

# Fusing multiple time varying tubes for robust MPC

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**Abstract:** We consider robust tube based model predictive control of discrete time, constrained, linear systems subject to additive disturbances. Standard tube based approaches utilize as an auxiliary control law a single, fixed feedback/gain to counteract the effect of the future disturbances in the predictions. The fictive - never applied - control law allows to bound the error between the real state and the nominal predictions by so called tubes. The tube control law strongly influences the shape and size of the tube. Consequently, the choice of the gain has a major impact on the domain of attraction and the control performance of the overall controller. The objective of this work is to overcome these limitations by combining multiple tubes online, each determined by a different controller gain. This reduces the conservatism and improves the closed loop performance. The computational demand for the resulting control law increases only marginally, compared to the standard case. We establish constraint satisfaction, robust recursive feasibility and robust stability. Moreover extension to the case of varying disturbance bounds are discussed. The proposed approach and its benefits are illustrated using simulations.

*Keywords:* Constrained control, Robust Model Predictive Control, Tube based MPC.

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## 1. INTRODUCTION

Model predictive control (MPC) is a control technique, which solves at each sampling instant an optimal control problem to determine the feedback. MPC enables the direct consideration of multi-variable systems and constraints, while optimizing the closed loop performance (Kouvaritakis and Cannon, 2016; Rawlings et al., 2017; Grüne and Pannek, 2017; Lucia et al., 2016). For real systems the prediction of the future often involves lots of uncertainty due to inaccurate system models or due to external disturbances. This results in a mismatch between the reality and the prediction utilized in the optimization, which needs to be considered appropriately to guarantee robustly stability, feasibility and constraint satisfaction.

Tube based MPC is an approach which allows to guarantee these critical closed loop properties, c.f. (Kouvaritakis and Cannon, 2016; Rawlings et al., 2017; Mayne, 2014, 2015, 2018; Mayne et al., 2005; Chisci et al., 2001; Langson et al., 2004). In tube based MPC the feedback and system dynamics are decomposed into two parts: a nominal dynamics without disturbances, which is utilized for the predictions and optimizations and the prediction error dynamics, which is stabilized/enforced towards an invariant set via a simple auxiliary control law.

Often robust predictive control combines an optimization/prediction based control law based on the nominal

behavior with a linear state feedback taking the error into account, see e.g. (Kouvaritakis and Cannon, 2016; Rawlings et al., 2017; Chisci et al., 2001; Mayne, 2014, 2015, 2018; Mayne et al., 2005; Langson et al., 2004). The resulting feedback has a computational effort similar to standard MPC. There are also approaches featuring more degrees of freedom, for example striped parameterized MPC (Munoz-Carpintero et al., 2014), elastic tube MPC (Raković et al., 2016) or fully parameterized MPC (Raković et al., 2011; Raković, 2012). Unfortunately, for these approaches the amount of optimization variables and the size of the optimization problem increases significantly, which consequently also leads to an increase of the computational demand rendering implementation on embedded systems challenging.

This work blends/interpolates the tube from multiple tubes. In MPC interpolation has been used, e.g. for fast, approximate MPC (Sui et al., 2008; Rossiter and Ding, 2010), to improve terminal sets (Balandat, 2011), robust MPC for system subject to multiplicative uncertainties based on linear matrix inequalities (Pluymers et al., 2005), for linear parameter varying systems (Hanema et al., 2020) or to determine robust positive invariant sets (Rakovic and Baric, 2010; Raković et al., 2007).

Besides stabilization model predictive control and tube based MPC has been tailored and used for many tasks. For example tube based MPC has been used for set-point tracking (Limon et al., 2010), robust output feedback (Mayne et al., 2006, 2009; Kögel and Findeisen, 2017b), decentralized/distributed/hierarchical MPC, see e.g. (Scattolini, 2009; Lucia et al., 2015; Farina and Scattolini, 2012; Rivero et al., 2014; Kögel and Findeisen, 2018; Blasi

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et al., 2018; Bätthge et al., 2018; Ibrahim et al., 2020) and resource aware MPC, compare (Heemels et al., 2012; Kögel et al., 2019; Kögel and Findeisen, 2016a; Brunner et al., 2016) to name just a couple of such results.

The contribution of this work is a tube based MPC approach utilizing a generalized control law, compared to existing approaches, such as (Chisci et al., 2001; Mayne, 2018). In a nutshell the control law determines at each sampling instant a suitable tube using a convex combination of predetermined sets/tubes. As shown, the resulting approach is less conservative and posses a larger domain of attraction. Still the optimization problem is convex and compared to the standard approach the computational complexity increases only slightly. We provide conditions to guarantee constraint satisfaction, feasibility and robust stability. Moreover, we discuss extensions of the proposed approach to handle time varying disturbances.

We note that this work uses tube formulations based on (Chisci et al., 2001) utilizing a “growing” tube, whereas in (Kögel and Findeisen, 2020) we considered a formulation based on (Langson et al., 2004; Mayne et al., 2005) utilizing a time invariant tube, see e.g. (Mayne, 2018) for a comparison between both approaches.

The remainder of this work is structured as follows: Section 2 outlines the considered problem. Section 3 presents the proposed tube blending MPC approach and discusses closed loop properties such as robust constraint satisfaction, robust recursive feasibility and robust stability. Section 4 outlines an extensions of the proposed approach to the case of online bounds on the disturbances. Examples are presented in Section 5 underlining its benefits and applicability, before concluding with a summary.

The notation is mainly standard. For sets  $A, B$  the operators  $A \oplus B$  and  $A \ominus B$  denote the Minkowski sum and Minkowski difference, respectively, see Blanchini and Miani (2008); Rawlings et al. (2017). For sets  $M_i: M_1 \oplus \dots \oplus M_Z = \bigoplus_{i=1}^Z M_i$ .  $x^*$  denotes the optimal value of an optimization variable  $x$ . For a vector  $x$  and a positive definite matrix  $M$   $\|x\|_M^2 = x^T M x$ .

We furthermore use the concept of input-to-state stability: *Definition 1.* (Input-to-state stability (ISS) (Limon et al., 2009; Khalil, 2002))

A system  $x(k+1) = f(x(k), w(k))$  is input-to-state stable if there exist a class  $\mathcal{KL}$  function  $\alpha$  and a class  $\mathcal{K}$  function  $\beta$  (we refer to (Khalil, 2002) for the definitions) such that for any admissible  $x(0)$  and any admissible disturbance sequences  $\{w(k)\}$ :

$$\|x(k+m)\| \leq \alpha(\|x(k)\|, k-m) + \max_{k \leq l \leq k+m-1} \beta(\|w(l)\|).$$

## 2. SYSTEM SETUP

We consider an uncertain system governed by

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad (1)$$

where  $x_k \in \mathbb{R}^n$  denotes the system state,  $u_k \in \mathbb{R}^p$  is the control input applied by actuators and  $w_k \in \mathbb{W}$  is an unknown, but bounded disturbance. This disturbance is restricted to the convex, compact polytope  $\mathbb{W}$ , which

contains origin. The matrices  $A$  and  $B$  are exactly known and the state  $x_k$  is known at  $k$ .

The system state  $x_k$  as well as the control input  $u_k$  need to satisfy constraints of the form

$$x_k \in \mathbb{X}, \quad u_k \in \mathbb{U}, \quad (2)$$

where  $\mathbb{X}$  and  $\mathbb{U}$  are compact, convex polytopes and contain the origin in their interiors.

## 3. TUBE BASED MPC

We want to control the system (1) such that for any admissible disturbance sequence the constraints (2) are always satisfied and the system state  $x_k$  is robustly stabilized. A commonly used control scheme to achieve these goals is tube based MPC (Chisci et al., 2001; Mayne, 2018).

The basic idea of tube based MPC is to use a nominal prediction model for the prediction and optimization subject to tightened constraints. The mismatch between predictions and the reality are handled by a fictive, never applied, control law. This decomposition allows to bound the effect of the disturbance onto the prediction error by sets. This guarantees constraint satisfaction and robust stability even though at every time instant  $k$  only a slightly more complex convex optimization problem needs to be solved as in the nominal case.

### 3.1 Basic idea of blending multiple tubes

In the following we first describe the basic idea of tube based MPC and second the proposed blending approach.

*Nominal system & Model predictive control* The model predictive controller optimizes at every time instant  $k$  a nominal state trajectory  $\hat{\mathbf{x}}_k = \{\hat{x}_{k|k}, \dots, \hat{x}_{k+N|k}\}$  by choosing an input sequence  $\hat{\mathbf{u}}_k = \{\hat{u}_{k|k}, \dots, \hat{u}_{k+N-1|k}\}$  subject to

$$\hat{x}_{k+i+1|k} = A\hat{x}_{k+i|k} + B\hat{u}_{k+i|k}, \quad \hat{x}_{k|k} = x_k, \quad (3)$$

where  $N$  denotes the horizon. Note that the dynamics (3) correspond to a nominal prediction, i.e. (1) with  $w_k \equiv 0$ , and starting from the current state  $x_k$ . The state sequence  $\hat{\mathbf{x}}_k$  and input sequence  $\hat{\mathbf{u}}_k$  are optimized with respect to convex constraints and a convex cost function (both discussed later). From the resulting optimal input sequence  $\hat{\mathbf{u}}_k$  only the first part is used as feedback:

$$u_k = u_{k|k}^*. \quad (4)$$

*Prediction mismatch* The nominal prediction model (3) ignores the presence of the disturbance  $w_k$  in the real system (1). So, the disturbance  $w_k$  results in a mismatch/error between the real state at time  $k+i$ ,  $i > 0$  and its prediction at  $k$ :

$$e_{k+i|k} = x_{k+i} - \hat{x}_{k+i|k}.$$

“Standard” Tube based approach (Chisci et al., 2001) To bound the error (Chisci et al., 2001) utilizes a fictitious, auxiliary control law in the prediction. The future input  $u_{k+i|k}$  is assumed to take the error  $e_{k+i|k} = x_{k+i} - \hat{x}_{k+i+1|k}$  into account using the simple, affine control law

$$u_{k+i} = u_{k+i|k}^* + Ke_{k+i|k} \quad (5)$$

where  $u_{k+i|k}^*$  is the optimal input determined at  $k$  and  $K$  is a controller gain - a design parameter. Clearly, for  $i = 0$

(5) is equal to the real control law (4). With the choice of the input as in (5) the resulting error dynamics are

$$\begin{aligned} e_{k+i+1|k} &= (A + BK)e_{k+i|k} + w_{k+i}, \quad w_{k+i} \in \mathbb{W}, \\ e_{k|k} &= 0. \end{aligned} \quad (6)$$

*Tube blending* We propose considering  $M$  different gains  $K^{[j]}$  for the disturbance affine feedback.

Each gain  $K^{[j]}$  results in an error  $\epsilon_{k+i|k}^{[j]}$  with the dynamics

$$\begin{aligned} \epsilon_{k+i+1|k}^{[j]} &= (A + BK^{[j]})\epsilon_{k+i|k}^{[j]} + w_{k+i}, \quad w_{k+i} \in \mathbb{W}, \\ \epsilon_{k|k}^{[j]} &= 0. \end{aligned} \quad (7)$$

Clearly, different gains  $K^{[j]}$  yield different error dynamics.

The idea of the proposed approach is to utilize a convex combination of the different bounds for the prediction error  $x_{k+i} - \hat{x}_{k+i}$  and the auxiliary control law.

We propose to combine them by

$$x_{k+i} - \hat{x}_{k+i|k} = e_{k+i|k} = \sum_{j=1}^M \lambda^{[j]} \epsilon_{k+i|k}^{[j]} \quad (8a)$$

$$u_{k+i|k} = u_{k+i|k}^* + \sum_{j=1}^M \lambda^{[j]} K^{[j]} \epsilon_{k+i|k}^{[j]}, \quad (8b)$$

where  $\lambda^{[j]}$  are interpolation parameters satisfying

$$\lambda^{[j]} \geq 0, \quad \sum_{j=1}^M \lambda^{[j]} = 1. \quad (9)$$

The combination (8) is affine in the nominal state  $\hat{x}_{k+i|k}$ , the nominal input  $\hat{u}_{k+i|k}$  and the parameter  $\lambda = (\lambda^{[1]} \dots \lambda^{[M]})$ . For  $i = 0$  (8b) is the real control law (4).

### 3.2 Bounding the prediction error

The error  $e_{k+i|k}$  for the approach (6) can be bounded in form of sets:  $e_{k+i|k} \in \mathbb{E}_i$  where

$$\mathbb{E}_{i+1} = (A + BK)\mathbb{E}_i \oplus \mathbb{W}, \quad \mathbb{E}_0 = \{0\}. \quad (10)$$

The future state trajectory  $x_{k+i}$  is thus bounded by a ‘‘tube’’ with diameter  $\mathbb{E}_i$  and center  $\hat{x}_{k+i|k}$  at time  $k$ .

For the proposed approach for each of the  $M$  error dynamics (7) sets  $\mathbb{E}_i^{[j]}$  can thus be determined similarly by

$$\mathbb{E}_{i+1}^{[j]} = (A + BK^{[j]})\mathbb{E}_i^{[j]} \oplus \mathbb{W}, \quad \mathbb{E}_0^{[j]} = \{0\}. \quad (11)$$

Thus, the ‘‘blended’’ prediction mismatch is bounded by

$$e_{k+i|k} \in \bigoplus_{j=1}^M \lambda^{[j]} \mathbb{E}_i^{[j]}, \quad (12)$$

which depends on the interpolation parameter  $\lambda$  chosen by the optimizer.

Note that the sets  $\mathbb{E}_i^{[j]}$  depend on the time  $i$ , in contrast to (Langson et al., 2004; Kögel and Findeisen, 2020) where sets independent of  $i$  are used.

### 3.3 Blended tube MPC setup

The proposed model predictive control law determines the input  $u_k$  (4) optimizing the nominal state sequence  $\hat{\mathbf{x}}_k$ , the nominal input sequence  $\hat{\mathbf{u}}_k$ , the interpolation parameter  $\lambda$

subject to the nominal dynamics (3), the constraints on the interpolation parameter  $\lambda$  (9) and

$$\hat{x}_{i|k} \in \bigoplus_{j=1}^M \lambda^{[j]} \mathbb{X}_i^{[j]}, \quad \hat{u}_{i|k} \in \bigoplus_{j=1}^M \lambda^{[j]} \mathbb{U}_i^{[j]}, \quad (13a)$$

$$\hat{x}_{k+N|k} \in \bigoplus_{j=1}^M \lambda^{[j]} \mathbb{T}^{[j]}, \quad (13b)$$

where  $\mathbb{X}_i^{[j]}$  and  $\mathbb{U}_i^{[j]}$  are tightened constraints given by

$$\mathbb{U}_i^{[j]} = \mathbb{U} \ominus K^{[j]} \mathbb{E}_i^{[j]}, \quad \mathbb{X}_i^{[j]} = \mathbb{X} \ominus \mathbb{E}_i^{[j]}, \quad (14)$$

and the terminal sets  $\mathbb{T}^{[j]}$  are convex polytopes as defined later. The optimization is done with respect to the quadratic cost function:

$$J(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k) = \|\hat{x}_{k+N|k}\|_P^2 + \sum_{i=k}^{k+N-1} \|\hat{x}_{i|k}\|_Q^2 + \|\hat{u}_{i|k}\|_R^2, \quad (15)$$

where  $Q \succeq 0$ ,  $R \succ 0$  and  $P \succeq 0$  are weighting matrices.

Consequently, the proposed approach is based on the repeated solution of the optimization problem  $\mathcal{P}(x_k)$

$$\min_{\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k, \lambda} J(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k) \text{ subject to (3), (9), (13),} \quad (16)$$

and the first part of the arising optimal input sequence is used as feedback, compare (4).

### 3.4 Closed loop properties

In the following we describe conditions to guarantee constraint satisfaction, recursive feasibility and robust stability of the proposed approach.

#### Recursive feasibility and constraint satisfaction

*Assumption 2. (Terminal sets)*

The terminal sets  $\mathbb{T}^{[j]}$  are convex polytopes, which satisfy for the terminal gain  $K^f$

$$\mathbb{T}^{[j]} \subseteq \overline{\mathbb{X}}_N^{[j]}, \quad K^f \mathbb{T}^{[j]} \subseteq \overline{\mathbb{U}}_N^{[j]},$$

$$\forall \hat{x} \in \mathbb{T}^{[j]}, w \in \mathbb{W} : (A + BK^f)\hat{x} + (\tilde{A}^{[j]})^N w \in \mathbb{T}^{[j]},$$

where  $\tilde{A}^{[j]} = A + BK^{[j]}$ .

The terminal sets  $\mathbb{T}^{[j]}$  are robust positive invariant sets, so  $A + BK^f$  needs to be Schur stable.

Given the setup the following results can be achieved:

*Proposition 3. (Feasibility)* Let Assumption 2 hold. If  $\mathcal{P}(x_0)$  is feasible, then for any admissible disturbance sequence  $\{w_k \in \mathbb{W}\}$  and  $k \geq 0$  the closed loop system (1), (4) and (16) satisfies

- constraint satisfaction, i.e.  $x_k \in \mathbb{X}$  and  $u_k \in \mathbb{U}$ ,
- recursive feasibility, i.e.  $\mathcal{P}(x_k)$  is feasible.

The proof can be found in Appendix A.

*Robust stability* To guarantee that the closed loop system is robustly stable in the sense of ISS the terminal penalty  $P$  has to be chosen suitably:

*Assumption 4. (Condition for terminal penalty  $P$ )*

The terminal penalty  $P \succ 0$  satisfies for the gain  $K^f$

$$P = (A + BK^f)^T P (A + BK^f) + Q + (K^f)^T R K^f. \quad (17)$$

Note that if  $K^f$  is chosen as LQR gain, then  $P$  can be computed also from the Riccati equation.

With a correctly chosen terminal penalty  $P$  input-to-state stability can be guaranteed:

*Corollary 5. (Input-to-state stability) Let Assumptions 2 and 4 hold. If  $\mathcal{P}(x_0)$  is feasible, then for any admissible disturbance sequence  $\{w_k, w_k \in \mathbb{W}\}$  the closed loop system (1), (4) and (16) is input-to-state stable.*

A detailed proof is avoided due to space limitations. Basically, the results can be verified using (Limon et al., 2009, Thm. 4), using Proposition 3 and Assumption 4.

#### Computational demand

The proposed approach requires to solve at each time instant the optimization problem (16). Similar to the standard approach (Chisci et al., 2001), which corresponds to  $M = 1$  with (16), it is also a convex quadratic program. So an efficient solution is possible, see e.g. (Lucia et al., 2016; Kögel and Findeisen, 2013; Zometa et al., 2013; Kögel and Findeisen, 2017a). Moreover, the computational demand increases only slightly: the only additional optimization variables are the interpolation parameters  $\lambda$ .

The sets  $\mathbb{X}_i^{[j]}$ ,  $\mathbb{U}_i^{[j]}$  and  $\mathbb{T}^{[j]}$   $j = 1, \dots, M$  can be computed offline; using e.g. Herceg et al. (2013); Rivero et al. (2013); Löfberg (2004). The amount of offline computations increases by about a factor of  $M$  compared to (Chisci et al., 2001). However, as these computations are done offline is not a major burden.

*Remark 6. (Choice of gains  $K^{[j]}$ ) The choice of the gains  $K^{[j]}$  often has a major impact on the feasibility and the control performance. The choice of  $K^{[j]}$  influences the size/shape of the tightened constraints  $\mathbb{X}^{[j]}$  and  $\mathbb{U}^{[j]}$ .*

In the standard approach (Chisci et al., 2001)/ $M = 1$  the choice of  $K^{[j]}$  can significantly limit the control authority available to the controller. This has usually a direct influence on the achievable control performance and onto the size of feasible initial states  $x_0$ . Moreover, it can influence the terminal set  $\mathbb{T}$  and (possibly) the choice of the terminal control gain  $K^f$ /terminal penalty matrix  $P$ . For example a “bad” choice of  $K$  might prevent the use of the LQR gain as terminal control gain.

The proposed approach allows the utilization of multiple gains  $K^{[j]}$ , which partly reduces this problems. However still the choice of (useful) gains  $K^{[j]}$  is important and the subject of future works.

*Remark 7. (Comparison to (Kögel and Findeisen, 2020)) In (Kögel and Findeisen, 2020) we sketched a similar idea using the tube based approach of (Langson et al., 2004). There we use a different nominal model, where the initial state can be selected from a set and thus not to be equal to the current state. This allows to utilize the same tightened constraints for all predictions steps, but it is challenging to guarantee ISS and also the extension presented in the next section is not possible.*

## 4. EXTENSION: TIME VARYING BOUNDS

Above a time invariant bound  $\mathbb{W}$  on the disturbance  $w_k$  was used. In certain applications online better time

varying bounds on  $\mathbb{W}$  are available, e.g. if the disturbances correspond to the influence of the neighboring systems, see e.g. (Lucia et al., 2016, 2015; Blasi et al., 2018) or are due to state estimation, compare (Kögel and Findeisen, 2016a).

Therefore we consider that time-varying bounds of the disturbance are available of the form

$$w_k \in \mathbb{W}_{k+i|k}.$$

We assume that these bounds are consistent, i.e. satisfy *Assumption 8. (Consistent bounds on disturbance  $w_k$ ) The sets  $\mathbb{W}_{k+i|k}$  are convex, compact polytopes satisfying:*

$$\mathbb{W}_{k+i|k+1} \subseteq \mathbb{W}_{k+i|k} \subseteq \mathbb{W}. \quad (18)$$

Using the online available bounds on the disturbance  $w_k$  tighter bounds on the mismatch are given by

$$\mathbb{E}_{i+1|k}^{[j]} = (A + BK^{[j]})\mathbb{E}_{i|k}^{[j]} \oplus \mathbb{W}_{i|k}, \quad \mathbb{E}_{k|k}^{[j]} = \{0\},$$

which results in the online tightened constraints:

$$\hat{x}_{i|k} \in \bigoplus_{j=1}^M \lambda^{[j]} \bar{\mathbb{X}}_{k+i|k}^{[j]}, \quad \hat{u}_{i|k} \in \bigoplus_{j=1}^M \lambda^{[j]} \bar{\mathbb{U}}_{k+i|k}^{[j]}, \quad (19)$$

where  $\bar{\mathbb{X}}_{k+i|k}^{[j]} = \mathbb{X} \ominus \mathbb{E}_{k+i|k}^{[j]}$  and  $\bar{\mathbb{U}}_{k+i|k}^{[j]} = \mathbb{U} \ominus K^{[j]}\mathbb{E}_{k+i|k}^{[j]}$ .

If the online bounds  $\mathbb{W}_{k+i|k}$  on the (future) disturbance  $w_{k+i}$  are more precise (smaller) than  $\mathbb{W}$ , then the error bounds  $\mathbb{E}_{k+i|k}^{[j]}$  are also smaller than the offline bounds  $\mathbb{E}_i^{[j]}$ . Consequently, the online constraints tightened (19) will be larger/less restrictive than the ones using the offline bounds (14). The adapted optimization problem becomes

$$\min_{\hat{x}_k, \hat{u}_k, \lambda} J(\hat{x}_k, \hat{u}_k) \text{ subject to (3), (9), (19)}. \quad (20)$$

*Corollary 9. (Feasibility properties - varying bounds)*

Let Assumptions 2, 4, 8 hold and let  $\bar{\mathbb{X}}_{k+i|k}^{[j]}$ ,  $\bar{\mathbb{U}}_{k+i|k}^{[j]}$  be given by (19). If (20) is feasible at  $k = 0$ , then for any admissible disturbance  $\{w_k, w_k \in \mathbb{W}_{k|k}\}$  and consistent bounds  $\mathbb{W}_{i+1|k}$  the controller (4) and (20) achieves for the system (1)

- constraint satisfaction, i.e.  $x_k \in \mathbb{X}$  and  $u_k \in \mathbb{U}$ ,
- recursive feasibility, i.e. (20) is feasible at  $k + 1$ ,
- input-to-state stability.

A proof is avoided here due to space limitations.

*Remark 10. (Computation of tightened constraints (19)) At every time instant  $k$  first the tightened constraints  $\bar{\mathbb{X}}_{k+i|k}^{[j]}$ ,  $\bar{\mathbb{U}}_{k+i|k}^{[j]}$  (19) are determined, before (20) can be solved. Computing the sets requires to solve linear programs. Fortunately, these optimizations can be done very efficiently exploiting the special structure. One can solve the (large) optimization by separating the overall problem into multiple smaller optimization problems. Moreover, if all  $\mathbb{W}_{k+i|k}$  are boxes (or zonotopes), then the online tightening can be calculated straightforwardly without actually solving optimization problems.*

## 5. EXAMPLES

In this section we illustrate the proposed approach. The set computations and simulations have been done in Matlab using the toolboxes Löfberg (2004); Herceg et al. (2013); Rivero et al. (2013).

### 5.1 Example 1

The first example illustrates the increased domain of attraction of the proposed approach.

*Dynamics and constraints* We consider the dynamics

$$A = \begin{pmatrix} 0.95 & 0 & 0.05 & 0 \\ 0 & 0.95 & 0 & 0.05 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix},$$

$$\mathbb{W} = \{w \text{ s.t. } \begin{pmatrix} -0.001 \\ -0.001 \\ -1 \\ -1 \end{pmatrix} \leq w \leq \begin{pmatrix} 0.001 \\ 0.001 \\ 1 \\ 1 \end{pmatrix}\},$$

and the constraints and disturbance bound

$$\mathbb{X} = \{x \text{ s.t. } \begin{pmatrix} -4.5 \\ -4.5 \\ -2.5 \\ -2.5 \end{pmatrix} \leq x \leq \begin{pmatrix} 4.5 \\ 4.5 \\ 2.5 \\ 2.5 \end{pmatrix}\},$$

$$\mathbb{U} = \{u \text{ s.t. } \|u\|_\infty \leq 5\}.$$

*Controller design* For the MPC we choose the horizon length as  $N = 30$ . We consider three different gains  $K^{[i]}$ :

$$K^{[1]} = \begin{pmatrix} 0.2862 & -0.2862 & 0.7854 & -0.7854 \\ 0.2862 & 0.2862 & 0.7854 & 0.7854 \\ 0.3717 & -0.1809 & 1.0007 & -0.5193 \\ 0.3717 & 0.1809 & 1.0007 & 0.5193 \end{pmatrix},$$

$$K^{[2]} = \begin{pmatrix} 0.3717 & -0.1809 & 1.0007 & -0.5193 \\ 0.3717 & 0.1809 & 1.0007 & 0.5193 \end{pmatrix},$$

$$K^{[3]} = \begin{pmatrix} 0.1809 & -0.3717 & 0.5193 & -1.0007 \\ 0.1809 & 0.3717 & 0.5193 & 1.0007 \end{pmatrix}.$$

The penalty matrices are  $Q = I$ ,  $R = I$  and  $K^f$  is obtained using LQR. The terminal sets  $\mathbb{T}^{[i]}$  are calculated to satisfy Assumption 2.

*Feasible regions* Figure 1 illustrates the sets for which the optimization problems (16) is feasible (with  $x(3) = x(4) = 0$ ) using a single gain  $K^{[i]}/M = 1$  (the approach of (Chisci et al., 2001)) and all three gains - the proposed approach ( $M = 3$ ). Note that the sets for the different gains  $K^{[i]}$  do not include each other, i.e. no gain is “optimal”. The proposed approach yields a set which is a convex combinations of the sets for the approach of (Chisci et al., 2001) with the different gains. Consequently, the set of feasible states is for the proposed approach significantly larger as it includes all points obtained of the other sets and some additional points.

### 5.2 Example 2

The second example illustrates the performance of the proposed approach and the sketched extension.

*Dynamics and constraints* As second example we consider the following system

$$A = \begin{pmatrix} 1 & 0.25 & 0.25 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0.25 & 0 \\ 0 & 0.25 \end{pmatrix},$$

$$\mathbb{W} = \{w \text{ s.t. } \begin{pmatrix} -0.01 \\ -0.1 \\ -0.1 \end{pmatrix} \leq w \leq \begin{pmatrix} 0.01 \\ 0.1 \\ 0.1 \end{pmatrix}\},$$

and the state and input constraints are given by

$$\mathbb{X} = \{x \text{ s.t. } \begin{pmatrix} -30 \\ -1 \\ -1 \end{pmatrix} \leq x \leq \begin{pmatrix} 30 \\ 1 \\ 1 \end{pmatrix}\},$$

$$\mathbb{U} = \{u \text{ s.t. } \|u\|_\infty \leq 1\}.$$

*Controller design* The MPC uses a horizon of  $N = 40$  and  $M = 3$  different gains  $K^{[i]}$ :

$$K^{[1]} = \begin{pmatrix} 2.1892 & 2.4918 & 0.6225 \\ 0.2049 & 0.1264 & 3.5659 \end{pmatrix},$$

$$K^{[2]} = \begin{pmatrix} 0.2049 & 3.5659 & 0.1264 \\ 2.1892 & 0.6225 & 2.4918 \end{pmatrix},$$

$$K^{[3]} = \begin{pmatrix} 1.7544 & 2.8982 & 0.7478 \\ 1.7544 & 0.7478 & 2.8982 \end{pmatrix}.$$

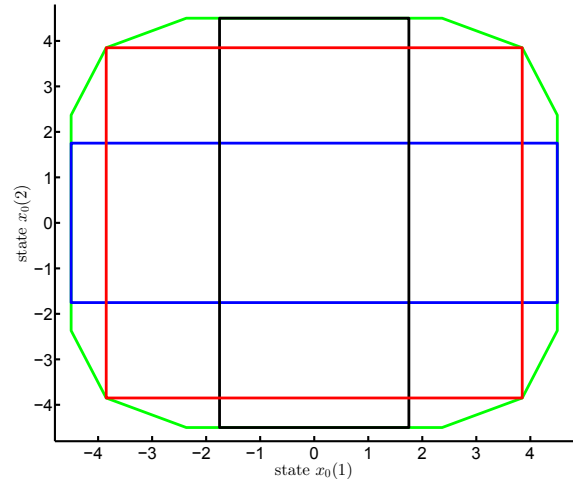


Fig. 1. Plots of boundaries of feasible states  $x_0$  using (Chisci et al., 2001) with single gains  $K^{[1]}$  (red),  $K^{[2]}$  (black) and  $K^{[3]}$  (blue) and using proposed approach (16) (green). Plot with varying  $x(1)-x(2)$ ,  $x_0(3) = x_0(4) = 0$ .

The penalty matrices are chosen as  $Q = \text{diag}(0.111)$ ,  $R = 100 \cdot I$ . The terminal penalty  $P$  and the terminal control gain  $K^f$  are determined using LQR techniques, i.e. Assumption 4 holds. The terminal sets  $\mathbb{T}^{[i]}$  are calculated such that Assumption 2 holds.

*Simulations* Figure 2 shows the closed loop response for the proposed approach for the nominal case and using for  $w_k$  uniform random noise, appropriately scaled by the bounds of  $\mathbb{W}$ . The simulations start from the initial state  $x_0 = (12.1 \ 0 \ 0)$ . Note that all constraints are satisfied and the system is robustly stabilized.

In Figure 3 we plotted the nominal closed loop response of the proposed approach (Section 3) and the existing approach (Chisci et al., 2001). We observe that all constraints are satisfied using all approaches. However the proposed approach chooses the input significantly less aggressive, which results in a better performance.

*Performance evaluation* Table 1 compares the performance obtained using the proposed approach and the extension sketched in Section 4. Therefore the performance using the stage cost in (15) is evaluated using simulations over 500 steps starting from the initial state  $x_0 = (12.1 \ 0 \ 0)$ . We compare three different cases: First, we use  $\mathbb{W}_{k+i|k} = \mathbb{W}$  and  $w_k \equiv 0$ , i.e. the performance obtained with the proposed approach in the absence of disturbances and without the extension. In this case the performance using only a single gain  $K^{[i]}$ , i.e. using approach (Chisci et al., 2001), is about 8.5% to 12.3% worse compared to the proposed approach.

Second the performance is evaluated using for  $w_k$  random noise as discussed above. The performance of 30 Monte Carlo simulations has been averaged with the same initial state  $x_0$  as above. Again the proposed approach delivers a better performance.

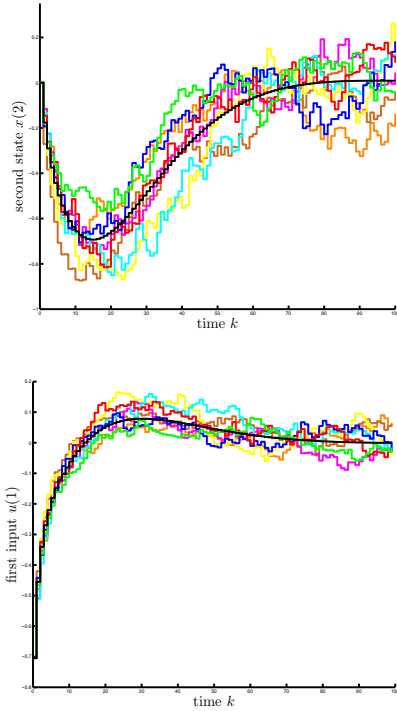


Fig. 2. Closed loop response using proposed approach for nominal case  $w_k \equiv 0$  (black) and different disturbance realizations (different colors).

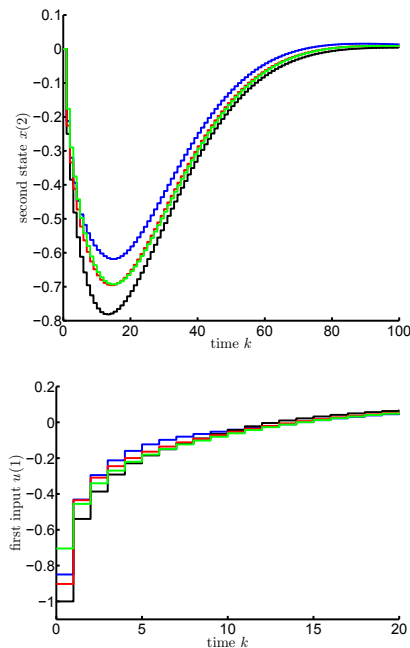


Fig. 3. Nominal closed loop response using proposed approach (green) and existing approach with gain  $K^{[1]}$  (blue),  $K^{[2]}$  (black) and  $K^{[3]}$  (red).

Finally, to illustrate the benefit of the extension we consider that online a tighter bound on the disturbance impacting the third state is available:  $\mathbb{W}_{k+i|k} = \tilde{\mathbb{W}}$ , where

$$\tilde{\mathbb{W}} = \left\{ w \text{ s.t. } \begin{pmatrix} -0.01 \\ -0.1 \\ -0.001 \end{pmatrix} \leq w \leq \begin{pmatrix} 0.01 \\ 0.1 \\ 0.001 \end{pmatrix} \right\}.$$

Note that this tighter bound and the proposed extension significantly improves the performance even in absence of a disturbance. This is due to the fact that the resulting tightened constraints are less restrictive and therefore the proposed controller has a larger control authority.

Table 1. Performance comparison

	$\mathbb{W}_{k+i k} = \mathbb{W}$ $w_k \equiv 0$	$\mathbb{W}_{k+i k} = \mathbb{W}$ $w_k$ random	$\mathbb{W}_{k+i k} = \tilde{\mathbb{W}}$ $w_k \equiv 0$
All 3 gains $K^{[i]}$	505.8	652.0	454.6
Only gain $K^{[1]}$ ,	567.8	715.9	465.4
Only gain $K^{[2]}$ ,	567.8	713.2	461.5
Only gain $K^{[3]}$ ,	548.5	694.9	462.3

## 6. SUMMARY AND OUTLOOK

This work proposed a tube based MPC approach, which determines online the shape of the tube by blending multiple tubes. The approach enables to use multiple gains and thus leads to less conservative results than (Chisci et al., 2001) with almost the same online computational complexity. For the closed loop we analyzed constraint satisfaction, robust recursive feasibility and robust stability. Examples outlined the approach and its benefits.

In future works include a more detailed analysis and investigation of the proposed approach as well as comparisons with other methods. Moreover, we aim to extend the proposed approach to nonlinear systems and output feedback MPC by combining it with the ideas of (Kögel and Findeisen, 2016b, 2015, 2017b).

## REFERENCES

- Balandat, M. (2011). Interpolation in output-feedback tube-based robust MPC. In *Proc. IEEE Con. Decision & Control & Euro. Control Conf.*, 1904–1909.
- Bäthge, T., Kögel, M., Cairano, S.D., and Findeisen, R. (2018). Contract-based predictive control for modularity in hierarchical systems. In *Proc. IFAC Conf. Nonlinear Model Predictive Control*, 499 – 504.
- Blanchini, F. and Miani, S. (2008). *Set-theoretic methods in control*. Springer.
- Blasi, S., Kögel, M., and Findeisen, R. (2018). Distributed model predictive control using cooperative contract options. In *Proc. IFAC Conf. Nonlinear Model Predictive Control*, 448–454.
- Brunner, F.D., Heemels, M., and Allgöwer, F. (2016). Robust self-triggered MPC for constrained linear systems: A tube-based approach. *Automatica*, 72, 73–83.
- Chisci, L., Rossiter, J.A., and Zappa, G. (2001). Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37(7), 1019–1028.
- Farina, M. and Scattolini, R. (2012). Distributed predictive control: A non-cooperative algorithm with neighbor-to-neighbor communication for linear systems. *Automatica*, 48(6), 1088–1096.
- Grüne, L. and Pannek, J. (2017). *Nonlinear model predictive control*. Springer.
- Hanema, J., Lazar, M., and Toth, R. (2020). Heterogeneously parameterized tube model predictive control for LPV systems. *Automatica*, 111, 108622.
- Heemels, W., Johansson, K.H., and Tabuada, P. (2012). An introduction to event-triggered and self-triggered

- control. In *Proc. IEEE Conf. Decision and Control*, 3270–3285.
- Herceg, M., Kvasnica, M., Jones, C., and Morari, M. (2013). Multi-Parametric Toolbox 3.0. In *Proc. European Control Conf.*, 502–510.
- Ibrahim, M., Kögel, M., Kallies, C., and Findeisen, R. (2020). Contract-based hierarchical model predictive control and planning for autonomous vehicle. IFAC World Congress 2020 (To appear).
- Khalil, H.K. (2002). *Nonlinear Systems*. Prentice-Hall, New Jersey.
- Kögel, M. and Findeisen, R. (2016a). Output feedback MPC with send-on-delta measurements for uncertain systems. In *Proc. IFAC Works. Distributed Estimation & Control of Networked Systems*.
- Kögel, M. and Findeisen, R. (2016b). Sampled-data, output feedback predictive control of uncertain, nonlinear systems. In *Proc. IFAC Symp. Nonlinear Control systems*.
- Kögel, M. and Findeisen, R. (2017a). Low latency output feedback model predictive control for constrained linear systems. In *Proc. IEEE Conf. Decision and Control*, 1925–1932.
- Kögel, M. and Findeisen, R. (2017b). Robust output feedback predictive control for uncertain linear systems with reduced conservatism. In *Proc. IFAC World Congress*, 11172–11177.
- Kögel, M., Quevedo, D., and Findeisen, R. (2019). Combined control and communication scheduling for constrained system using robust output feedback MPC. In *Proc. Euro. Control Conf.*, 1778–1783.
- Kögel, M. and Findeisen, R. (2013). On efficient predictive control of linear systems subject to quadratic constraints using condensed, structure-exploiting interior point methods. In *Proc. IEEE Conf. Decision & Control and European Control Conf.*, 27–34.
- Kögel, M. and Findeisen, R. (2015). Discrete-time robust model predictive control for continuous-time nonlinear systems. In *Proc. American Control Conf.*, 924–930.
- Kögel, M. and Findeisen, R. (2018). Improved Robust Decentralized MPC. In *Proc. Euro. Control Conf.*, 312–318.
- Kögel, M. and Findeisen, R. (2020). Robust MPC with reduced conservatism by blending multiples tubes. American Control Conference 2020 (To appear).
- Kouvaritakis, B. and Cannon, M. (2016). *Model predictive control*. Springer.
- Langson, W., Chrysochoos, I., Raković, S., and Mayne, D.Q. (2004). Robust model predictive control using tubes. *Automatica*, 40(1), 125–133.
- Limon, D., Alamo, T., Raimondo, D., De La Peña, D.M., Bravo, J., Ferramosca, A., and Camacho, E. (2009). Input-to-state stability: a unifying framework for robust model predictive control. In *Nonlinear model predictive control*, 1–26. Springer.
- Limon, D., Alvarado, I., Alamo, T., and Camacho, E. (2010). Robust tube-based MPC for tracking of constrained linear systems with additive disturbances. *Journal of Process Control*, 20(3), 248–260.
- Löfberg, J. (2004). YALMIP : A Toolbox for Modeling and Optimization in MATLAB. In *Proc. CACSD Conference*. Taipei, Taiwan.
- Lucia, S., Kögel, M., and Findeisen, R. (2015). Contract-based predictive control of distributed systems with plug and play capabilities. In *Proc. IFAC Conf. Nonlinear Model Predictive Control*, 205–211.
- Lucia, S., Kögel, M., Zometa, P., Quevedo, D., and Findeisen, R. (2016). Predictive control, embedded cyber-physical systems and systems - A perspective. *Annual Reviews in Control*, 41, 193 – 207.
- Mayne, D.Q. (2014). Model predictive control: Recent developments and future promise. *Automatica*, 50(12), 2967 – 2986.
- Mayne, D.Q. (2018). Competing methods for robust and stochastic MPC. In *Proc. IFAC NMPC Conf.*, 169–174.
- Mayne, D.Q., Raković, S., Findeisen, R., and Allgöwer, F. (2006). Robust output feedback model predictive control of constrained linear systems. *Automatica*, 42(7), 1217–1222.
- Mayne, D.Q., Raković, S., Findeisen, R., and Allgöwer, F. (2009). Robust output feedback model predictive control of constrained linear systems: Time varying case. *Automatica*, 45(9), 2082 – 2087.
- Mayne, D.Q., Seron, M.M., and Raković, S. (2005). Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41(2), 219–224.
- Mayne, D. (2015). Robust and stochastic MPC: Are we going in the right direction? In *Proc. IFAC NMPC Conf.*, 1–8.
- Munoz-Carpintero, D., Kouvaritakis, B., and Cannon, M. (2014). Striped parameterized tube model predictive control. In *Proc. IFAC World Congress*, 11998–12003.
- Pluymers, B., Rossiter, J., Suykens, J., and De Moor, B. (2005). Interpolation based MPC for LPV systems using polyhedral invariant sets. In *Proc. American Control Conf.*, 810–815.
- Raković, S.V. (2012). Invention of prediction structures and categorization of robust MPC syntheses. In *Proc. IFAC NMPC Conf.*, 245–273.
- Rakovic, S.V. and Baric, M. (2010). Parameterized robust control invariant sets for linear systems: Theoretical advances and computational remarks. *IEEE Trans. Automatic Control*, 55(7), 1599–1614.
- Raković, S.V., Kouvaritakis, B., Cannon, M., Panos, C., and Findeisen, R. (2011). Fully parameterized tube MPC. *IFAC Proceedings Volumes*, 44(1), 197–202.
- Raković, S.V., Levine, W.S., and Açikmeşe, B. (2016). Elastic tube model predictive control. In *Proc. American Control Conf.*, 3594–3599.
- Raković, S., Kerrigan, E.C., Mayne, D.Q., and Kouramas, K.I. (2007). Optimized robust control invariance for linear discrete-time systems: Theoretical foundations. *Automatica*, 43(5), 831–841.
- Rawlings, J.B., Mayne, D.Q., and Diehl, M.M. (2017). *Model Predictive Control: Theory, Computation, and Design*. Nob Hill Publishing, Madison, WI, 2nd edition.
- Riverso, S., Battocchio, A., and Ferrari-Trecate, G. (2013). PnPMPc toolbox. URL <http://sisdin.unipv.it/pnmpc/pnmpc.php>.
- Riverso, S., Farina, M., and Ferrari-Trecate, G. (2014). Plug-and-play model predictive control based on robust control invariant sets. *Automatica*, 50(8), 2179–2186.
- Rossiter, J.A. and Ding, Y. (2010). Interpolation methods in model predictive control: an overview. *International*

*Journal of Control*, 83(2), 297–312.

Scattolini, R. (2009). Architectures for distributed and hierarchical model predictive control - a review. *Journal of Process Control*, 19(5), 723 – 731.

Sui, D., Feng, L., and Hovd, M. (2008). Robust output feedback MPC for linear systems via interpolation technique. In *Proc. IFAC World Congress*, 12504–12509.

Zometa, P., Kögel, M., and Findeisen, R. (2013). muo-mpc: A free code generation tool for embedded real-time linear model predictive control. In *Proc. American Control Conf.*, 5320–5325.

### Appendix A. PROOF OF PROPOSITION 3

We will show that if  $\mathcal{P}(x_k)$  (16) is feasible, then the constraints (2) are satisfied at  $k$  and  $\mathcal{P}(x_{k+1})$  is feasible for any  $w_k \in \mathbb{W}$ .

If  $\mathcal{P}(x_k)$  (16) is feasible, then from (4), (13a) we have

$$x_k = \hat{x}_{k|k}^* \in \mathbb{X}, \quad u_k = \hat{u}_{k|k}^* \in \mathbb{U},$$

so the constraints (2) are satisfied for  $k$ .

To verify that  $\mathcal{P}(x_{k+1})$  is feasible for any  $w_k \in \mathbb{W}$  we claim that the following initial guess (based on the solution of  $\mathcal{P}(x_k)$  and  $w_k$ ) is feasible for  $\mathcal{P}(x_{k+1})$

$$\Lambda = \Lambda^*, \quad (\text{A.1a})$$

$$\hat{u}_{k+i|k+1} = \hat{u}_{k+i|k}^* + \sum_{j=1}^M \lambda^{[j]} K^{[j]} (\tilde{A}^{[j]})^{i-1} w_k, \quad (\text{A.1b})$$

$$\hat{x}_{k+l|k+1} = \hat{x}_{k+l|k}^* + \sum_{j=1}^M \lambda^{[j]} (\tilde{A}^{[j]})^{l-1} w_k, \quad (\text{A.1c})$$

where  $\tilde{A}^{[j]} = A + BK^{[j]}$ , and  $i = 1, \dots, N-1, l = 1, \dots, N$

$$\hat{u}_{k+N|k+1} = K^f \hat{x}_{k+N|k}^* + \sum_{j=1}^M \lambda^{[j]} K^{[j]} (\tilde{A}^{[j]})^{N-1} w_k, \quad (\text{A.1d})$$

$$x_{k+N+1|k+1} = \tilde{A}^f \hat{x}_{k+N|k}^* + \sum_{j=1}^M \lambda^{[j]} (\tilde{A}^{[j]})^N w_k, \quad (\text{A.1e})$$

where  $\tilde{A}^f = A + BK^f$ . Note that  $w_k$  is known at  $k+1$ ; it can be computed from  $w_k = x_{k+1} - Ax_k - Bu_k$ .

First let us verify the right part of (3); we have:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + w_k = A \underbrace{\hat{x}_{k|k}}_{=x_k} + B \underbrace{\hat{u}_{k|k}}_{=u_k} + w_k.$$

One can verify that the initial guess (A.1) satisfies the left part of (3) for  $i = 0, \dots, N-2$  as follows

$$\begin{aligned} \hat{x}_{k+i+1|k+1} &= \hat{x}_{k+i+1|k}^* + \sum_{j=1}^M \lambda^{[j]} (\tilde{A}^{[j]})^i w_k \quad (\text{A.2}) \\ &= A(\hat{x}_{k+i|k}^* + \sum_{j=1}^M \lambda^{[j]} (\tilde{A}^{[j]})^{i-1} w_k) \\ &\quad + B(\hat{u}_{k+i|k}^* + \sum_{j=1}^M \lambda^{[j]} K^{[j]} (\tilde{A}^{[j]})^{i-1} w_k) \\ &= A\hat{x}_{k+i|k+1} + B\hat{u}_{k+i|k+1}, \end{aligned}$$

and for  $i = N-1$  one can verify that

$$\begin{aligned} \hat{x}_{k+N+1|k+1} &= \tilde{A}^f \hat{x}_{k+N|k}^* + \sum_{j=1}^M \lambda^{[j]} (\tilde{A}^{[j]})^N w_k \quad (\text{A.3}) \\ &= A(\hat{x}_{k+N|k}^* + \sum_{j=1}^M \lambda^{[j]} (\tilde{A}^{[j]})^{N-1} w_k) \\ &\quad + B(K^f \hat{x}_{k+N|k}^* + \sum_{j=1}^M \lambda^{[j]} K^{[j]} (\tilde{A}^{[j]})^{N-1} w_k) \\ &= A\hat{x}_{k+N|k+1} + B\hat{u}_{k+N|k+1}. \end{aligned}$$

In summary, (3) holds for the choice of (A.1). Clearly  $\lambda \in \Lambda$  holds, as  $\lambda = \lambda^* \in \Lambda$ .

Note that using (11) for  $i = 1, \dots$  one can verify that

$$\mathbb{E}_i^{[j]} = \mathbb{E}_{i-1}^{[j]} \oplus (\tilde{A}^{[j]})^{i-1} \mathbb{W}. \quad (\text{A.4})$$

This implies that

$$\bar{\mathbb{X}}_i^{[j]} = \bar{\mathbb{X}}_{i-1}^{[j]} \oplus (\tilde{A}^{[j]})^{i-1} \mathbb{W}, \quad (\text{A.5a})$$

$$\bar{\mathbb{U}}_i^{[j]} = \bar{\mathbb{U}}_{i-1}^{[j]} \oplus K^{[j]} (\tilde{A}^{[j]})^{i-1} \mathbb{W}. \quad (\text{A.5b})$$

With this result one can verify that (13a) holds using

$$\hat{x}_{k+i|k+1} = \sum_{j=1}^M \lambda^{[j]} \underbrace{(\hat{x}_{k+i|k}^* + (\tilde{A}^{[j]})^{i-1} w_k)}_{\in \bar{\mathbb{X}}_i^{[j]} \oplus (\tilde{A}^{[j]})^{i-1} \mathbb{W} \subseteq \bar{\mathbb{X}}_{i-1}^{[j]}}$$

$$\hat{u}_{k+i|k+1} = \sum_{j=1}^M \lambda^{[j]} \underbrace{(\hat{u}_{k+i|k}^* + K^{[j]} (\tilde{A}^{[j]})^{i-1} w_k)}_{\in \bar{\mathbb{U}}_i^{[j]} \oplus K^{[j]} (\tilde{A}^{[j]})^{i-1} \mathbb{W} \subseteq \bar{\mathbb{U}}_{i-1}^{[j]}}$$

for  $i = 1, \dots, N-1$  and using Assumption 2 and (14) we have

$$\hat{x}_{k+N|k+1} = \sum_{j=1}^M \lambda^{[j]} \underbrace{(\hat{x}_{k+N|k}^* + (\tilde{A}^{[j]})^{N-1} w_k)}_{\in \mathbb{T}^{[j]} \oplus (\tilde{A}^{[j]})^{N-1} \mathbb{W} \subseteq \bar{\mathbb{X}}_{N-1}}$$

$$\hat{x}_{k+N+1|k+1} = \sum_{j=1}^M \lambda^{[j]} \underbrace{((A + BK^f) \hat{x}_{k+N|k}^* + (\tilde{A}^{[j]})^N w_k)}_{\in \mathbb{T}^{[j]}}$$

$$\hat{u}_{k+N|k+1} = \sum_{j=1}^M \lambda^{[j]} \underbrace{(K^f \hat{x}_{k+N|k}^* + K^{[j]} (\tilde{A}^{[j]})^{N-1} w_k)}_{\in \bar{\mathbb{U}}_{N-1}^{[j]} \oplus K^{[j]} (\tilde{A}^{[j]})^{N-1} \mathbb{W} \subseteq \bar{\mathbb{U}}_{N-1}}$$

This verifies that (13a) holds for the considered initial guess (A.1). In summary  $\mathcal{P}(x_{k+1})$  is feasible. ■