + O(u^3).$$

As a consequence of the form (2), given any smooth enough map $\lambda(\cdot): \mathbb{R}^n \to \mathbb{R}$, one gets

$$\lambda(x^+(u)) = h(F_0(x)) + \int_0^u \mathrm{L}_{G(\cdot,w)} \lambda(x^+(v))\mathrm{d}v \tag{6}$$

or, when exploiting the flow associated with the solution to the differential equation (2b)

$$\lambda(x^+(u)) = e^{u\mathcal{G}(\cdot,u)}\lambda\big|_{x^+} = \lambda(F_0(x)) + u\mathrm{L}_{G_1}\lambda(F_0(x)) + \frac{u^2}{2}(\mathrm{L}^2_{G_1} + \mathrm{L}_{G_2})\lambda(F_0(x))) + O(u^3).$$

For further discussion on the equivalence among (1), (4) and (2), the interested reader is referred to Monaco and Normand-Cyrot (1998).

### *2.2 Average passivity and passivation in discrete-time*

The concept of average passivity has been introduced in Monaco and Normand-Cyrot (2011) to relax the necessity of a direct input/output link that is unavoidable in discrete time to invoke passivity. Denoting by $\Sigma(h)$ the discrete-time system composed with the dynamics (1) and an output map $h(\cdot,u): \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, the following definition is recalled.

*Definition 2.1.* ($u$-average passivity). The dynamics (1) with output $y = h(x,u)$ is said to be *$u$-average passive* if there exists a positive semi-definite function $S(\cdot): \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ (the *storage function*) such that, for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}$

$$S(x^+(u)) - S(x) \le \int_0^u h(x^+(v), v)\mathrm{d}v = uh^{\mathrm{av}}(x,u). \tag{7}$$

with

$$h^{\mathrm{av}}(x,u) := \frac{1}{u}\int_0^u h(x^+(v),v)\mathrm{d}v \tag{8}$$

being the *$u$-average output* verifying $h^{\mathrm{av}}(x,0) = h(F_0(x),0)$.

It is worth mentioning that $u$-average passivity with respect to $h(\cdot,u)$ is equivalent to usual passivity with respect to the averaged output $h^{\mathrm{av}}(\cdot,u)$.
Assuming that the discrete-time dynamics (1) possesses a stable equilibrium (say at the origin) with Lyapunov function $S(\cdot): \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ verifying, for $u = 0$, $S(F_0(x)) - S(x) \le 0$, from Definition 2.1 one can immediately deduce

$$\begin{aligned}\Delta S(x) &:= S(F(x,u)) - S(x) \\ &= S(F_0(x)) - S(x) + \int_0^u \mathrm{L}_{G(\cdot,v)} S(x^+(v))\mathrm{d}v \\ &\le \int_0^u \mathrm{L}_{G(\cdot,u)} S(x^+(v))\mathrm{d}v, \quad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}.\end{aligned}$$

Hence, $u$-average passivity with respect to the output $h(x,u) = \mathrm{L}_{G(\cdot,u)} S(x)$ holds. Since the average output approximately rewrites in $O(|u|)$ as

$$h^{av}(x,u) = \mathrm{L}_{G(\cdot,0)} S(F_0(x)) + O(|u|) \tag{9}$$

the passivating output for (1) can be immediately computed starting from real time measures of the state (that is $x(k)$). The following definition is recalled as instrumental for stabilization purposes.

*Definition 2.2.* (ZSD). Consider the system (1) with output $h(x,u)$ and let $\mathcal{Z} \subset \mathbb{R}^n$ be the largest invariant set contained into $\{x \in \mathbb{R}^n \text{ s.t. } h(x,0) = 0\}$. Then $\Sigma(h)$ is zero-state detectable (ZSD) if the origin is asymptotically stable conditionally to $\mathcal{Z}$.

Stabilization via average damping can be deduced as recalled here below.

*Theorem 2.1.* Let the system (1) with output $h(x,u)$ be $u$-average passive and ZSD. Then, the feedback $u = u(x)$ solution to the damping equality

$$u + \kappa h^{\mathrm{av}}(x,u) = 0, \quad \kappa > 0 \tag{10}$$

makes the origin of (1) globally asymptotically stable (GAS).

For the existence of a solution to (10) and computational aspects, the interested reader can refer to Monaco and Normand-Cyrot (2011); Mattioni et al. (2019).

### *2.3 Problem statement*

In the following, assuming that measurements of the states of (1) are delayed over a time window of length $N \geq 0$ (i.e., $y_x(k) = x(k-N)$) we look for an output mapping making the delayed system $u$-average passive. Standard notations and assumptions for discrete-time delayed systems are adopted. As in Stojanović et al. (2007); Fridman (2014); Pepe et al. (2018), $\mathcal{C}$ (resp. $\mathcal{C}_u$) denotes the space of functions mapping the set $\{-N, \dots, -1\}$ into $\mathbb{R}^n$ (resp. $\mathbb{R}$); $x_0 = x_0(\theta) \in \mathcal{C}$ represents the initial condition of the system; $u_k = u_k(\theta) = u(k+\theta) \in \mathcal{C}_u$ for $\theta \in \{-N, \dots, -1\}$ denotes the story of the control signal. From now on, for the sake of compactedness, we denote $x_k = (x(k-N), u_k)$, with the following standing assumption.

*Assumption 2.1.* The dynamics (1) possesses a stable equilibrium the origin (that is $x_\star = 0$) with Lyapunov function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and is ZSD with respect to the output $h(x,u) = \mathrm{L}_{G(\cdot,u)} S(x)$.

Hereinafter, we address the problem of computing a stabilizing passivity-based controller for discrete-time dynamics of the form (1) based on delayed measurements of the state; namely, we seek a class of outputs $y_r(k) = h_r(x_k, u(k))$ (depending on $x(k-N)$ and a finite buffer of past values of the control signal) that makes the system (1) $u$-average passive with a suitable storage functional $S_r(\cdot) : \mathcal{C} \to \mathbb{R}_{\geq 0}$ verifying

$$\Delta S_r(x_k) = S_r(x_{k+1}) - S_r(x_k) \leq u(k) h_r^{\mathrm{av}}(x_k, u(k)) \quad (11)$$

and in such a way that the corresponding damping feedback makes the origin of (1) GAS. The solution we propose relies upon the definition of the so-called *output reduction* variable $\nu(\cdot) : \mathcal{C} \times \mathcal{C}_u \to \mathbb{R}^n$ (as the counterpart of input reduction proposed by Mattioni et al. (2018a)) whose passivity properties will be shown to be equivalent to the ones of (1) in presence of delayed state measurements.

*Remark 2.1.* All the results do apply to nonlinear systems issued from sampling along the lines of Mattioni et al. (2017) also covering the case of non-entire delays.

*Remark 2.2.* Assumption 2.1 is applicable to a large varieties of discrete-time dynamics as, for instance, issued from sampling of passive systems.

## 3. DAMPING THROUGH OUTPUT REDUCTION

Starting from the dynamics (1) and assuming $F_0(\cdot) = F(\cdot, 0)$ invertible [1], we introduce the *output reduction variable* $\nu(k) = r(x_k)$ as, for all $k \geq 0$,

$$\begin{aligned} \nu(k) &= r(x_k) \\ &= F_0^{-N}(x(k)) = F_0^{-N}(\cdot) \circ F^N(x(k-N), u_k) \end{aligned} \quad (12)$$

with

$$F_0^{-N}(\cdot) = \underbrace{F_0^{-1}(\cdot) \circ \dots \circ F_0^{-1}(\cdot)}_{N \text{ times}}$$

$$F^N(\cdot, u_k) = F(\cdot, u(k-1)) \circ \dots \circ F(\cdot, u(k-N))$$

for initial condition $u_0 \equiv 0$, $r(x_0, u_0) = F_0^{-N}(\cdot) \circ F^N(x(-N), 0) = x(-N)$ and verifying $r(x(k-N), 0) = x(k-N)$ when $u \equiv 0$. It is a matter of computations to prove the following result.

[1] There exists $F_0^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $F_0^{-1}(F_0(x)) = F_0(F_0^{-1}(x)) = x$.

*Proposition 3.1.* Consider the dynamics (1) under Assumption 2.1 and state measurements affected by a delay of length $N \geq 0$. Let the output reduction variable be (12), then the *output reduced dynamics* (or simply reduced dynamics) is free of delay and evolves as

$$\nu(k+1) = F_{-N}(\nu(k), u(k)) \quad (13)$$

with $F_{-N}(\nu, u) := F_0^{-N}(\cdot) \circ F(\cdot, u) \circ F_0^N(\nu)$. Additionally, (13) admits the differential-difference representation

$$\nu^+ = F_0(\nu) \quad (14a)$$

$$\frac{\mathrm{d}\nu^+(u)}{\mathrm{d}u} = G_{-N}(\nu^+(u), u) \quad \text{with} \quad \nu^+(0) = \nu^+ \quad (14b)$$

with, by definition

$$G_{-N}(\cdot, u) := Ad_{F_0^{-N}} G(\cdot, u). \quad (15)$$

The operator $Ad_{F_0^{-N}} G(\cdot, u)$ in (15) denotes the transport of a vector field $G(\cdot, u)$ along the mapping $F_0^{-N}(\cdot)$ (see Monaco et al. (2007)) so getting

$$\begin{aligned} G_{-N}(\cdot, u) &:= \left( \mathrm{L}_{G(\cdot,u)} F_0^{-N}(\cdot) \right) \circ F_0^N(\cdot) \\ &= \left( \nabla[F_0^{-N}(\cdot)] G(\cdot, u) \right)\Big|_{x = F_0^N(\cdot)} \\ &= \sum_{i \geq 0} \frac{u^i}{i!} G_{[-N], i+1}(\cdot) \end{aligned}$$

with analogously $G_{[-N],i}(\cdot) := Ad_{F_0^{-N}} G_i(\cdot)$ for $i \geq 1$. Exploiting the exponential representation in (4), one gets through composition that

$$\begin{aligned} \nu(k) =& e^{u(k-1)\mathcal{G}_{-N}(\cdot, u(k-1))} \circ e^{u(k-2)\mathcal{G}_{-N+1}(\cdot, u(k-2))} \\ & \circ \dots \circ e^{u(k-N)\mathcal{G}_{-1}(\cdot, u(k-N))} I_d \big|_{x(k-N)} \end{aligned}$$

with according to (5)

$$\begin{aligned} u\mathcal{G}_\ell(\cdot, u) =& u G_{[\ell],1} + \frac{u^2}{2} G_{[\ell],2} + \frac{u^3}{3!}(G_{[\ell],3} \\ & + \frac{1}{2}[G_{[\ell],1}, G_{[\ell],2}]) + O(u^4) \end{aligned}$$

for all $\ell \in \mathbb{Z}$. Hence, (12) approximately rewrites as

$$\begin{aligned} \nu(k) =& x(k-N) + \sum_{i=1}^{N} u(k-i) G_{[-N+i-1],1}(x(k-N)) \\ & + O(|u_k|^2). \end{aligned}$$

*Remark 3.1.* For the construction of the reduction (12), invertibility of $F_0(\cdot) = F(\cdot, 0)$ can be weakened to require submersivity only (Monaco et al. (2007)).

*Remark 3.2.* Differently from the continuous-time case (Artstein (1982); Mattioni et al. (2018b)), the reduction variable and the reduced model can be exactly computed through successive composition of the involved mappings.

*Remark 3.3.* The reduced dynamics (13) is free of delays and the reduction (12) is computed as a prediction of the state at time $k$ starting from the measures at $k - N$ brought backwards through the inverse drift of the system. Moreover, the computation of (12) requires only the knowledge of the state at time $k-N$ (that is measured) plus a finite number of past control signals over the time window $[k-N, k)$ which can be stored into a finite-dimensional buffer. As a consequence, the initial condition problem (that is typical of prediction-based control) is overcome by reduction as at the initial step $k = 0$ the state $x_0 = x(-N)$ is measured and the control past story can be fixed by the designer (e.g., at $u_0 = 0$). Moreover, As $N \to 0$, the reduction variable recovers the state measure that is $\nu(k) = x(k)$.

*Remark 3.4.* The reduction (12) and the reduced dynamics (13) are different from their input-delay counterparts (Mattioni et al. (2018a)). In the present case, (12) is capturing the effect of the delay over the state measurements and not directly onto the evolution of the dynamics (1) which is free of delay in open loop. Indeed, the delayed measures affect the system evolutions once a state-feedback is applied.

As the reduced dynamics (13) preserves the same control-free component as (1) the following result can be proven.

*Proposition 3.2.* Consider the dynamics (1) under Assumption 2.1 and state measurements affected by a delay of length $N \geq 0$. Consider the output reduction variable (12) evolving according to (13). Then, the reduced dynamics (13) is $u$-average passive with respect to the output $y(k) = \hat{h}_r(\nu(k), u(k))$ with

$$\hat{h}_r(\nu, u) = \mathrm{L}_{G_{-N}(\cdot,u)} S(\nu) \tag{16a}$$

$$\hat{h}_r^{\mathrm{av}}(\nu, u) = \frac{1}{u}\int_0^u \hat{h}_r(\nu^+(v), v)\mathrm{d}v \tag{16b}$$

and the same storage function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

*Proof.* Using the Lyapunov function in Assumption 2.1 as storage function for (13), one has that

$$\begin{aligned}\Delta S(\nu) &= S(F_{-N}(\nu, u)) - S(\nu)\\ &= \underbrace{S(F_0(\nu)) - S(\nu)}_{\leq 0} + \int_0^u \mathrm{L}_{G_{-N}(\cdot,v)} S(\nu^+(v))\mathrm{d}v\\ &\leq \int_0^u \mathrm{L}_{G_{-N}(\cdot,u)} S(\nu^+(v))\mathrm{d}v = u\hat{h}_r^{\mathrm{av}}(\nu, u)\end{aligned}$$

so getting the result. ∎

*Remark 3.5.* The average output (16b) can be approximated in $O(|u|^2)$ as

$$\begin{aligned}\hat{h}_r^{\mathrm{av}}(\nu, u) =& \mathrm{L}_{G_{[-N],1}} S(F_0(\nu)) +\\ & \frac{u}{2}\Big(\mathrm{L}^2_{G_{[-N],1}} + \mathrm{L}_{G_{[-N],2}}\Big) S(F_0(\nu)) + O(|u|^2).\end{aligned}$$

From Proposition 3.2, because (13) evolves with the same drift as (1), average passivation of the reduced dynamics can be inferred from the delay-free case. Moreover, the new output (16a) making the reduced system (13) average passive is explicitily parameterized by the delay $N$ through the new transported vector field $G_{-N}(\cdot, u)$ in (15). It turns out, as proved in the following result, that $u$-average passivation of (13) allows to infer average passivation of the original system (1) with the new reduction output (16a).

*Corollary 3.1.* Consider the dynamics (1) under Assumption 2.1 and state measurements affected by a delay of length $N \geq 0$. Then, the dynamics (1) is $u$-average passive with output functional $y_r(k) = h_r(x_k, u(k))$ where

$$\begin{aligned}h_r(x_k, u(k)) =& \hat{h}_r(\nu(k), u(k))\Big|_{\nu(k)=r(x_k)}\\ =& \mathrm{L}_{G_{-N}(\cdot,u(k))} S_r(x_k)\end{aligned} \tag{17}$$

and storage functional $S_r(x_k) := S(r(x_k)) > 0$ verifying the dissipativity inequality (11) for all $k \geq 0$.

*Proof.* The average passivating output (17) rewrites as

$$\mathrm{L}_{G_{-N}(\cdot,u(k))} S_r(x_k) = \mathrm{L}_{G(\cdot,u(k))} S(F_0^{-N}(x(k)))$$

that is, the delay-free passivating output computed over the reduction $\nu(k) = F_0^{-N}(x(k))$ with $x(k)$ being the predicted state brought $N$-step backward through the drift of (1) only. The average reduced output reads

$$h_r^{\mathrm{av}}(x_k, u(k)) = \frac{1}{u(k)}\int_0^{u(k)} \mathrm{L}_{G(\cdot,v)} S(F_0^{-N}(x^+(v)))\mathrm{d}v$$

so yielding

$$\begin{aligned}\Delta_k S_r(x_k) =& \Delta_k S(\nu(k)) \leq \int_0^{u(k)} \mathrm{L}_{G_{-N}(\cdot,v)} S(\nu^+(v))\mathrm{d}v\\ =& \int_0^{u(k)} \mathrm{L}_{G(\cdot,v)} S(F_0^{-N}(x^+(v)))\mathrm{d}v.\end{aligned}$$

Accordingly, average passivity of the reduced dynamics (13) with output (16a) implies passivity of the original dynamics (1) with output (17). ∎

*Remark 3.6.* As one might have expected, Corollary 3.1 states that for passivation (1) to be achieved despite delayed state measurements, the output and the storage need to be functional of the state $x_k = (x(k-N), u_k)$ of the overall time-delay system.

*Remark 3.7.* When $N \to 0$ one recovers the delay-free passivating output.

*Remark 3.8.* Based on Remark 3.5 one has

$$\begin{aligned}&h_r^{\mathrm{av}}(x_k, u(k)) = \mathrm{L}_{G_{[-N],1}} S(F_0(x(k-N)))\\ &+\sum_{i=1}^{N} u(k-i)\mathrm{L}_{G_{[-N+i-1],1}} S(F_0(x(k-N))) + \frac{u(k)}{2}\times\\ &\Big(\mathrm{L}^2_{G_{[-N],1}} + \mathrm{L}_{G_{[-N],2}}\Big) S(F_0(x(k-N))) + O(|(u_k, u(k))|^2).\end{aligned}$$

From the arguments above, stabilization through delayed averaged output damping can be achieved provided that ZSD holds.

*Theorem 3.1.* Consider the dynamics (1) under Assumption 2.1 and state measurements affected by a delay of length $N \geq 0$. Then, the damping reduction-based feedback $u = u_r(x_k)$ solution to the damping equality

$$u + \kappa_r h_r^{\mathrm{av}}(x_k, u) = 0, \quad \kappa_r > 0 \tag{18}$$

with $h_r^{\mathrm{av}}(x_k, u(k)) = \hat{h}_r^{\mathrm{av}}(\nu(k), u(k))$ and $\nu(k) = r(x_k)$ as in (12) and (16a) makes the origin GAS for (1).

*Proof.* For proving the result it is enough to show that ZSD of the output delay-free system (1) with $h(x, u) = \mathrm{L}_{G(\cdot,u)} S(x)$ implies ZSD of (1) under the delayed output (17). To this end, Let $\mathcal{Z}_r$ be the largest invariant set contained in $\{x_k \in \mathcal{C} \text{ s.t. } \mathrm{L}_{G_{-N}(\cdot,0)} S_r(x_k) = \mathrm{L}_{G(\cdot,0)} S(x(k-N)) = 0\}$. As $F_0(\cdot)$ is invertible (with non singular jacobian by definition) and for $u \equiv 0$, $\nu(k) = x(k-N)$ and $G_{-N}(\cdot, 0) = Ad_{F_0^{-N}} G(\cdot, 0)$, one gets $\{x_k \in \mathcal{C} \text{ s.t. } \mathrm{L}_{G_{-N}(\cdot,0)} S_r(x_k) = 0\} \equiv \{x \in \mathbb{R}^n \text{ s.t. } \mathrm{L}_{G(\cdot,0)} S(x) = 0\}$ and thus the result. ∎

*Remark 3.9.* Exploiting the definition of (12), the averaged passivating output rewrites as

$$h_r^{\mathrm{av}}(x_k, u(k)) = \frac{1}{u(k)}\int_0^{u(k)} \mathrm{L}_{G(\cdot,v)} S(F_0^{-N}(x^+(v)))\mathrm{d}v$$

computed on the basis of the reduced dynamics (13). By approximating the expression above as

$$h_r^{\text{av}}(x_k, u(k)) = \mathrm{L}_{G(\cdot,0)}S(F_0(x(k-N))) + O(|(u_k, u(k))|)$$

one gets that (3.2) gets the form an expansion around the delay-free (9) computed at the measured retarded state $x(k-N)$ plus higher terms taking into account the effect of the delay and vanishing as $u \to 0$. This is not the same as in the prediction approach where the passivating output

$$h_p^{\text{av}}(x_k, u(k)) = \frac{1}{u(k)} \int_0^{u(k)} \mathrm{L}_{G(\cdot,v)}S(x^+(v))\mathrm{d}v$$

is the prediction of the delay-free (8) $N$-step forward (i.e., $h_p^{\text{av}}(x_k) = h^{\text{av}}(F^N(x(k-N), u_k)$ and $x(k) = F^N(x(k-N), u_k))$). By approximating the above expression one gets

$$h_p^{\text{av}}(x_k, u(k)) = \mathrm{L}_{G(\cdot,0)}S(F_0^{N+1}(x(k-N))) + O(|(u_k, u(k))|)$$

that is the delay-free output map (9) computed over the predicted state $x(k) = F_0^N(x(k-N))$.

*Remark 3.10.* On the basis of Remark 3.8, an approximate solution to (18) rendering the origin of (1) locally asymptotically stable is given by

$$\begin{aligned} u_r^{\text{ap}}(x_k) &= -\tilde{\kappa}_r(x(k-N))\ell(x_k) \\ \ell(x_k) &= \mathrm{L}_{G_{[-N],1}}S(F_0(x(k-N))) \\ &\quad + \sum_{i=1}^{N} u(k-i)\mathrm{L}_{G_{[-N+i-1],1}}S(F_0(x(k-N)) \\ \tilde{\kappa}_r(\cdot) &= 2\kappa_r\Big(2 + \kappa_r\big(\mathrm{L}^2_{G_{[-N],1}} + \mathrm{L}_{G_{[-N],2}}\big)S(F_0(\cdot))\Big)^{-1}. \end{aligned}$$

Starting from the above approximations, a bounded globally asymptotically stabilizing feedback can be computed along the lines of Mattioni et al. (2019).

### 3.1 The LTI case

Consider the case in which the discrete-time dynamics (1), is represented as a LTI system,

$$x(k+1) = Ax(k) + Bu(k) \tag{19}$$

with $A$ marginally Schur stable and $|A| \neq 0$ (that is no eigenvalues in zero), with Lyapunov function $S(x) = \frac{1}{2}x^\top Px$ and state measurements affected by a delay of length $N \geq 0$. By applying Proposition 3.2, the output reduction variable (12) specifies as

$$\nu(k) = x(k-N) + \sum_{i=k-N}^{k-1} A^{k-1-N-i}Bu(i) \tag{20}$$

and the reduced dynamics (13) as

$$\nu(k+1) = A\nu(k) + B_{-N}u(k) \tag{21}$$

with $B_{-N} := A^{-N}B$. As a consequence of Proposition 3.2, the reduced dynamics (21) is $u$-average passive with respect to $y(k) = B_{-N}^\top P\nu(k)$ with storage function $S(\nu) = \frac{1}{2}\nu^\top P\nu$ and average output $\hat{h}_r^{\text{av}}(\nu, u) = B_{-N}^\top PA\nu + \frac{1}{2}B_{-N}^\top PB_{-N}u$. Accordingly, one rewrites

$$S_r(x_k) = \|x(k-N) + \sum_{i=k-N}^{k-1} A^{k-1-N-i}Bu(i)\|_P^2 > 0$$

$$h_r(x_k, u) = B_{-N}^\top P(x(k-N) + \sum_{i=k-N}^{k-1} A^{k-1-i-N}Bu(i))$$

$$\begin{aligned} h_r^{\text{av}}(x_k, u) = B_{-N}^\top PA(x(k-N) + \sum_{i=k-N}^{k-1} A^{k-1-i-N}Bu(i)) \\ + \frac{1}{2}B_{-N}^\top PB_{-N}u(k) \end{aligned}$$

so that the stabilizing feedback solution to (18) is given by

$$u(k) = -\frac{2\kappa_r PA\nu(k)}{2 + \kappa_r B_{-N}^\top PB_{-N}} = -\frac{2\kappa_r PA^{-(N-1)}x(k)}{2 + \kappa_r B_{-N}^\top PB_{-N}}.$$

## 4. AN ACADEMIC EXAMPLE

Consider a discrete-time bilinear system described as

$$x(k+1) = Ax(k) + (Mx(k) + B)u(k) \tag{22}$$

with matrices

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix},\ M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The system (22) satisfies Assumption 2.1 with associated maps $F_0(x) = Ax$, $G(x, u) = (1,\ x_1 - u)^\top$, $x = (x_1, x_2)^\top$, and storage function

$$S(x) = \frac{1}{2}x^\top Px \quad \text{with} \quad P = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}.$$

When a delay of length $N \geq 0$ affects the state measurement, the reduction variable for (22) reads

$$\begin{aligned} \nu(k) = &A^{-N}\Psi(k-1, k-N)x(k-N) \\ &+ \sum_{j=k-N}^{k-1} A^{-N}\Psi(k-1, j)Bu(j), \end{aligned}$$

with $\Psi(k, k_0) = \prod_{i=k_0+1}^{k}(A + Mu(i))$ and $\Psi(k,k) = I$. Accordingly, one defines the reduced dynamics

$$\nu(k+1) = A\nu(k) + (M_{-N}\nu(k) + B_{-N})u(k),$$

with $M_{-N} = A^{-N}MA^N$, $B_{-N} = A^{-N}B$ and, moreover, $G_{-N}(\nu, u) = M_{-N}(A + M_{-N}u)^{-1}(\nu - B_{-N}u) + B_{-N}$. Thus, the damping feedback in Theorem 3.1 gets the form

$$u(k) = \frac{-\kappa_r(B_{-N} + M_{-N}\nu(k))^\top PA\nu(k)}{1 + \frac{\kappa_r}{2}(B_{-N} + M_{-N}\nu(k))^\top P(B_{-N} + M_{-N}\nu(k))} \tag{23}$$

making the origin of (22) GAS with $\kappa_r > 0$.

*Simulation.* To validate the effectiveness of the proposed method, let us compare the reduction-based controller (23) with the prediction-based control, namely

$$u_p(k) = \frac{-\kappa_p(B + Mx_p(k))^\top PAx_p(k)}{1 + \frac{\kappa_p}{2}(B + Mx_p(k))^\top P(B + Mx_p(k))}, \tag{24}$$

with $\kappa_p > 0$, $x_p(k) = x(k) = A^N\nu(k)$ both applied to the system (22) (Remark 3.9). In Figure 1, simulations of the closed-loop system are reported when considering both the reduction-based feedback (23) and the prediction-based feedback (24) under condition $x_0(-N) = (-1, 1)^\top$ and a delay of length $N = 1$. From those figures, it is shown that, although comparable, the reduction-based controller provides smoother trajectories and zero decay faster than the prediction one so highlighting the anticipating effect of the reduction. We note that, as $N$ increases significantly, reduction still ensures closed-loop stabilization albeit with degrading performances in the first $N$ time steps. In Figure 2, robustness with respect to delay mismatch is tested. Indeed, the design (for both prediction and reduction) is

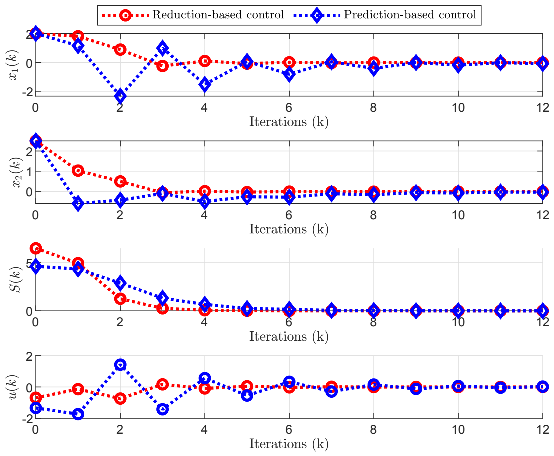


Fig. 1. $N = 2$ and $\kappa_r = \kappa_p = 10$.

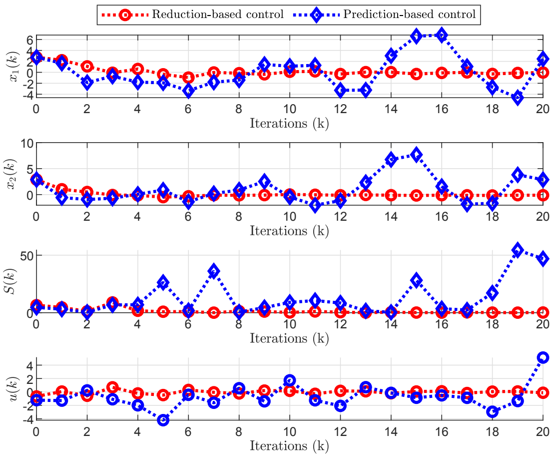


Fig. 2. Delay mismatch with $\kappa_r = \kappa_p = 1$.

based on a nominal value of the delay $N = 2$ whereas the actual delay is set to $N_r = 4 > N$. In this case, the reduction-based controller still guarantees stabilization of the closed loop with acceptable performances whereas the prediction-based one fails.

## 5. CONCLUSIONS AND PERSPECTIVES

A new passivity-based stabilizing control strategy is proposed for discrete-time nonlinear systems affected by delayed state measurements. Based on the concept of output-reduction, a new variable is computed from the delayed state measure at time $k \geq 0$ and the story of the control signal over the delayed time window with no need of prediction. The MIMO case follows the same lines as Mattioni et al. (2018a). The results are promising also in terms of robustness with respect to uncertainties on the delay for which a formal study is undergoing. Current work is addressing the case of stabilization through reduction for nonlinear systems with general delayed output measurements (that may not be the state) also under large delays.

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