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Overcoming the dissipation obstacle with Bicomplex Port-Hamiltonian Mechanics

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Abstract: The dissipation obstacle refers to the problem that there is no general solution to shape the energy of dissipative port-Hamiltonian (pH) systems with the method of Casimir functions. This paper argues that it is caused by lack of a strictly symplectic structure of a dissipative port-Hamiltonian system. We develop a method of bicomplex pH systems that is strictly symplectic and we show how it overcomes the obstacle and allows one to systematically use Casimir functions to shape the energy.

Keywords: dissipation obstacle, energy-Casimir method, port-Hamiltonian systems, Hamiltonian mechanics, energy shaping,

1. INTRODUCTION

In engineering, Newtonian mechanics is favored over the more fundamental analytical mechanics of Lagrange and Hamilton (see e.g. the introduction of Allison et al. (2014)). This is mostly due to the inability of analytical mechanics to model non-conservative systems. Virtually all engineering systems are non-conservative, hence the preference for Newtonian mechanics.

In the mathematics literature, much progress has been made of late to expand the theory of Hamiltonian mechanics to deal with systems that cannot be dealt with using Newtonian methods (see e.g. Arnol'd (1989); Singer (2001); Marsden and Ratiu (2002)). Much of the effort consists of developing the symplectic-geometric methods that underlie Hamiltonian mechanics, but no specific results are available for non-conservative systems.

In the physics literature, several methods have been proposed to apply analytical mechanics to non-conservative systems. These methods include the use of the *ad hoc* Rayleigh dissipation function (Goldstein, 1980, p. 23) and the use of fractional calculus to generalize frictional energy in Lagrangian mechanics (Riewe (1997); Allison et al. (2014)). But there exists no systematic means to apply these methods in engineering. Another line of research consists of Hamiltonians defined on complex spaces which are mainly used for quantum-mechanical systems, but neither these have been connected to dissipative systems (Shankar, 1994, p. 203).

In systems and control engineering, port-Hamiltonian (pH) systems theory is a recent development that builds on the mathematical tradition and aims to put energy back in controls (Ortega et al. (2001)). Non-conservative systems are modeled as pH systems by separating the dissipative elements from the energy-storing elements and connecting those through Dirac structures. The so-called energy-Casimir method is used to design stabilizing controllers for pH systems. However, there is an important class of dissipative systems that are not amenable to be controlled using this methods. This issue has come to be known as the dissipation obstacle (see e.g. Ortega et al. (2001); Van der Schaft and Jeltsema (2014); Van der Schaft (2017)).

This paper draws on both the symplectic theory of the mathematical tradition and the complex theory of quantum mechanics, to develop complex Hamiltonians that systematically overcome the dissipation obstacle in pH systems theory. The paper is organized as follows. In Section 2 we overview the relevant pH systems theory and identify the problems connected with the dissipation obstacle. In Section 3 we develop our strictly symplectic, bicomplex Hamiltonian system together with its Casimir elements. In Section 4 we apply it to the simple damped harmonic oscillator and analyze the symplectic geometry of the solution. Finally, in Section 5 we show how our bicomplex theory can be used in the context of control-by-interconnection and deal with the stability, Lyapunov and Casimir functions, and constructively show how a stabilizing controller can be found for any damped harmonic oscillator, irrespective of the placing of the resistive elements, thus overcoming the dissipation obstacle.

2. STRICTLY SYMPLECTIC HAMILTONIAN MECHANICS AND THE DISSIPATION OBSTACLE

2.1 Hamiltonian Systems

In order to be a Hamiltonian system, a system needs to contain the following (Arnol'd, 1989, p. 161):

(i) an even-dimensional manifold: the state space,
(ii) a symplectic structure on it, and
(iii) a function on it: the Hamiltonian.

The prototypical example of such a strictly symplectic Hamiltonian system is the harmonic oscillator with inertance I, compliance C, and a state vector $x = \begin{bmatrix} q & p \end{bmatrix}^\intercal$, representing the position and the momentum respectively. In the Hamiltonian formalism, the dynamics of this system are expressed as

$$\dot{x} = J\nabla H(x) \tag{1}$$

Here $\nabla H(x)$ is the gradient of the Hamiltonian $H(x) = \frac{p^2}{2I} + \frac{q^2}{2C}$, $H(x) : \mathbb{R}^2 \to \mathbb{R}$. Matrix J is a 2×2 symplectic matrix, i.e. a matrix with determinant ± 1 satisfying

$$J^\dagger \Omega J = \Omega, \tag{2}$$

where $J^\dagger$ is the (conjugate) transpose of J and Ω is any 2×2 nonsingular, skew-symmetric matrix (see Rim (2015)). For the system in (1), the symplectic structure is represented by the matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3}$$

This system clearly contains the three requirements for a system to be a Hamiltonian system: (i) the state space of x, (ii) the structure represented by J, and (iii) the Hamiltonian function $H(x)$.

2.2 Port-Hamiltonian Systems

Port-Hamiltonian (pH) systems theory employs the Hamiltonian formalism in an input-output-based analysis of engineering systems. In addition, dissipative elements are incorporated by using a positive semi-definite dissipation matrix R. The input-state-output description of a pH system with input u and output y is (Ortega et al. (2001))

$$\begin{aligned} \dot{x} &= [J - R]\nabla H(x) + gu \\ y &= g^\intercal \nabla H(x) \end{aligned} \tag{4}$$

Here, the matrix g represents the input structure of the system and its transpose, $g^\intercal$, represents the output structure of the system.

However, this system does not satisfy (ii) due to the positive semi-definiteness of R. Therefore, dissipative pH systems are not strictly symplectic Hamiltonian systems.

2.3 Dissipation obstacle

In the pH framework (see e.g. Ch. 15 of Van der Schaft and Jeltsema (2014)), a controller is modeled as another pH system with state x_c, Hamiltonian $H_c(x_c)$, input structure g_c that interfaces the system with an input u_c and output y_c. The input-state-output description of the controller is of the same form as (4). The plant and the controller are interconnected in a negative feedback loop. This results in the closed-loop system

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} J - R & -gg_c^\intercal \\ g_c^\intercal & J_c - R_c \end{bmatrix} \begin{bmatrix} \nabla H(x) \\ \nabla H_c(x_c) \end{bmatrix} \\ \begin{bmatrix} y \\ y_c \end{bmatrix} &= \begin{bmatrix} g^\intercal & 0 \\ 0 & g_c^\intercal \end{bmatrix} \begin{bmatrix} \nabla H(x) \\ \nabla H_c(x_c) \end{bmatrix} \end{aligned} \tag{5}$$

A problem arises when designing controllers that stabilize the closed-loop system at a non-trivial equilibrium point x^* using the energy-Casimir method (See Ortega et al. (2001)). This problem is known as the "dissipation obstacle" and we briefly sketch how this comes about for comparison with our solution.

The strategy consists of finding a Lyapunov function of the closed-loop system

$$V(x, x_c) = H(x) + H_c(x_c) + C(x, x_c) \tag{6}$$

that has a minimum at the desired equilibrium point x^*. This is achieved with a Casimir function $C = C(x, x_c)$. One shows that any such Casimir needs to satisfy

$$\begin{bmatrix} \nabla_x^\intercal C(x, x_c) & \nabla_{x_c}^\intercal C(x, x_c) \end{bmatrix} \begin{bmatrix} J - R & -gg_c^\intercal \\ g_c g^\intercal & J_c - R_c \end{bmatrix} = 0 \tag{7}$$

This implies that when $R \succeq 0$ and $R_c \succeq 0$ that

$$R\nabla_x C(x, x_c) = 0 \quad \text{and} \quad R_c \nabla x_c C(x, x_c) = 0 \tag{8}$$

Effectively, this prescribes that a candidate Casimir function cannot depend on any states that are influenced by the dissipation and, hence, the energy-Casimir method is inapplicable to a vast collection of engineering systems that have dissipation acting on the controlled states. This is known as the dissipation obstacle (see e.g. Ortega et al. (2001); Van der Schaft and Jeltsema (2014)).

We see that the dissipation obstacle indeed arises from the positive semi-definiteness of the dissipation matrix R. From a mathematical perspective, the problem appears to be that R is a metric tensor rather than a 2-form, endowing the state space with a Riemannian structure rather than a symplectic one.

In the following section we show how dissipation can be modeled using a symplectic structure, albeit a bicomplex one.

3. OUR PROPOSAL: STRICTLY SYMPLECTIC BICOMPLEX HAMILTONIAN SYSTEMS

In this section, we introduce what we call "bicomplex Hamiltonian systems". These will be seen to adhere strictly the three requirements ((i)-(iii)) for a Hamiltonian system, while incorporating any dissipation that may be present.

To this end, we write the dynamics of a bicomplex Hamiltonian systems as

$$\dot{\chi} = \mathcal{J}\nabla\mathcal{H}(\chi), \tag{9}$$

which is manifestly in strict Hamiltonian form. The state variable χ will take values in $\chi \in \mathbb{C}^2$, hence our choice of bicomplex. Section 3.1 shows how the bicomplex state χ relates to the usual state $x \in \mathbb{R}^2$. In Section 3.2 we introduce the symplectic structure matrix $\mathcal{J}$. Finally, in Section 3.3 we derive a complex Hamiltonian function $\mathcal{H}(\chi) : \mathbb{C}^2 \to \mathbb{C}$ on the bicomplex state space.

3.1 Bicomplex State-Space

Let the state $x = \begin{bmatrix} q & p \end{bmatrix}^\intercal$ be a point in the 2n-dimensional phase space spanned by the generalized positions $q = \begin{bmatrix} q_1, \dots, q_n \end{bmatrix}^\intercal$ and momenta $p = \begin{bmatrix} p_1, \dots, p_n \end{bmatrix}^\intercal$. Define complex states $\mathbf{x}_j$ and $\bar{\mathbf{x}}_j$ as follows: [1]

$$\mathbf{x}_j = \sqrt{\frac{I\omega}{2}} \left(q_j + \frac{i}{I\omega} p_j \right), \tag{10}$$

[1] This is inspired by the ladder operators developed by Paul Dirac for the quantum harmonic oscillator (Shankar, 1994, p 203).

and

$$\bar{\mathbf{x}}_j = \sqrt{\frac{I\omega}{2}}\left(q_j - \frac{i}{I\omega}p_j\right) \tag{11}$$

Here $\omega = \sqrt{1/IC}$ is the generalized natural frequency, I is the generalized inertance, and C is the generalized compliance. Some domain specific examples of inertances and compliances are listed in Table 1.

Table 1. Elements

Generalized	**Mechanical**	**Electrical**
Compliance, C	Spring, $1/k$	Capacitor, C
Inertance, I	Mass, m	Inductor, L
Resistance, r	Damper, b	Resistor, R

To define the bicomplex state χ, let $\mathbf{x} = \left[\mathbf{x}_1, \dots, \mathbf{x}_n\right]^\mathsf{T}$ and $\bar{\mathbf{x}} = \left[\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_n\right]^\mathsf{T}$ and set

$$\chi = \left[\mathbf{x}\ \bar{\mathbf{x}}\right]^\mathsf{T} \tag{12}$$

We can consider the bicomplex state as a conjugate pair, for instance by thinking of $\mathbf{x}$ as the generalized complex positions and $\bar{\mathbf{x}}$ as the conjugate generalized complex momenta. The bicomplex state is an element of $\mathbb{C}^{2n}$ and clearly even-dimensional, thus meeting requirement (i).

3.2 Bicomplex Symplectic Structure

Such bicomplex states are used in Quantum Mechanics where they are known as ladder operators (See e.g. (Shankar, 1994, p 203)).[2] In the quantum-mechanical literature, the bicomplex state space is endowed with a bicomplex Poisson bracket (see e.g. Novaes (2004))

$$\{\mathcal{U}, \mathcal{V}\} = -i\left[\frac{\partial \mathcal{U}}{\partial \mathbf{x}}\frac{\partial \mathcal{V}}{\partial \bar{\mathbf{x}}} - \frac{\partial \mathcal{V}}{\partial \mathbf{x}}\frac{\partial \mathcal{U}}{\partial \bar{\mathbf{x}}}\right] \tag{13}$$

Here $\mathcal{U}(\chi)$ and $\mathcal{V}(\chi)$ are holomorphic functions on the bicomplex state space.

From this Poisson bracket we can reverse-engineer the symplectic structure matrix $\mathcal{J}$ (see (Marsden and Ratiu, 2002, p. 65)). In general,

$$\{\mathcal{U}, \mathcal{V}\} = \begin{bmatrix}\frac{\partial \mathcal{U}}{\partial \mathbf{x}} & \frac{\partial \mathcal{U}}{\partial \bar{\mathbf{x}}}\end{bmatrix} \mathcal{J} \begin{bmatrix}\frac{\partial \mathcal{V}}{\partial \mathbf{x}} \\ \frac{\partial \mathcal{V}}{\partial \bar{\mathbf{x}}}\end{bmatrix}, \tag{14}$$

and comparing this with (13), we see that

$$\mathcal{J} = \begin{bmatrix}0 & -i \\ i & 0\end{bmatrix} \tag{15}$$

We can check that $\mathcal{J}$ is symplectic by taking the skew-symmetric

$$\Omega = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix},$$

and verifying that

$$\mathcal{J}^\dagger \Omega \mathcal{J} = \Omega, \tag{16}$$

and that $\det(\mathcal{J}) = -1$. Here $\mathcal{J}^\dagger$ is the conjugate transpose of $\mathcal{J}$. This completes requirement (ii).

[2] Interestingly, it was Paul Dirac who developed them, the namesake of the Dirac structures in pH systems.

3.3 Complex Hamiltonian

For a function $\mathcal{H}$ to be a Hamiltonian, it should give the time derivative of a function under the Poisson bracket. In particular, using the bracket (13) for the complex position $\mathbf{x}$ requires that

$$\dot{\mathbf{x}} = \{\mathbf{x}, \mathcal{H}\} = -i\frac{\partial \mathcal{H}}{\partial \bar{\mathbf{x}}} \tag{17}$$

Although the complex conjugate momentum $\bar{\mathbf{x}}$ is not holomorphic, it is nevertheless readily found by complex conjugating the above expression that

$$\dot{\bar{\mathbf{x}}} = \overline{\{\mathbf{x}, \mathcal{H}\}} = i\frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{x}} \tag{18}$$

If we collect $\dot{\mathbf{x}}$ and $\dot{\bar{\mathbf{x}}}$ into a single expression for $\dot{\chi}$ and express the bicomplex gradient as

$$\nabla \mathcal{H}(\chi) \triangleq \left[\frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{x}}\ \frac{\partial \mathcal{H}}{\partial \bar{\mathbf{x}}}\right]^\mathsf{T}, \tag{19}$$

we can summarize the dynamics in a strictly symplectic manner as $\dot{\chi} = \mathcal{J}\nabla\mathcal{H}(\chi)$. This reproduces (9) and completes our determination of a strictly symplectic bicomplex Hamiltonian System.

3.4 Casimirs

The complex Hamiltonian itself is finally found by integrating expression (19). However, this determines the Hamiltonian up to an integrating constant, so there will be freedom of choice in choosing a Hamiltonian.

To determine the nature of such an integrating constant, notice that evidently a function $\mathcal{C}$ that gives 0 under the Poisson bracket can be added to the Hamiltonian for an alternate Hamiltonian $\mathcal{K} = \mathcal{H} + \mathcal{C}$. This addition does not affect the motion since

$$\{\mathbf{x}, \mathcal{K}\} = \{\mathbf{x}, \mathcal{H}\} + \{\mathbf{x}, \mathcal{C}\} = \dot{\mathbf{x}} + 0 = \dot{\mathbf{x}} \tag{20}$$

Functions such that $\{\mathcal{U}, \mathcal{C}\} = 0$ for any function $\mathcal{U}$ are known as Casimir functions. This can be used to advantage in closed-loop systems as we show in Section 5 for bicomplex systems.

4. APPLICATION: THE DAMPED HARMONIC OSCILLATOR

In practice, the real-valued dynamics of a system in the A-matrix representation are known. In this section we show how this is used to determine the complex Hamiltonian (up to a Casimir) for the case of the damped harmonic oscillator (DHO).

Our strategy is to determine the bicomplex $\mathcal{A}$-matrix corresponding to the usual A-matrix and then to integrate $\mathcal{A}\chi = \mathcal{J}\nabla\mathcal{H}$.

4.1 DHO

Consider the system in Figure 2. We start with the familiar state-space representation on $\mathbb{R}^2$ spanned by the position (or charge) q and momentum (or flux linkage) p

$$\underbrace{\begin{bmatrix}\dot{q} \\ \dot{p}\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix}0 & 1/I \\ -1/C & -r/I\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}q \\ p\end{bmatrix}}_{x} \tag{21}$$

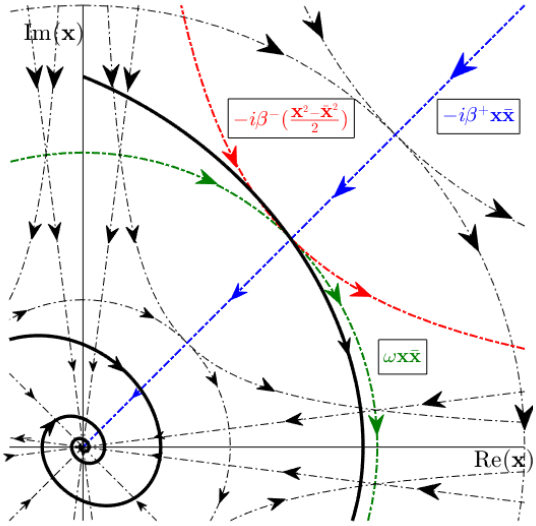

Fig. 1. The Bicomplex Hamiltonian flow consists of three parts: a rotation with orbits by $\omega\mathbf{x}\bar{\mathbf{x}}$ (green), a contraction with orbits $-i\beta^+\mathbf{x}\bar{\mathbf{x}}$ (blue), and a squeeze with orbits $-i\beta^-(\frac{\mathbf{x}^2-\bar{\mathbf{x}}^2}{2})$ (red). The solid line represents a composite flow. The direction of the squeeze flow reverses with the sign of β^-.

To convert this into a state-space representation on $\mathbb{C}^2$

$$\dot{\chi} = \mathcal{A}\chi, \tag{22}$$

invert the expressions for the complex states (10) and (11) to recover

$$q = \sqrt{\frac{1}{2I\omega}}(\bar{\mathbf{x}} + \mathbf{x}), \tag{23}$$

$$p = i\sqrt{\frac{I\omega}{2}}(\bar{\mathbf{x}} - \mathbf{x}), \tag{24}$$

substitute these into (21) and rearrange to obtain

$$\underbrace{\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\bar{\mathbf{x}}}\end{bmatrix}}_{\dot{\chi}} = \underbrace{\begin{bmatrix}-(\beta+i\omega) & \beta\\ \beta & -(\beta-i\omega)\end{bmatrix}}_{\mathcal{A}}\underbrace{\begin{bmatrix}\mathbf{x}\\ \bar{\mathbf{x}}\end{bmatrix}}_{\chi} \tag{25}$$

Here $\beta = r/(2I)$ is the damping coefficient and $\omega = \sqrt{1/(LC)}$ the natural frequency.

To express the dynamics in bicomplex pH form (9), we set

$$\mathcal{J}\nabla\mathcal{H} = \mathcal{A}\chi,$$

substitute for the $\mathcal{A}$-matrix in (25) and evaluate the left-hand side using the bicomplex symplectic matrix (15) and gradient (19), to obtain

$$-i\frac{\partial\mathcal{H}}{\partial\bar{\mathbf{x}}} = -(\beta + i\omega)\mathbf{x} + \beta\bar{\mathbf{x}}, \tag{26}$$

$$i\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{x}} = \beta\mathbf{x} - (\beta + i\omega)\bar{\mathbf{x}}, \tag{27}$$

conform the expressions for the Poisson brackets in (17) and (18).

Integrating, we find the following expression for the complex Hamiltonian

$$\mathcal{H} = \omega\mathbf{x}\bar{\mathbf{x}} - i\beta\mathbf{x}\bar{\mathbf{x}} - i\beta(\frac{\mathbf{x}^2 - \bar{\mathbf{x}}^2}{2}) \tag{28}$$

Its validity is readily checked by substituting into (26) and (27).

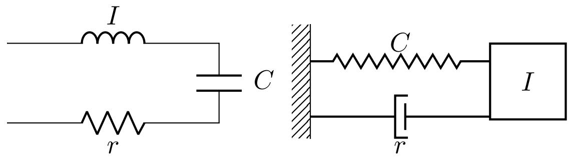

Fig. 2. Damped harmonic oscillator with series configuration. The symbols are listed in Table 1. This system can be modeled as a Hamiltonian system with complex Hamiltonian in (28)

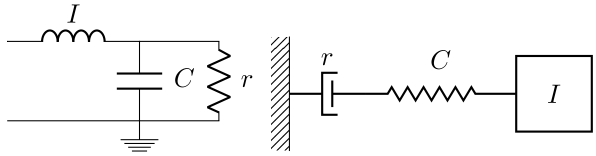

Fig. 3. Damped harmonic oscillator with parallel configuration. The symbols are defined in Table 1. This system can be modeled as a Hamiltonian system with complex Hamiltonian in (29)

4.2 Resistive Configurations

For the system in Figure 3 with the damper in series with the spring (or resistor in parallel with the capacitor), we can follow the same procedure as in Section 4.1 to find:

$$\mathcal{H} = \omega\mathbf{x}\bar{\mathbf{x}} - i\beta\mathbf{x}\bar{\mathbf{x}} + i\beta(\frac{\mathbf{x}^2 - \bar{\mathbf{x}}^2}{2}) \tag{29}$$

For this case the damping coefficient $\beta = 1/(2rC)$. A comparison of this Hamiltonian with (28) shows that the only modification is the sign of the rightmost term. Physically, this comes about because the damping is now proportional to the position, instead of to the momentum.

In general, we can consider systems with two resistive elements as combinations of these two extremes. Combining the Hamiltonians of these extremes, (28) and (29), we obtain the general complex Hamiltonian of a DHO

$$\mathcal{H} = \underbrace{\overset{\text{rotation}}{\omega\mathbf{x}\bar{\mathbf{x}}}}_{\text{Storage Function } \mathcal{S}} - \underbrace{\left(\overset{\text{contraction}}{i\beta^+\mathbf{x}\bar{\mathbf{x}}} + \overset{\text{squeeze}}{i\beta^-\left(\frac{\mathbf{x}^2 - \bar{\mathbf{x}}^2}{2}\right)}\right)}_{\text{Resistive Function } \mathcal{R}} \tag{30}$$

Here $\beta^+ = (\beta_s + \beta_p)$ and $\beta^- = (\beta_s - \beta_p)$ for $\beta_s = r/(2I)$ and $\beta_p = 1/(2rC)$. This Hamiltonian is valid for the simple DHO with any resistive configuration.

4.3 Geometry of the Complex Hamiltonian Flow

In equation (30) we indicate how the complex Hamiltonian of the DHO can be decomposed into the sum of two functions: an energy storage function $\mathcal{S}$ and a resistive function $\mathcal{R}$. Figure 1 shows the Hamiltonian flows corresponding to these two motions.

Consider first the storage function. In the absence of damping when $\beta^+ = \beta^- = 0$, the complex Hamiltonian reverts to the regular undamped Hamiltonian, since

$$\omega \mathbf{x}\bar{\mathbf{x}} = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kq^2, \tag{31}$$

after substituting the expressions for the complex states. This corresponds to the energy stored in the C and I elements of the system, hence the choice of storage function for this function. In the phase plane, the level lines of $\mathcal{S}$ appear as circles, shown in green in Figure 1. These circles are precisely the orbits of the rotation group acting on the phase plane giving the familiar Hamiltonian flow of the harmonic oscillator indicated by the green arrow heads. This flow is symplectic even on $\mathbb{R}^2$.

The resistive function can be viewed as the sum of two parts: one that appears as a contraction in the phase plane (in blue) and one that appears as a squeeze (in red). Physically, the contraction models the average dissipation over a cycle, whereas the squeeze provides the dynamics of the damping within a cycle. Analyzing the general Hamiltonian shows us that systems with different resistive configurations all share the direction of contraction, but differ in the squeeze. In fact, by choosing $\beta_s = \beta_p$, the squeeze disappears altogether and the system follows a smooth hyperbolic spiral down to the equilibrium state.

The orbits of the squeeze map are shown as hyperbolae and these are in fact symplectic on the real phase plane. The contraction however compresses the phase volume and therefore cannot be symplectic in $\mathbb{R}^2$. This underscores the necessity of using the bicomplex phase space $\mathbb{C}^2$ to model the dissipation in a symplectic fashion.

5. BICOMPLEX PORT-HAMILTONIAN SYSTEMS

In this section we employ the bicomplex Hamiltonian formalism in a port-based approach. We first connect our bicomplex system to real inputs and outputs, then we establish the passivity of the system, and finally control any DHO by using Control-by-Interconnection (CbI), including those that otherwise suffer from the dissipation obstacle.

5.1 Input-State-Output Description

To connect the bicomplex internal-system dynamics to real input and output signals $u, y \in \mathbb{R}$, we use a bicomplex-valued matrix $\mathcal{G} : \mathbb{R} \to \mathbb{C}^2$. This leads to the following input-state-output form:

$$\begin{aligned} \dot{\chi} &= \mathcal{J}\nabla\mathcal{H}(\chi) + \mathcal{G}u \\ y &= \mathcal{G}^\intercal \nabla\mathcal{H}(\chi) \end{aligned} \tag{32}$$

We will refer to a system of this form as a "bicomplex port-Hamiltonian system". Comparing it to the expression (4) for its real counterpart, one sees that the bicomplex approach obviates the need for the dissipation matrix R. Otherwise the bicomplex system appears as a complexified version of the real system.

In port-based approaches the input and output form a power-conjugate pair, consisting of an effort (force or voltage) and a flow (velocity or current). There are two possible choices for $\mathcal{G}$ matrix, either the structure matrix for an effort input:

$$\mathcal{G}_e = \left[i\sqrt{\frac{1}{2I\omega}} \;\; -i\sqrt{\frac{1}{2I\omega}} \right]^\intercal, \tag{33}$$

or the structure matrix for a flow input:

$$\mathcal{G}_f = \left[\sqrt{\frac{I\omega}{2}} \;\; \sqrt{\frac{I\omega}{2}} \right]^\intercal \tag{34}$$

5.2 Passivity and Stability

In order to check whether a bicomplex pH system can be stabilized around an equilibrium point, we must show that the system is passive, i.e., that the rate of change of the stored energy $\mathcal{S}$ at an equilibrium does not exceed the power supplied to the system:

$$\frac{\mathrm{d}\mathcal{S}(\chi)}{\mathrm{d}t} \leq y^\intercal u$$

To check this, we evaluate

$$\frac{\mathrm{d}\mathcal{S}(\chi)}{\mathrm{d}t} = \nabla^\intercal \mathcal{S}\dot{\chi} = -\underbrace{\nabla^\intercal \mathcal{S}\mathcal{J}\nabla\mathcal{R}}_{\delta(\chi)} + \underbrace{\nabla^\intercal \mathcal{R}\mathcal{G}}_{\phi^\intercal(\chi)} u + y^\intercal u \tag{35}$$

For the second equality we have substituted (32) and the fact that $\nabla^\intercal \mathcal{S}\mathcal{J}\nabla\mathcal{S} = 0$.

This expresses the rate of change of the stored energy as the sum of three real-valued terms: the dissipation rate $\delta(\chi)$, the power drawn by the feedthrough signal $\phi^\intercal(\chi)$ and the power $y^\intercal u$ supplied at the port. Using the stored energy and resistive functions of the DHO (30), one checks that the dissipation rate is positive and the feedthrough power $\phi^\intercal u$ non-positive. It follows that the DHO is passive.

In fact, one can check that the value of the bicomplex dissipation rate $\delta(\chi)$ equals the dissipation rate $d(x)$ of the real pH systems theory:

$$\delta(\chi) = \nabla^\intercal \mathcal{S}\mathcal{J}\nabla\mathcal{R} = \nabla H(x)^\intercal R \nabla H(x) = d(x)$$

Notice, however, that $\delta(\chi)$ determines this value as the area spanned by the vector gradients of the storage and resistive functions consistent with the symplectic structure, whereas the $d(x)$ determines it as a length of the vector gradient of the Hamiltonian consistent with the metric structure implied by the R matrix. This distinction is essential to overcome the dissipation obstacle.

5.3 Overcoming the Dissipation Obstacle

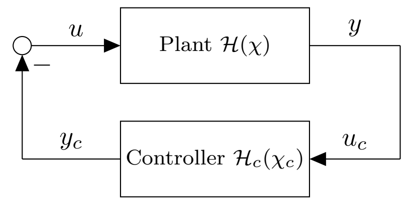

Fig. 4. Control-by-Interconnection scheme of a plant and controller in a negative feedback loop. The controller is another bicomplex Hamiltonian system with symplectic structure $\mathcal{J}$ in (15) and complex Hamiltonian $\mathcal{H}_c(\chi_c)$. The closed-loop system is also a bicomplex Hamiltonian system with the interconnection matrix $\mathcal{J}_{cl} \in \mathbb{C}^4$ as symplectic structure and the closed-loop Hamiltonian $\mathcal{H}_{cl} = \mathcal{H} + \mathcal{H}_c : \mathbb{C}^4 \to \mathbb{C}$. The closed-loop system can be stabilized at a non-trivial equilibrium by generating Casimirs of the closed-loop Hamiltonian, without suffering from the dissipation obstacle.

We now show how the dissipation obstacle can be overcome for a general DHO. This includes the RLC circuit in Figure

3 with voltage input that is known to suffer from it (See e.g. Ortega et al. (2001)).

The objective is to stabilize this system at a non-trivial desired equilibrium point using control by interconnection; see Figure 4. The input-state-output description of the DHO plant with effort input is as in (32) with $\mathcal{G}_e$ to accommodate the effort input and $\mathcal{H}$ as in (30). The controller is also as in (32), but with $\mathcal{G}_f$ to accommodate the output flow as input to the controller. We pick a DHO controller, so $\mathcal{H}_c$ is also as in (30).

Interconnecting the plant and the controller in the negative feedback loop of Figure 4 yields the closed-loop system

$$\underbrace{\begin{bmatrix} \dot{\chi} \\ \dot{\chi}_c \end{bmatrix}}_{\dot{\chi}_{cl}} = \underbrace{\begin{bmatrix} \mathcal{J} & -\mathcal{G}_e\mathcal{G}_f^{\intercal} \\ \mathcal{G}_f\mathcal{G}_e^{\intercal} & \mathcal{J} \end{bmatrix}}_{\mathcal{J}_{cl}} \underbrace{\begin{bmatrix} \nabla\mathcal{H}(\chi) \\ \nabla\mathcal{H}_c(\chi_c) \end{bmatrix}}_{\nabla\mathcal{H}_{cl}} \tag{36}$$

This closed-loop system is a 4-complex pH system $\dot{\chi}_{cl} = \mathcal{J}_{cl}\nabla\mathcal{H}_{cl}$ with state $\chi_{cl} \in \mathbb{C}^4$ and Hamiltonian that is the sum of the plant and controller Hamiltonian

$$\mathcal{H}_{cl}(\chi, \chi_c) = \mathcal{H}(\chi) + \mathcal{H}_c(\chi_c)$$

The structure matrix is indeed symplectic, i.e.,

$$\mathcal{J}_{cl}^{\dagger}\Omega\mathcal{J}_{cl} = \Omega$$

with Ω a skew-symmetric matrix and $\det(\mathcal{J}_{cl}) = \pm 1$.

The system in (36) has its equilibrium point at the origin. To stabilize it around an equilibrium point χ^*, we simply choose a complex Casimir $\mathcal{C}(\chi, \chi_c)$ in the manner outlined in Section 3.4 to obtain a new Hamiltonian

$$\mathcal{V}(\chi, \chi_c) = \mathcal{H}_{cl}(\chi, \chi_c) + \mathcal{C}(\chi, \chi_c) \tag{37}$$

This new Hamiltonian $\mathcal{V}$ takes the place of the Lyapunov function in the real pH theory. Notice how the complex theory gives a natural interpretation for the Lyapunov function in terms of energy and arises naturally due to the freedom of choice in a Casimir for the complex Hamiltonian. The existence of such a Lyapunov function is guaranteed for the DHO.

An obvious candidate Lyapunov function that achieves equilibrium at χ^* is the "shifted" Hamiltonian

$$\mathcal{V}(\chi, \chi_c) = \mathcal{H}(\chi - \chi^*) + \mathcal{H}_c(\chi_c - \chi_c^*), \tag{38}$$

that leads to a Casimir function

$$\mathcal{C}(\chi, \chi_c) = \mathcal{H}(\chi - \chi^*) - \mathcal{H}(\chi) + \mathcal{H}_c(\chi_c - \chi_c^*) - \mathcal{H}_c(\chi_c) \tag{39}$$

Substituting the expression for the $\mathcal{H}$ Hamiltonians of the DHO plant and controller gives the explicit expression.

There is no further need to verify that Casimir found this way is indeed a Casimir, since it is one by construction. However, it is perhaps instructive to make an explicit complex comparison with the real requirement as in (7). Expanding the complex Poisson bracket in terms of the symplectic structure matrix, the requirement that the bracket vanishes can be written as

$$\begin{bmatrix} \nabla^{\intercal}\mathcal{C}(\chi) & \nabla^{\intercal}\mathcal{C}(\chi_c) \end{bmatrix} \begin{bmatrix} \mathcal{J} & -\mathcal{G}_e\mathcal{G}_f^{\intercal} \\ \mathcal{G}_f\mathcal{G}_e^{\intercal} & \mathcal{J} \end{bmatrix} = 0 \tag{40}$$

which is the complex analog to (7). One can substitute (39) in this equality to check.

To complete the specification of the equilibrium, we find the point χ_c^* of the controller by setting $\dot{\chi}_c = 0$ in (36) to obtain

$$\chi_c^* = -(\mathcal{J}\mathbf{H}_c)^{-1}\mathcal{G}_f\mathcal{G}_e^{\intercal}\nabla\mathcal{H}(\chi^*) \tag{41}$$

Here $\mathbf{H}_c = \chi_c^{-1}\nabla\mathcal{H}_c$ and $\mathcal{J}\mathbf{H}_c$ is invertible for every $\omega_c > 0$.

The equilibrium point (χ^*, χ_c^*) is stable since the plant and the controller are both passive as was shown for the general DHO in Section 5.2: both $\mathrm{d}\mathcal{S}(\chi)/\,\mathrm{d}t \leq y^{\intercal}u$ and $\mathrm{d}\mathcal{S}_c(\chi_c)/\,\mathrm{d}t \leq y_c^{\intercal}u_c$.

We see that bicomplex pH systems theory allows us to control any DHO using the control-by-interconnection method without being impeded by the dissipation obstacle. The resulting closed-loop systems will be guaranteed to be stable. If needed, the controller can be further tuned by adjusting its natural frequency ω_c to achieve some desired speed and its damping coefficient β_c achieve some desired damping rate.

6. CONCLUSIONS

In the preceding section we have shown how the dissipation obstacle of pH systems theory can be overcome for any DHO using bicomplex pH systems. We have shown how this can be done constructively in a CbI context, allowing one to systematically identify the requisite Lyapunov and Casimir functions. A fortunate by-product is that these obtain a direct physical meaning in terms of energy and that the mathematics is much simplified, at least on a formal level.

Although the bicomplex theory in the paper is quite general, the CbI applications are limited to linear time-invariant damped harmonic oscillators. We hope to explore the opportunities to expand the bicomplex theory to a broader class of systems.

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