Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Consensus of Nonlinear Systems with Data-Rate Constraints ⋆

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Abstract: In this paper, consensus for a network of dynamical systems which communicate over data-rate constrained communication channels is considered. Each system in the network is equipped with a sensor and an actuator which are at locations remote from one another. In order to transmit the state of any system to any of the actuators, the sensors use data-rate constrained communication channels. The actuators then use the messages to determine control inputs such that the systems achieve a particular type of consensu. Sensor/actuator pairs that achieve that particular type of consensus are called consensus protocols. In this contribution, an efficient in terms of required data-rates consensus protocol is presented. For the protocol, a theorem proving conditions on the sufficient minimal data-rates to implement it is provided. The sufficient data-rate is proven to depend on the singular values of the linear part of the mapping of the systems in the network. Finally, an example is provided in the form of consensus for a network of harmonically forced bouncing ball systems, for which an analytical bound is provided on the sufficient outgoing channel rates.

Keywords: Synchronization, Networks, Limited data rate, Discrete-time systems, Nonlinear systems

1. INTRODUCTION

Ever since the introduction of wireless technologies in many different real-world applications, a new subfield has appeared in the dynamics and control world: dynamical systems and data-rate constraints. This subfield is dedicated to studying the interactions between dynamical systems and communication technologies. The common setups involve one or several dynamical systems where sensors, controllers and actuators are placed at locations remote from one another and are connected via wireless communication channels. Examples of such systems can be found in cooperative driving of connected cars, trajectory planning for swarms of drones, distributed networks of wirelessly connected sensors etc,... Although there have been improvements in the wireless communication technologies over the years, these technologies still suffer from two major drawbacks: limited data-rates and packet losses. In this paper, we will focus on the first issue: limited data-rates. This feature becomes problematic when it is combined with a system that possesses some source of uncertainty. According to Shannon, this uncertainty is information which can be in the form of noise, parametric uncertainty or uncertainties in the initial conditions. In those cases, it is necessary to find efficient communication strategies to guarantee the proper control/estimation of the systems over the constrained communication networks.

⋆ This paper was elaborated in the UCoCoS project which received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 675080. A. Matveev's work is supported by the Russian Science Foundation under grant 19-19-00403.

In the early years of the subfield, most of the work focused on two major aspects: the design of observers and the design of controllers over data-rate constrained channels. The earliest work (see e.g. Wong and Brockett (1997) or Elia and Mitter (2001) and references therein), mainly focused on linear systems for which most control and estimation problems have been solved. One can find broad overviews of the results for linear systems in Nair et al. (2007), Baillieul and Antsaklis (2007) and Andrievsky et al. (2010).

Only a few years after the first results for linear systems, result for nonlinear systems appeared. The earliest results were obtained for systems with specific structures as e.g. in De Persis (2003) and Baillieul (2004). Later, generalizations were obtained in Nair et al. (2004) and Liberzon and Hespanha (2005). In the first of these two papers, the concept of feedback entropy was used to provide bounds on the necessary channels rates. In the second paper, the authors generalized concepts from linear systems to nonlinear systems to provide bounds. After that, several notions of entropy were introduced to provide bounds on the sufficient / necessary data-rates to observe/control dynamical systems over data-rate constrained channels (see Kawan (2009), Colonius et al. (2013), Matveev and Savkin (2009), Kawan (2017), Sibai and Mitra (2017), Liberzon and Mitra (2016), and Pogromsky and Matveev (2016)).

The work in this paper is part of a logical continuation of the estimation and control problems: consensus of dynamical systems over communication networks with limited data-rates. This problem was first studied in Fradkov et al. (2008b) and

Fradkov et al. (2008a) where master/slave synchronization of two nonlinear systems was considered. In Li et al. (2011) the problem of average consensus in networks of linear systems with fixed topologies and limited data-rates was tackled. In You and Xie (2011) the specific effects of network topology and data-rate constraints were studied. In Dong (2019), consensus for networks of nonlinear systems with data-rate constraints was considered.

We focus on the problem of consensus for a network of identical dynamical systems. Each system is equipped with a sensor and an actuator which are placed at locations remote from one another. In order to transmit estimates of their state to their actuators, the sensors have to send messages via the communication channels of the communication network. These channels are subject to data-rate constraints. The sum of the ingredients that are necessary for the systems to be in consensus is called a consensus protocol. The main contribution of this paper is providing a consensus protocol that leads to consensus of the systems whilst functioning over the channels with limited data-rates. The consensus protocol that is developed in this paper employs ideas from symbolic set dynamics in order to construct the alphabets that are used for communication (see Morse and Hedlund (1938)). For this consensus protocol, we provide a theorem that describes the minimum sufficient data-rate in order to reach consensus. The rate is proven to depend on the larger-than-one singular values of the linear part of the mapping of the systems.

The paper is organized as follows: in Section 2, the setting of the problem to be solved is described. A notion of consensus is also introduced. In Section 3, the communication strategy that is going to be used to generate the alphabets for communication are presented. In Section 4 the actual procedure that leads to consensus is described. In Section 5 a theorem which provides sufficient conditions on the minimum channel capacities to implement the aforementioned procedure is proposed. Finally, the Section 6, an example of how this theorem can be applied on the problem of consensus for a network of bouncing ball systems is presented.

2. PROBLEM STATEMENT

We consider k discrete-time dynamical systems of the following form

$$x_i(t+1) = f(x_i(t), u_i(t)), \qquad x_i(0) = x_{i0}, \tag{1}$$

for $i \in \{1, \ldots, k\}$, where $f : X \times U \to X$ is a nonlinear mapping, $X \subset \mathbb{R}^n$ is the state-space, $u_i \in U \subset \mathbb{R}^m$ are the control inputs and $x_{i0} \in X$ are initial states. We impose the following regularity assumptions on the mapping f and the sets X and U.

Assumption 1. The function f is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m$. The set U is compact. The set X is bounded. Moreover, for all inputs $u_i(t) \in U$, the solutions $x_i(t)$ of (1) remain within the set X.

All initial conditions of the systems are within a distance $\delta > 0$ from each other, where δ is a parameter that is part of the consensus protocol. This leads the following assumption.

Assumption 2. The following condition holds for all initial states

$$\left\| x_{i0} - x_{j0} \right\| \le 2\delta,$$

$\forall i, j \in \{1, \ldots, k\}$.

In this paper, the notation $\|\cdot\|$ refers to any appropriate vector norm. The interactions between the systems happen in the form of messages which are sent over communication channels. The messages are sent by the sensors to the actuators. Each system has a sensor and an actuator. The sensor is connected to some of the actuators by means of discrete communication channels. The connections in the network are described by a communication adjacency matrix $\mathfrak{A} \in \mathbb{R}^{k \times k}$. The entries of the communication adjacency matrix $\mathfrak{a}_{ij}$ are 1 if sensor i can communicate with actuator j and 0 otherwise. Each non-zero entry of the communication adjacency matrix thus corresponds to a communication channel. Each sensor is composed of several coders $\mathscr{C}_{ij}$ which generate messages to be sent over the different communication channels. All messages are sent simultaneously over the network. The time interval between consecutive messages $\bar{s} > 0$ is fixed. The set of communication instants is $S = \{0, \bar{s}, 2\bar{s}, \ldots\}$. The coder keeps in memory the past messages $E_{ij}(t)$,

$$E_{ij}(t) := \{e_{ij}(s) : s \in S, s < t\},$$

$\forall i, j \in \{1, \ldots, k\} : \mathfrak{a}_{ij} = 1$. As a part of the consensus protocol, it is assumed that the sensors and actuators have an initial estimate of their own state $\hat{x}_i(0)$ as well as an initial target r_0 which verify

$$\|\hat{x}_i(0) - x_i(0)\| \le \delta, \qquad \hat{x}_i(0) = \hat{x}_j(0) = r_0, \tag{2}$$

$\forall i, j \in \{1, \ldots, k\}$, where δ is the initial mismatch. The coder equations are

$$e_{ij}(t) = \mathscr{C}_{ij}(f, r_0, \bar{s}, \delta, x_i(t), \hat{x}_i(0), E_{ij}(t)) \tag{3}$$

for $i, j \in \{1, \ldots, k\} : \mathfrak{a}_{ij} = 1,\ t \in S$. The messages travel from the sensors to the actuators via the communication channels which are limited in terms of data-rate. Each channel has its own alphabet function $\mathscr{A}_{ij}$ which determines what messages it can transmit. For simplicity, we assume that the messages are received at the same time instant as they are sent (the time between two consecutive communications $\bar{s} > 0$ prevents the channel from transmitting infinite amounts of data instantaneously). The data-rate constraints on the communication channels imply that the messages $e_{ij}(t)$ that are sent have to be part of finite-sized alphabets. These alphabets are lists of symbols which are indexed from 1 to $l_{ij}(t) < \infty$ where the last index $l_{ij}(t)$ and thus the size of the alphabet is determined by the alphabet functions $\mathscr{A}_{ij}$. The alphabet function equations are

$$l_{ij}(t) = \mathscr{A}_{ij}(f, r_0, \bar{s}, \delta, x_i(t), \hat{x}_i(0), E_{ij}(t)), \tag{4}$$

for $i, j \in \{1, \ldots, k\} : \mathfrak{a}_{ij} = 1,\ t \in S$. The restriction on the choice of the messages is then simply

$$e_{ij}(t) \in \{1, ..., l_{ij}(t)\}, \tag{5}$$

for $i, j \in \{1, \ldots, k\} : \mathfrak{a}_{ij} = 1,\ t \in S$. Each of the channels has its own maximum number of bits b^+_{ij} that can be transmitted per time interval . This quantity depends on the chosen length of the time interval. For any choice of $\bar{s}$, the following should hold

$$\log_2(l_{ij}(t)) \le b^+_{ij}(\bar{s}), \tag{6}$$

for $i, j \in \{1, \ldots, k\} : \mathfrak{a}_{ij} = 1,\ t \in S$. The previous equation links communication instants t with communication intervals $\bar{s}$. In particular, at every communication instant, the logarithm of the number of different messages that can be transmitted should always be inferior or equal to the maximum number of bits that can be transmitted during any time interval between communications (otherwise the information in the messages cannot be encoded).

On the other side of the communication channels, actuators receive the messages sent by the coders.

They keep in memory all previous received messages $F_i(t) = \{e_{ji}(s) : s \in S, s \le t, j \in \{1,\dots,k\} : \mathfrak{a}_{ji} = 1\}$ which they use to generate an appropriate control input for each system

$$u_i(t) = \mathscr{U}_i(f, r_0, \bar{s}, \delta, \hat{x}_i(0), F_i(t)) \tag{7}$$

$i \in \{1,\dots,k\}$, $\forall t \ge 0$. Fig. 1 depicts how the different elements interact for a configuration of two systems with $\mathfrak{A} = 1_{2\times 2}$ (where $1_{n\times m}$ is the all-ones matrix of size n times m).

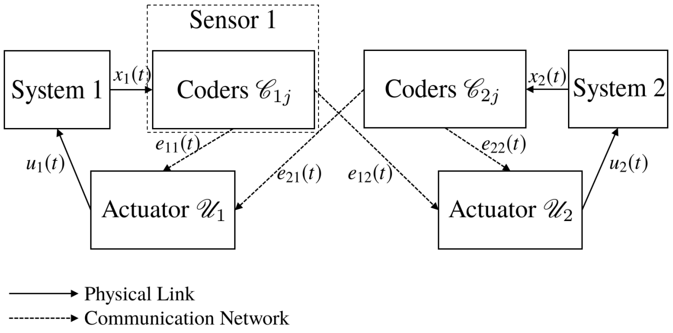

Fig. 1. Systems, sensors and actuators setup for two systems ($k = 2$)

Through interactions of the sensors and actuators, and suitably chosen control laws, the objective will be to keep the states of the systems within a certain distance $G\delta$ of each other, where G is a positive constant. Consensus in this paper should thus be understood as a local property: given that the systems are δ-close to each other initially, we look for a suitable consensus protocol such that the systems remain $(G\delta)$-close to each other at all times. This consensus property should be verified with the same G for all δ, as long as δ is chosen small enough to begin with. The constant G will be referred to as a **consensus factor** and is defined as follows.

Definition 1. Let the solutions of (1) exist for some $u_i \in U$ and let Assumption 2 hold for k systems with a particular $\delta > 0$. Then the quantity $G < \infty$ is called a consensus factor if the following holds

$$\left\| x_i(t) - x_j(t) \right\| \le G\delta, \qquad \forall t \ge 0,\ \forall i, j \in \{1,\dots,k\}.$$

In order to properly define the objective of this paper, we will need a quantity that measures the sum of the channel capacities from all channels that each sensor is connected to. This quantity is the **outgoing communication capacity** c_i which is defined as

$$c_i := \min_{j:\mathfrak{a}_{ij}=1} \limsup_{\bar{s}\to\infty} \frac{b^+_{ij}(\bar{s}) \sum_{l=1}^{k} \mathfrak{a}_{il}}{\bar{s}}. \tag{8}$$

The objective of this paper is to design a consensus protocol, consisting of coders, alphabets, and actuators such that the protocol keeps a number of systems in consensus, given that the systems are close enough to each other at the beginning. Moreover, this consensus protocol should function with limited outgoing communication capacities. If a particular consensus protocol achieves both these features, it is said to lead to consensus which is a property defined as follows.

Definition 2. A consensus protocol (3), (4), (7) leads to consensus of k systems with outgoing channel rates c_i as defined in (8) if both of the following conditions hold

(1) There exists $G < \infty$, $\delta^* > 0$, such that for all $\delta : \delta^* \ge \delta > 0$, G is a consensus factor as defined in Definition 1 with those particular δ;
(2) The messages exchanged by the consensus protocol respect the channel bit-rate constraints (6).

3. AN ALPHABET FOR COMMUNICATION

Now that the problem statement has been posed, the natural question is: "What protocol leads to consensus?", "What are the minimum outgoing channel rates necessary to implement such a protocol?", and "How do these quantities depend on the system's equations?". A first fundamental part of a functioning consensus protocol is how the sensors communicate estimates of the states or reference trajectories to the actuators over the data-rate constrained channels, i.e. how to encode estimates of the states by using alphabet functions. In order to make the consensus protocol more graspable, we start by developing that part. Note that the alphabets only describe a possible method to encode the information about the state/reference into messages. Which sensor should send what information to what actuator is another part of the consensus protocol and will be discussed in the next section.

The procedure we describe in this section is valid as long as δ is strictly positive and chosen small enough, and $\bar{s} \ge 1$. How these two parameters should be chosen will be discussed later in this paper.

The idea behind the communication protocol, which is based on ideas from Matveev and Pogromsky (2016), Pogromsky and Matveev (2011), and Voortman et al. (2019) is to cover the state-space X with balls of size δ, where δ is the initial distance (the left part of Fig. 2 depicts such a covering). We will use the notation $B_\delta(x)$ for a ball of radius δ, centered in x. For any fixed initial distance δ, a covering V of the state-space X with balls with indexes l, centers $v_l \in X$ and radii δ is built. Since, from Assumption 1 the state-space of the system is bounded, this covering will be of finite size. The set V contains all centers v_l. This covering is assumed to be minimal in the sense that $X \subseteq \bigcup_{l=1}^{m} B_\delta(v_l)$ and m is the smallest possible number of balls verifying this property.

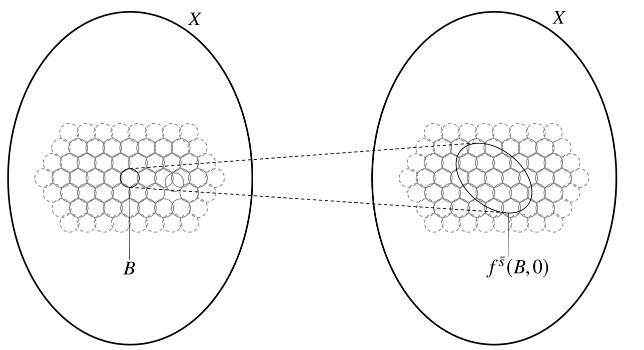

Fig. 2. Every ball B_δ of the covering of the state-space X is mapped into an ellipsoid after $\bar{s}$ time-steps.

When computing the image of a ball of size δ in the absence of input and if δ is chosen small enough, the linear part of the mapping $f(\cdot,0)$ is predominant and hence the image of a balls through the mapping applied $\bar{s}$ times ($f^{\bar{s}}(\cdot,0) = \underbrace{f(\dots f(\cdot,0),0)}_{\bar{s}\text{ times}}$), is an ellipsoid whose semi-axes corresponds to the right-singular vectors of the linear part of the mapping. This situation is illustrated in Fig. 2. The intersection of this ellipsoid and the state-space is a set which can be covered by selecting the minimum number of balls in the original covering such that the union of those balls covers the intersection of the ellipsoid and the state-space. During the initialization phase of the consensus protocol an alphabet is built for each of the balls in the original covering. The symbols of this alphabet each correspond to one of the balls contained in its image's intersection with the original covering (See Fig. 3). For any fixed communication interval $\bar{s}$, the image I_l of each of the balls l through the mapping f applied $\bar{s}$ times with no input is computed. This image is then covered by selecting balls with centers $v_p^l \in V$. Each ball thus has a covering with centers $V_l \subseteq V$. For each of the coverings, the balls $v_p^l \in V_l$ are indexed from 1 till m_l where m_l is the number of balls in V_l. These coverings are constructed such that m_l is as small as possible and

$$X \cup f^{\bar{s}}(B_\delta(v_l), 0) \subseteq \bigcup_{v_p^l \in V_l} B_\delta(v_p^l).$$

Because they constitute the possible alphabets for communication, the coverings V and $\{V_l\}$ are known by all sensors and actuators.

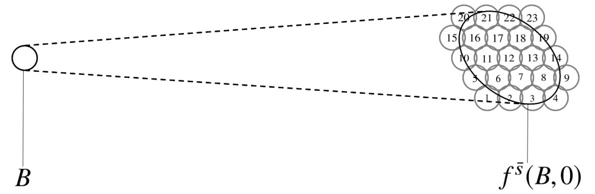

Fig. 3. Alphabet for the ball B_δ of Fig. 2

The alphabets are then used to communicate in the following way. Let an actuator have an estimate of the current position of its system (in the form of a ball of radius δ in which the current state is contained and whose center is the estimate). Because the coder has access to the full state, it can easily compute the future state of the system with no input. In order to transmit an estimate of the future state to the actuator, the coder then simply transmits the index of the ball in which the state will be in $\bar{s}$ time-steps (Fig. 3 presents this idea schematically). Coders can send estimates of their state to actuators of other systems by proceeding similarly.

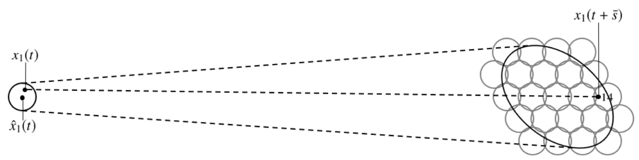

Fig. 4. Particular situation with the state, estimate, and message "14".

4. THE CONSENSUS PROTOCOL

In this section, the actual consensus protocol, in the form of coders, alphabets, and actuators is presented. For simplicity, we will consider mappings of the following form

$$f(x_i(t), u_i(t)) = \varphi(x_i(t)) + u_i(t).$$

Configurations with mappings which are nonlinear in the input are left for further research. We introduce the notation $\varphi^s(\cdot)$ for the mapping φ applied s times ($\varphi^s(\cdot) = \varphi(\ldots\varphi(\cdot))$).

Because the actuation will occur at different instants than the communication instants, the consensus protocol functions in periods $\bar{s} \geq 2$.

We will note $l_t \in \{1, \ldots, \#V\}$ the index of the ball in which all states are contained at time $t \in S$, where $\#V$ refers to the cardinality of the set V. The consensus protocol will guarantee that at each communication instant, i.e. at $t \in S$, all states are within radius δ of v_{l_t}. By proceeding this way, at the beginning of each communication interval all systems end in the same configuration as they were initially. In that case, the index l_t of the ball in which all systems are is known by all sensors and actuators. We will use the notation $\bar{t}$ for the last communication instant and $l_{\bar{t}}$ thus refers to the index of the ball in which all states were at the beginning of the communication interval. For brevity, we will not repeat the arguments of each function and instead use the abbreviated notation: $\mathscr{A}_{ij}(\ldots), \mathscr{C}_{ij}(\ldots)$, and $\mathscr{U}_i(\ldots)$.

The consensus protocol we will present functions as follows: part or all of the systems decide on a common trajectory. Out of the k systems, k_m systems ($k \geq k_m \geq 2$) are decision-makers and thus exchange information in order to decide on a trajectory to follow and follow that trajectory while $k_s = k - k_m$ systems are followers which follow the trajectory without participating in the decision of which trajectory to follow. Without any loss of generality, we will assume that the systems numbered from 1 till k_m are decision-makers and the rest are followers. Note that it is possible that $k_m = k$ in which case there are no followers, only decision-makers.

The target trajectory is determined as the average of the state that the decision-maker systems would be in if they were left unactuated during $\bar{s}$ timesteps after the last communication instant. Because of data-rate limitations, two approximations are made in determining the target trajectory: firstly it is not the exact states that are transmitted but only estimates of the state and secondly, the point that the systems use as a target is the point in the covering closest to the average of the estimates, as opposed to simply the average of the estimates.

The communication adjacency matrix, is determined as follows. Among the group of decision-makers, all systems exchange estimates of their states with each other and they also send an estimate of their states to their own actuator, which implies that the k_m first rows of the communication adjacency matrix are all ones. They as well as the followers. The followers only need to send an estimate of their own state to their actuator in order to be actuated. The communication adjacency matrix is

$$\mathfrak{A} = \mathfrak{A}_{\text{tc}} := \begin{bmatrix} 1_{k_m \times k_m} & 1_{k_m \times k_s} \\ 0_{k_s \times k_m} & I_{k_s \times k_s} \end{bmatrix}. \tag{9}$$

The consensus protocol is

Procedure 1. $\forall t \in S$,

$$\begin{aligned}
&\mathscr{A}_{ij}(\cdot) = m_{l_t}, && \forall i \leq k_m, j \in \{1, \ldots, k\}, \\
&\mathscr{A}_{ii}(\cdot) = m_{l_t}, && \forall i > k_m, \\
&\mathscr{C}_{ij}(\cdot) = \arg \min_{p \in \{1, \ldots, m_{l_t}\}} \left\| \varphi^{\bar{s}}(x_i(t)) - v_p^{l_t} \right\|, && \\
& && \forall i \leq k_m, j \in \{1, \ldots, k\}, \\
&\mathscr{C}_{ii}(\cdot) = \arg \min_{p \in \{1, \ldots, m_{l_t}\}} \left\| \varphi^{\bar{s}}(x_i(t)) - v_p^{l_t} \right\|, && \forall i > k_m,
\end{aligned}$$

$\forall t \geq 0$,

$$\mathscr{U}_i(\cdot) = \begin{cases} 0, & \text{if } (t+1) \notin S, \\ \underset{v \in V_{l_{\bar{t}}}}{\arg\min} \left\| \sum_{j=1}^{k_m} \frac{v_{e_{ji}(\bar{t})}}{k_m} - v \right\| - v^{l_{\bar{t}}}_{e_{ii}(\bar{t})}, & \text{if } (t+1) \in S. \end{cases}.$$

Note that the actuator only applies a control input at times t such that $(t+1) \in S$. Since $\bar{s}$ is chosen larger or equal to 2, this is always possible. The input that is applied is the difference between the target that was computed as the point in the covering closes to the average and the center of the ball in which each system would have ended up, had no actuation been applied. With this choice of actuation and with the linearity of the input, all systems end up in the same ball at the communication instants. Since at the beginning of every communication instant all systems are in the same configuration as they were in $t = 0$ (namely, all systems in the same ball with center v_{l_t}), this procedure is endlessly repeatable.

Remark 1. Although the problem statement allows for the coder and actuator to rely on the full history of messages, our consensus protocol does not require us to store all previous messages: storing only v_{l_t} is sufficient.

5. RESULTING RATES

The consensus protocol that was described in the previous section functions as long as $\delta > 0$ and $\bar{s} \geq 2$. No conditions that guarantee that the outgoing communication rates will be below certain thresholds have yet been given. In this section, we present these results. More precisely, we provide a theorem for the consensus protocol which gives sufficient conditions on the communication adjacency matrix and outgoing communication capacities c_i to implement the consensus protocol. The sufficient communication rate depends on how a ball of radius δ expands/contracts into an ellipsoid under the mapping φ. It is well-known that the image of a ball under a linear mapping is an ellipsoid whose semi-axis are the right-singular vectors of the matrix associated with this linear map and the length of the axis are the singular values multiplied by the original radius of the ball. In the nonlinear case, the singular values of the Jacobian of the mapping φ have the same effect as long as the original ball is small enough such that the higher order terms are neglectable. For nonlinear systems, the singular values are generally state-dependent which means that it is difficult to provide upper bounds on the expansion/contraction rate depending on those singular values.

One possibility to get rid of the state-dependency in the singular values of the Jacobian, is to use the an assumption from Matveev and Pogromsky (2016). We first introduce the following notations

$$A^s(x) := \frac{\partial \varphi^s}{\partial x}(x), \qquad A(x) := A^1(x).$$

We then have the following assumption.

Assumption 3. Matveev and Pogromsky (2016) There exist constant $\Lambda_d \geq 0$, $d \in \{1,\dots,n\}$, and a positive definite $n \times n$ matrix $P = P^\intercal$ such that

$$\sum_{i=1}^{d} \log_n \lambda_i(x) \leq \Lambda_d, \qquad \forall x \in X \tag{10}$$

for all $d \in \{1,\dots,n\}$, where $\log_2(0) := -\infty$ and $\lambda_1(x) \geq \cdots \geq \lambda_n(x) \geq 0$ are the roots of

$$\det\left(A^\intercal(x) P A(x) - \lambda P\right) = 0 \tag{11}$$

repeated according to their algebraic multiplicities.

From this assumption, we define $\bar{\Lambda} = \max_{i \in \{1,\dots,n\}} \Lambda_i$. The $\lambda_i(x)$ of the previous assumption are in fact the squares of the singular-values of $A(x)$ expressed in a different coordinate basis. Indeed, decomposing the matrix P as $P = U^\intercal U$ where U is non-singular (since P is positive definite and symmetric such a decomposition always exists) allows us to rewrite (11) as

$$\det\left(A^\intercal(x) U^\intercal U A(x) - \lambda U^\intercal U\right) = 0$$

which, since U is non-singular, has the same solutions as

$$\det\left(U^{-\intercal} A^\intercal(x) U^\intercal U A(x) U^{-1} - \lambda I_n\right) = 0.$$

The solutions of the previous equation are thus the squares of the singular values of $U^{-\intercal} A^\intercal(x) U^\intercal U A(x) U^{-1}$, which is equivalent to saying that they are the square of the singular values of $A(x)$ in a different coordinate basis. One important consequence is that for the norm $\|x\|_P = \sqrt{x^\intercal P x}$, the following inequality holds $\|Mx\|_P \leq \sqrt{\lambda_1(M)}\, \|x\|_P$. With the previous assumption in mind, we now present the main contribution of this paper: a theorem that provides sufficient conditions on the communication adjacency matrix and outgoing communication rates to implement the consensus protocol.

Theorem 1. Let Assumptions 1, 2 and 3 hold for k systems (1). Then Procedure 1 leads to consensus of those systems over any channels with outgoing channel rates

$$c_i > k \frac{\bar{\Lambda}}{2}, \qquad \forall i \in \{1,\dots,k_m\},$$
$$c_i > \frac{\bar{\Lambda}}{2}, \qquad \forall i \in \{k_m+1,\dots,k\},$$

and communication adjacency matrix $\mathfrak{A} = \mathfrak{A}_{\mathrm{tc}}$ as defined in (9).

The proof of this theorem will be provided in the full version of this paper. Note that the previous theorem implies that there exists a consensus factor G. It is possible to find analytical values for G but for brevity, the details of these computations are omitted from this paper.

6. EXAMPLE

In this section, we illustrate the use of the previous theorem by computing the bounds on the outgoing channel rates for the consensus of a network of harmonically forced bouncing ball systems. For this system, we will apply Theorem 1 in order to find the minimum required outgoing communication capacities to implement the consensus protocol.

6.1 Harmonically Forced Bouncing Ball System

The harmonically forced bouncing ball system is a simple discrete-time system with complex behavior. It is a discrete-time realization of a ball which bounces on a harmonically forced table. The system has been studied extensively in Mello and Tufillaro (1987), Tufillaro et al. (1992), Clark et al. (1995), Cao et al. (1997). The bouncing ball map is Tufillaro et al. (1992):

$$\varphi_{\mathrm{BB}} : \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_1 + x_2 \\ \alpha x_2 - \beta \cos(x_1 + x_2) \end{pmatrix} \right\},$$

where $\alpha \in (0,1)$ is the energy restitution coefficient and $\beta = 2\omega^2(1+\alpha)A/g > 0$ where ω is the angular frequency of the table, A the amplitude of the table, and g the gravity of earth. This map is invariant under the coordinate change $x_1 \to x_1 + 2\pi j$, $j \in \mathbb{N}$. The state-space of the system is thus the cylinder $\mathfrak{C} = S^1_0 \times \mathbb{R}$ where $S^1_0 \subset \mathbb{R}^2$ is the unit circle centered around

$(0,0)$. The cylinder is a smooth Riemannian manifold with the standard Riemannian metric.

We consider k bouncing ball systems with input:

$$x_i(t+1) = \varphi_{\mathrm{BB}}(x_i(t)) + u_i(t).$$

The system has a positively invariant set $X = S_0^1 \times [-\beta(1-\alpha)^{-1}; \beta(1-\alpha)^{-1}]$. Applying Theorem 1 to these bouncing ball systems gives the following proposition

Proposition 1. Theorem 1 holds for k bouncing ball systems with

$$\bar{\Lambda} = 2\log_2\left(1+\alpha+\beta+\sqrt{(1+\alpha+\beta)^2-4\alpha}\right) - 2.$$

The proof of this proposition will be presented in the full version of this paper.

7. CONCLUSION

In this paper, the problem of consensus in networks of nonlinear dynamical systems has been posed. A specific type of consensus was introduced. The question that were then answered are: "What protocols lead to consensus?", "What are the minimum outgoing channel rates necessary to implement such a protocol?", and "How do these quantities depend on the system's equations?". An answer, in the form of a consensus protocol, was provided, together with a theorem that gives sufficient conditions on the outgoing rates. The rates were proven to depend on the singular values of the linear part of the mapping. The theory was applied to compute the rate required to keep bouncing ball systems in consensus for which an analytical bound on the sufficient outgoing communication rates was provided.

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