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A Bias-Correction Approach for the Identification of Piecewise Affine Output-Error Models ⋆

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Abstract: The paper presents an algorithm for the identification of *PieceWise Affine Output-Error* (PWA-OE) models, which involves the estimation of the parameters defining affine submodels as well as a partition of the regressor space. For the estimation of affine submodel parameters, a bias-correction scheme is presented to correct the bias in the least squares estimates which is caused by the output-error noise structure. The obtained bias-corrected estimates are proven to be consistent under suitable assumptions. The bias-correction method is then combined with a recursive estimation algorithm for clustering the regressors. These clusters are used to compute a partition of the regressor space by employing linear multi-category discrimination. The effectiveness of the proposed methodology is demonstrated via a simulation case study.

Keywords: Hybrid systems, PWA regression, bias corrected least-squares, output-error models

1. INTRODUCTION

PieceWise Affine (PWA) models can be used to describe the behavior of hybrid dynamical systems, which are governed by both continuous and discrete states. One of the main advantage of the PWA modelling paradigm is that tools developed for analysis and control of hybrid systems can also be applied to systems represented in a PWA form (Bemporad and Morari, 1999).

PWA models are defined by a set of affine functions, each associated to a polyhedral region of the regressor space. The problem of identifying these models consists of estimating the parameters defining the affine sub-models as well as a partition of the regressor space The main methods proposed in the literature for learning PWA models from data, include, among others, bounded-error or set-membership approaches (Bemporad et al., 2005); mixed-integer programming methods (Roll et al., 2004; Naik et al., 2017); clustering-based two-stage approaches (Nakada et al., 2005; Juloski et al., 2005; Ferrari-Trecate et al., 2003; Bemporad et al., 2018; Breschi et al., 2016).

The underlying assumption in these methods is that the output is generated by *AutoRegressive with eXogenous inputs* (ARX) submodels. With this assumption, standard linear *Least Squares* (LS) can be used to obtain consistent estimates of the submodel parameters, given a partition. In this paper, we relax this assumption and develop an identification algorithm for PWA *output-error* (PWA-OE) models. PWA-OE models make the identification problem more challenging, as the affine functions of the PWA map depend on the (unmeasured) "noise-free" regressors.

Very few works in the literature have addressed the identification of PWA-OE models. In (Canty et al., 2012), PWA-OE algorithm is proposed with an *Instrumental Variable* (IV) scheme. The correct choice of IVs is critical as they need to be uncorrelated with the system noise and correlated with the inputs. A *Prediction Error Method* (PEM) is proposed in (Rosenqvist and Karlström, 2005) for identifying piecewise linear models under the assumption that a partition is known. In (Breschi et al., 2019), PEM is used to estimate more general Box-Jenkins models, along with the estimation of the hidden mode sequence, indicating the active sub-model at each time instant. However, the PEM approach leads to a non-convex optimization problem which requires an accurate initial guess. In (Juloski and Weiland, 2006), PWA-OE identification problem is treated in a Bayesian framework, and a suboptimal algorithm is derived for estimating the model parameters along with the unknown mode sequence.

In this work, we propose a *bias-correction* approach, combined with a two-stage clustering based algorithm (Breschi et al., 2016), for the identification of PWA-OE models. Specifically, the bias-corrected least squares estimates of the affine submodel parameters

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are derived and are proven to be consistent under the assumption that the active mode sequence is known. This assumption is then relaxed and mode sequence is estimated from data using an iterative clustering-based algorithm. Bias-correction methods have been proposed in the literature for the identification of LTI systems (Hong et al., 2007), LPV models (Piga et al., 2015; Mejari et al., 2018) and for non-linear systems (Piga and Tóth, 2014). To the best of our knowledge, this paper presents the first contribution adapting the bias-correction method in a hybrid modelling framework.

The paper is organized as follows. The identification problem of PWA-OE models is formalized in Section 2. Bias-corrected least-squares estimates for the affine submodels parameters are derived in Section 3. Section 4 presents the two-stage clustering-based iterative algorithm for PWA-OE model identification. A numerical case study is reported in Section 5 to show the effectiveness of the presented algorithm. Finally, conclusions are given in Section 6.

2. PROBLEM FORMULATION

Consider the following discrete-time, single-input single-output nonlinear system $\mathcal{S}_{\rm o}$, with *Output-Error* (OE) structure,

$$y_{\rm o}(k) = f(x_{\rm o}(k)), \tag{1a}$$
$$y(k) = y_{\rm o}(k) + e_{\rm o}(k), \tag{1b}$$

where $y_{\rm o}(k) \in \mathbb{R}$ and $y(k) \in \mathbb{R}$ are the *noise-free* and noise-corrupted output of the system at time k respectively, $x_{\rm o}(k)$ denotes the *noise-free regressor*, namely,

$$x_{\rm o}(k) = [y_{\rm o}(k-1) \cdots y_{\rm o}(k-n_a)\ u(k-1) \cdots u(k-n_b)]^\top, \tag{1c}$$

where n_a, n_b are the parameters defining the model order, with $x_{\rm o}(k) \in \mathcal{X}$ and $e_{\rm o}(k) \in \mathbb{R}$ is a zero-mean additive white Gaussain noise with variance σ_e^2, statistically independent of the regressor $x_{\rm o}(k)$.

The map $f : \mathcal{X} \to \mathbb{R}$ is *PieceWise Affine* (PWA), namely,

$$f(x_{\rm o}) = \begin{cases} (\theta_1^{\rm o})^\top \begin{bmatrix} x_{\rm o} \\ 1 \end{bmatrix} & \text{if } x_{\rm o} \in \mathcal{X}_1, \\ \vdots & \\ (\theta_s^{\rm o})^\top \begin{bmatrix} x_{\rm o} \\ 1 \end{bmatrix} & \text{if } x_{\rm o} \in \mathcal{X}_s, \end{cases} \tag{1d}$$

where $s \in \mathbb{N}$ is the number of *modes*, and $\theta_i^{\rm o} \in \mathbb{R}^{(n_a+n_b+1)\times 1}$ is the parameter vector associated to the i-th affine submodel, and the set $\mathcal{X}_i \subseteq \mathcal{X}$ is a polyhedron defined as:

$$\mathcal{X}_i \doteq \{x_{\rm o} \in \mathbb{R}^{n_x} :\ \mathcal{H}_i x_{\rm o} \leq \mathcal{D}_i\}, \tag{1e}$$

with $\mathcal{H}_i$ and $\mathcal{D}_i$ being real matrices, for $i = 1, \ldots, s$, and $\{\mathcal{X}_i\}_{i=1}^s$ form a *complete* polyhedral partition [1] of the regressor space $\mathcal{X}$. Note that, unlike PWA-ARX models, the polyhedral partition $\{\mathcal{X}_i\}_{i=1}^s$ is defined over the space of *noise-free* regressors. As $e_{\rm o}$ does not affect the evolution of the output $y_{\rm o}$ and the function f is piecewise affine, $\mathcal{S}_{\rm o}$ in (1) is a *PieceWise Affine-Output Error* (PWA-OE) representation. Note that, the noise-free output $y_{\rm o}$ and consequently the noise-free regressor $x_{\rm o}$ are *not* available for measurements.

In order to describe the true PWA-OE system $\mathcal{S}_{\rm o}$ in (1), the following parameterized model structure $\mathcal{M}_\theta$ is introduced,

$$y(k) = \begin{cases} \theta_1^\top \begin{bmatrix} x(k) \\ 1 \end{bmatrix} + \epsilon(k), \text{ if } x(k) \in \mathcal{X}_1, \\ \vdots \\ \theta_s^\top \begin{bmatrix} x(k) \\ 1 \end{bmatrix} + \epsilon(k), \text{ if } x(k) \in \mathcal{X}_s, \end{cases} \tag{2a}$$

where $\epsilon(k)$ is the residual term (not necessarily white) modeling the mismatch between the true system and model output, $x(k)$ is the measured regressor vector, *i.e.*,

$$x(k) = [y(k-1) \cdots y(k-n_a)\ u(k-1) \cdots u(k-n_b)]^\top. \tag{2b}$$

We remark that, the problem of model structure selection (*i.e.*, estimation of the model orders n_a, n_b and the number of modes s) is not addressed in this paper. Thus, we assume that both the true system $\mathcal{S}_{\rm o}$ and the model $\mathcal{M}_\theta$ share the same parameters n_a, n_b and s, which are assumed to be known.

The identification problem addressed in this paper is formalized as follows,

Problem 1. Given a set of N input-output observations $\{u(k), y(k)\}_{k=1}^N$, generated by $\mathcal{S}_{\rm o}$ in (1), compute *consistent* estimates of the true parameters $\{\theta_i^{\rm o}\}_{i=1}^s$ characterizing the affine submodels of the PWA map f, and find a polyhedral partition $\{\mathcal{X}_i\}_{i=1}^s$ of the regressor space $\mathcal{X}$.

To this aim, a novel identification algorithm based on *bias-corrected* least squares is presented in the next sections.

3. BIAS-CORRECTED LEAST SQUARES

For an output-error model structure, ordinary LS give an asymptotically biased estimate of the model parameters (Ljung, 1999). To overcome this issue, we quantify the bias in the LS estimates and show how to eliminate it, in order to compute a *bias-corrected* estimate of the model parameters $\{\theta_i\}_{i=1}^s$. For now, we assume the true polyhedral partition of the regressor space $\mathcal{X}$ is known. Under this assumption, we prove that the bias-corrected least square estimates are *consistent*.

Let us define the mode function $\sigma : \mathbb{N} \to \mathbb{N}$, such that,

$$\sigma(k) = i \quad \Leftrightarrow \quad x_{\rm o}(k) \in \mathcal{X}_i,$$

i.e., the *active* mode $\sigma(k) \in \{1, \cdots s\}$, at time k, represents the partition which the regressor $x_{\rm o}(k)$ belongs to. In this section, we assume that the sequence of active modes $\{\sigma(k)\}_{k=1}^N$ is known. This assumption is relaxed later.

Let N_i be the number of regressor/ouput data points associated to the i-th affine submodel, with $\sum_{i=1}^s N_i = N$. Given the mode sequence $\{\sigma(k)\}_{k=1}^N$, we define $\mathbb{Y}_i \in \mathbb{R}^{N_i}$ as the output vector associated to the i-th affine submodel, which is constructed from the output sequence $\{y(k)\}_{k=1}^N$ such that,

$$y(k) \text{ is a row of } \mathbb{Y}_i \Leftrightarrow \sigma(k) = i, \tag{3}$$

and, analogously, let $\mathbb{X}_i \in \mathbb{R}^{N_i \times (n_a+n_b+1)}$ be the regressor matrix constructed from the sequence $\{x(k)\}_{k=1}^N$ by

[1] The collection $\{\mathcal{X}_i\}_{i=1}^s$ is a complete partition of $\mathcal{X}$ if $\bigcup_{i=1}^s \mathcal{X}_i = \mathcal{X}$ and $\mathring{\mathcal{X}}_i \cap \mathring{\mathcal{X}}_j = \emptyset$, $\forall i \neq j$, with $\mathring{\mathcal{X}}_i$ denoting the interior of $\mathcal{X}_i$.

stacking the extended regressors associated to the i-th affine submodel, *i.e.*,

$$\begin{bmatrix} x(k) \\ 1 \end{bmatrix}^\top \text{ is a row of } \mathbb{X}_i \Leftrightarrow \sigma(k) = i. \tag{4}$$

Using the notation introduced above, we define the noise-free output vector $\mathbb{Y}_{i,\mathrm{o}} \in \mathbb{R}^{N_i}$, the noise-free regressor matrix $\mathbb{X}_{i,\mathrm{o}} \in \mathbb{R}^{N_i \times (n_a+n_b+1)}$ and the measurement noise vector $\mathbb{E}_{i,\mathrm{o}} \in \mathbb{R}^{N_i}$, by stacking the noise-free outputs, noise-free extended regressors and the samples of the measurement noise e_o associated to the i-th mode, respectively. Note that, $\mathbb{Y}_{i,\mathrm{o}}$, $\mathbb{X}_{i,\mathrm{o}}$ and $\mathbb{E}_{i,\mathrm{o}}$ cannot be constructed in practice, since the sequence of noise-free output/regressor pairs $\{y_\mathrm{o}(k), x_\mathrm{o}(k)\}_{k=1}^N$ and the sequence of measurement noise $\{e_\mathrm{o}(k)\}_{k=1}^N$ are *not* available.

3.1 Computation of the bias in the least squares estimate

Consider the least squares estimate of the i-th affine submodel parameter, which is given by,

$$\theta_i^{\mathrm{LS}} = \underbrace{\left(\frac{\mathbb{X}_i^\top \mathbb{X}_i}{N_i}\right)^{-1}}_{\Gamma_{N_i}} \frac{\mathbb{X}_i^\top \mathbb{Y}_i}{N_i}, \tag{5}$$

where Γ_{N_i} is assumed to be invertible. In order to quantify the bias in the least squares estimate, we compute the difference between the LS estimate θ_i^{LS} and the true model parameter θ_i^o. To this aim, let us rewrite the output vector $\mathbb{Y}_i$ using (1a), (1b) and (1d) as follows:

$$\begin{aligned} \mathbb{Y}_i &= \mathbb{Y}_{i,\mathrm{o}} + \mathbb{E}_{i,\mathrm{o}} = \mathbb{X}_{i,\mathrm{o}}\theta_i^\mathrm{o} + \mathbb{E}_{i,\mathrm{o}}, \\ &= \mathbb{X}_i\theta_i^\mathrm{o} + \Delta\mathbb{X}_i\theta_i^\mathrm{o} + \mathbb{E}_{i,\mathrm{o}}, \end{aligned} \tag{6}$$

where $\Delta\mathbb{X}_i$ is defined as

$$\Delta\mathbb{X}_i = \mathbb{X}_{i,\mathrm{o}} - \mathbb{X}_i. \tag{7}$$

By substituting (6) in (5), the difference between the LS estimate θ_i^{LS} and the true parameter θ_i^o is expressed as follows:

$$\theta_i^{\mathrm{LS}} - \theta_i^\mathrm{o} = \underbrace{\left(\frac{\mathbb{X}_i^\top \mathbb{X}_i}{N_i}\right)^{-1} \frac{\mathbb{X}_i^\top \Delta\mathbb{X}_i}{N_i}\theta_i^\mathrm{o}}_{B_\Delta(\theta_i^\mathrm{o}, \mathbb{X}_i, \Delta\mathbb{X}_i)} + \underbrace{\left(\frac{\mathbb{X}_i^\top \mathbb{X}_i}{N_i}\right)^{-1} \frac{\mathbb{X}_i^\top \mathbb{E}_{i,\mathrm{o}}}{N_i}}_{B_{e_\mathrm{o}}}. \tag{8}$$

As the measurement noise e_o is assumed to be zero-mean white noise statistically independent of u, and since regressor $x(k)$ depends only on *past* outputs, the noise vector $\mathbb{E}_{i,\mathrm{o}}$ is uncorrelated with the regressors $\mathbb{X}_i$. Thus, the term B_{e_o} in (8) asymptotically (as $N_i \to \infty$) converges to 0 with probability 1 (w.p. 1), *i.e.*,

$$\lim_{N_i\to\infty} \left(\frac{\mathbb{X}_i^\top \mathbb{X}_i}{N_i}\right)^{-1} \frac{\mathbb{X}_i^\top \mathbb{E}_{i,\mathrm{o}}}{N_i} = 0 \quad \text{w.p. 1.}$$

However, the term $B_\Delta(\theta_i^\mathrm{o}, \mathbb{X}_i, \Delta\mathbb{X}_i)$ in (8) is not guaranteed to converge to 0, inducing a non-zero bias in the LS estimate. Thus, the LS estimate θ_i^{LS} in (5) is not consistent, *i.e.*,

$$\lim_{N_i\to\infty} \theta_i^{\mathrm{LS}} \neq \theta_i^\mathrm{o},$$

even for the known *true* mode sequence.

3.2 Elimination of the bias in the least squares estimate

Since the bias term $B_\Delta(\theta_i^\mathrm{o}, \mathbb{X}_i, \Delta\mathbb{X}_i)$ depends on the true model parameter θ_i^o as well as on the noise-free regressor matrix $\mathbb{X}_{i,\mathrm{o}}$ (see (7)), it can not be computed based on the observed input/output data. Therefore, the bias cannot be simply eliminated from the LS estimate θ_i^{LS}.

Following the same rationale used in (Piga and Tóth, 2014), we define the corrected LS estimate θ_i^{CLS} as follows:

$$\theta_i^{\mathrm{CLS}} = \theta_i^{\mathrm{LS}} - B_\Delta(\theta_i^{\mathrm{CLS}}, \mathbb{X}_i, \Delta\mathbb{X}_i), \tag{9}$$

where,

$$B_\Delta(\theta_i^{\mathrm{CLS}}, \mathbb{X}_i, \Delta\mathbb{X}_i) = \left(\frac{\mathbb{X}_i^\top \mathbb{X}_i}{N_i}\right)^{-1} \frac{\mathbb{X}_i^\top \Delta\mathbb{X}_i}{N_i}\theta_i^{\mathrm{CLS}} \tag{10}$$

The main idea behind the definition of θ_i^{CLS} in (9) is that, the LS estimate is corrected by eliminating the bias term B_Δ, which is evaluated at the estimate θ_i^{CLS}, rather than at the *unknown* true model parameter θ_i^o. By substituting equations (5) and (10) in (9) and with simple algebraic manipulations, we obtain

$$\theta_i^{\mathrm{CLS}} = \left(\frac{\mathbb{X}_i^\top \mathbb{X}_i + \mathbb{X}_i^\top \Delta\mathbb{X}_i}{N_i}\right)^{-1} \frac{\mathbb{X}_i^\top \mathbb{Y}_i}{N_i}. \tag{11}$$

Proposition 1. Under the assumption that the following limit exists:

$$\lim_{N_i\to\infty} \left(\frac{\mathbb{X}_i^\top \mathbb{X}_i + \mathbb{X}_i^\top \Delta\mathbb{X}_i}{N_i}\right)^{-1},$$

the corrected LS estimate θ_i^{CLS} in (11) is a *consistent* estimate of the true model parameter θ_i^o, *i.e.*,

$$\lim_{N_i\to\infty} \theta_i^{\mathrm{CLS}} = \theta_i^\mathrm{o}, \quad \text{w.p. 1}$$

See (Mejari et al., 2020, Proposition 1) for the proof.

Since $\Delta\mathbb{X}_i$ depends on the noise-free regressors $\mathbb{X}_{i,\mathrm{o}}$ (see (7)), the estimates θ_i^{CLS} in (11) cannot still be computed based on the available measurements. To overcome this issue, the term $\mathbb{X}_i^\top \Delta\mathbb{X}_i$ in (11) is replaced by a *bias-eliminating matrix* Ψ_i, which is constructed using the available information so to satisfy the following property:

$$\mathbf{C1}: \lim_{N_i\to\infty} \frac{1}{N_i}\mathbb{X}_i^\top \Delta\mathbb{X}_i = \lim_{N_i\to\infty} \frac{1}{N_i}\Psi_i \quad \text{w.p. 1.}$$

Such a bias-eliminating matrix Ψ_i can be obtained (following similar ideas used in (Mejari et al., 2018)) by evaluating the expected value of the matrix $E\left\{\mathbb{X}_i^\top \Delta\mathbb{X}_i\right\}$, such that $E\left\{\Psi_i\right\} = E\left\{\mathbb{X}_i^\top \Delta\mathbb{X}_i\right\}$ is satisfied by construction.

Proposition 2. The bias-eliminating matrix Ψ_i is given by,

$$\Psi_i = -\sigma_e^2 N_i \left[\begin{array}{c|c} I_{n_a\times n_a} & \mathbf{0}_{n_a\times(n_b+1),} \\ \hline \mathbf{0}_{(n_b+1)\times n_a} & \mathbf{0}_{(n_b+1)\times(n_b+1),} \end{array}\right] \tag{12}$$

where σ_e^2 is the variance of e_o and N_i is the number of training data points associated with the i-th mode. The matrix Ψ_i in (12) satisfies condition **C1**.

See (Mejari et al., 2020, Proposition 2) for the proof.

Note that, the bias-eliminating matrix in (12) depends on the noise variance σ_e^2 which is assumed to be known.

3.3 Bias-corrected estimate

The bias-corrected LS estimate is obtained by replacing $\mathbb{X}_i^\top \Delta\mathbb{X}_i$ in (11) with the bias-eliminating matrix Ψ_i (eq. (12)), *i.e.*,

$$\theta_i^{\mathrm{BC}} = \left(\frac{\mathbb{X}_i^\top \mathbb{X}_i + \Psi_i}{N_i}\right)^{-1} \frac{\mathbb{X}_i^\top \mathbb{Y}_i}{N_i}. \tag{13}$$

Proposition 3. Assume that the following limit exists:

$$\lim_{N_i \to \infty} \left(\frac{\mathbb{X}_i^\top \mathbb{X}_i + \Psi_i}{N_i}\right)^{-1},$$

the bias-corrected estimate θ_i^{BC} in (13) is a *consistent* estimate of the true model parameter θ_i^{o}, *i.e.*,

$$\lim_{N_i \to \infty} \theta_i^{\mathrm{BC}} = \theta_i^{\mathrm{o}}, \quad \text{w.p. } 1$$

See (Mejari et al., 2020, Proposition 3) for the proof.

4. PWA-OE IDENTIFICATION ALGORITHM

In the previous section, consistent bias-corrected estimates of the affine submodel parameters have been derived under the assumption that the underlying discrete mode sequence is known. In this section, we relax this assumption and we estimate the mode sequence along with the bias-corrected parameters in an iterative manner. Using the estimated mode sequence, a partition of the regressor space $\mathcal{X}$ is computed using multi-category discrimination techniques as in (Breschi et al., 2016).

4.1 Iterative clustering and parameter estimation

Algorithm 1 summarizes the main ideas of the proposed approach which involves the computation of the bias-corrected estimates $\{\theta_i\}_{i=1}^s$ as well as the unknown discrete state $\{\sigma(k)\}_{k=1}^N$ and the clusters $\{\mathcal{C}_i\}_{i=1}^s$ characterizing the regressor space partition. The cluster $\mathcal{C}_i$ is constructed by stacking all (estimated) regressors $\hat{x}(k)$ associated to mode i and its centroid c_i is defined as $c_i = \frac{1}{N_i}\sum_{\hat{x}(k)\in\mathcal{C}_i} \hat{x}(k)$.

Given an initial guess $\{\sigma^0(k)\}_{k=1}^N$ of the mode sequence, at each iteration $m \geq 1$, Algorithm 1 alternates between the computation of the bias-corrected model parameters $\{\theta_i^m\}_{i=1}^s$, for a fixed mode sequence $\{\sigma^{m-1}(k)\}_{k=1}^N$ obtained at iteration $(m-1)$ (see Step 1.1), and the estimation of the mode sequence $\{\sigma^m(k)\}_{k=1}^N$, for fixed bias-corrected model parameters $\{\theta_i^m\}_{i=1}^s$ obtained at the m-th iteration (Step 1.3). Specifically, at Step 1.1, the estimates $\{\theta_i^m\}_{i=1}^s$ are updated by computing the matrices $\{\mathbb{X}_i\}_{i=1}^s$, $\{\mathbb{Y}_i\}_{i=1}^s$ (eqs. (3) and (4)) and the bias-correcting matrices $\{\Psi_i\}_{i=1}^s$ (eq. (12)), based on the estimated mode sequence $\{\sigma^{m-1}(k)\}_{k=1}^N$ at the previous iteration. Then, the model parameters $\{\theta_i^m\}_{i=1}^s$ are used to update the mode sequence at Step 1.3.

In particular, at Step 1.3, for each time index $k \in [\max(n_a, n_b)+1, N]$, the simulated regressor $\hat{x}(k)$ is computed based on the simulated outputs $\{\hat{y}(k-i)\}_{i=1}^{n_a}$ (Step 1.3.1). The simulated output is initialized as $\hat{y}(k) = y(k)$ for $k \in [1, \max(n_a, n_b)]$. At Step 1.3.2, the prediction-error $\{e_i(k)\}_{i=1}^s$ are computed. Note that, $e_i(k)$ is the output-error computed based on the bias-corrected estimate θ_i^m and simulated regressor $\hat{x}(k)$. Step 1.3.3 selects the *best* mode $\sigma^m(k)$ to which the regressor $\hat{x}(k)$ is associated with by minimizing the prediction-error $e_i(k)$ and the distance between $\hat{x}(k)$ and the centroid c_i of the cluster $\mathcal{C}_i$ which contains all the regressors already assigned to mode i. A positive hyper-parameter λ is used to weigh these two terms. The centroid penalty $||\hat{x}(k) - c_i||^2$ takes into account the assumption that regressors "close" to each other are likely to belong to the same cluster while the term $e_i^2(k)$ penalizes the model mismatch.

Algorithm 1 Iterative bias-corrected parameter estimation and clustering.

Input: Observations $\{x(k), y(k)\}_{k=1}^N$; noise variance σ_e^2; number of modes s; tuning parameter λ; initial guess of the mode sequence $\{\sigma^0(k)\}_{k=1}^N$; number of iterations M.

1. **iterate for** $m = 1, \ldots, M$ **do**
   1.1. **compute** $\{\theta_i^m\}_{i=1}^s$ for fixed $\{\sigma^{m-1}(k)\}_{k=1}^N$:
   $$\theta_i^m = \left(\frac{\mathbb{X}_i^\top \mathbb{X}_i + \Psi_i}{N_i}\right)^{-1} \frac{\mathbb{X}_i^\top \mathbb{Y}_i}{N_i};\ \forall i = 1, \ldots, s.$$
   1.2. **set** $\mathcal{C}_i^m = \emptyset$, $c_i^m = 0$, $N_i^m = 0\ \ \forall i = 1, \ldots, s$;
   1.3. **for** $k = \max(n_a, n_b) + 1, \ldots, N$ **do**
      1.3.1. **let**
      $$\hat{x}(k) = [\hat{y}(k-1) \cdots \hat{y}(k-n_a)\ u(k-1) \cdots u(k-n_b)]^\top$$
      1.3.2. **let** for $i = 1, \ldots, s$
      $$e_i(k) \leftarrow y(k) - (\theta_i^m)^\top \begin{bmatrix} \hat{x}(k) \\ 1 \end{bmatrix};$$
      1.3.3. **let**
      $$\sigma^m(k) \leftarrow \arg\min_{i=1,\ldots,s} \lambda e_i^2(k) + \|\hat{x}(k) - c_i^m\|_2^2;$$
      1.3.4. **let** $\hat{y}(k) = (\theta_{\sigma^m(k)}^m)^\top \begin{bmatrix} \hat{x}(k) \\ 1 \end{bmatrix}$;
      1.3.5. **let** $\mathcal{C}_{\sigma^m(k)}^m \leftarrow \mathcal{C}_{\sigma^m(k)}^m \cup \{\hat{x}(k)\}$;
      1.3.6. **let** $N_{\sigma^m(k)}^m \leftarrow N_{\sigma^m(k)}^m + 1$;
      1.3.7. **update** centroid
      $$c_{\sigma^m(k)}^m \leftarrow \frac{1}{N_{\sigma^m(k)}^m} \sum_{\hat{x}(k)\in\mathcal{C}_{\sigma^m(k)}^m} \hat{x}(k)$$
   1.4. **end for**;
2. **end for**;

Output: Estimated parameters $\{\theta_i^M\}_{i=1}^s$; clusters $\{\mathcal{C}_i^M\}_{i=1}^s$; sequence of active modes $\{\sigma^M(k)\}_{k=1}^N$.

Based on the active mode $\sigma^m(k)$ selected at Step 1.3.3, the output $\hat{y}(k)$ (used to construct the regressor) is updated at Step 1.3.4, the regressors $\hat{x}(k)$ is assigned to cluster $\mathcal{C}_{\sigma^m(k)}$ at Step 1.3.5 and the cardinality $N_{\sigma^m(k)}$ of the cluster $\mathcal{C}_{\sigma^m(k)}$ is increased by 1 (Step 1.3.6). The cluster's centroid $c_{\sigma^m(k)}$ is finally updated at Step 1.3.7.

4.2 Partitioning the regressor space

Given the clusters $\{\mathcal{C}_i\}_{i=1}^s$ obtained from Algorithm 1, the partition $\{\mathcal{X}_i\}_{i=1}^s$ of the regressor space $\mathcal{X}$ can be computed using the computationally efficient linear multicategory discrimination algorithm proposed in (Breschi et al., 2016) and briefly discussed in this subsection.

In order to separate the clusters $\mathcal{C}_1, \ldots, \mathcal{C}_s$, we search for a piecewise-affine separator function $\phi : \mathbb{R}^{n_x} \to \mathbb{R}$ defined as

$$\phi(\hat{x}) = \max_{i=1,\ldots,s} \left(\begin{bmatrix} \hat{x}^\top & -1 \end{bmatrix} \begin{bmatrix} \omega^i \\ \gamma^i \end{bmatrix} \right), \tag{14}$$

where $\omega^i \in \mathbb{R}^{n_x}$ and $\gamma^i \in \mathbb{R}$ are unknown parameters to be computed.

Let N_i denote the cardinality of the i-th cluster $\mathcal{C}_i$ and $M_i \in \mathbb{R}^{N_i \times n_x}$ be the matrix obtained by stacking the regressors $\hat{x}^\top(k)$ belonging to $\mathcal{C}_i$ in its rows. The parameters $\{\omega^i, \gamma^i\}_{i=1}^s$, are computed by solving the following convex optimization problem (Breschi et al., 2016):

$$\min_{\xi} \frac{\kappa}{2} \sum_{i=1}^{s} \left(\|\omega^i\|_2^2 + (\gamma^i)^2\right) + \sum_{i=1}^{s} \sum_{\substack{j=1 \\ j \neq i}}^{s} \frac{1}{N_i} \left\| \left(\begin{bmatrix} M_i & -\mathbf{1}_{N_i} \end{bmatrix} \begin{bmatrix} \omega^j - \omega^i \\ \gamma^j - \gamma^i \end{bmatrix} + \mathbf{1}_{N_i} \right)_+ \right\|_2^2, \quad (15)$$

where $\xi = \begin{bmatrix} (\omega^1)^\top & \dots & (\omega^s)^\top & \gamma^1 & \dots & \gamma^s \end{bmatrix}^\top$, and for a given $x \in \mathbb{R}^n$, $(x)_+$ denotes a vector whose i-th element is $\max\{x_i, 0\}$. The regularization parameter $\kappa > 0$ makes sure that the optimization problem (15) is strongly convex.

5. NUMERICAL EXAMPLE

The effectiveness of the proposed identification algorithm is shown via a benchmark example. All computations are carried out on an i7 1.9-GHz Intel core processor with 32 GB of RAM running MATLAB R2019a.

We consider the data-generating system $\mathcal{S}_\text{o}$, introduced in (Bemporad et al., 2005), modified to the output-error structure in (1) as follows:

$$y_\text{o}(k) = \begin{cases} -0.4y_\text{o}(k-1) \\ +u(k-1)+1.5, & \text{if } 4y_\text{o}(k-1)-u(k-1)+10<0, \\ \\ 0.5y_\text{o}(k-1) \\ -u(k-1)-0.5, & \text{if } 4y_\text{o}(k-1)-u(k-1)+10 \geq 0, \\ & \text{and } 5y_\text{o}(k-1)+u(k-1)-6 \leq 0, \\ -0.3y_\text{o}(k-1) \\ +0.5u(k-1)-1.7, & \text{if } 5y_\text{o}(k-1)+u(k-1)-6 > 0, \end{cases}$$

$$y(k) = y_\text{o}(k) + e_\text{o}(k). \quad (16)$$

The system is characterized by $s = 3$ discrete modes. The input signal u is generated from a uniform random distribution taking values in the interval $[-4, 4]$. The noise e_o corrupting the output signal is generated by a zero-mean white Gaussian process with variance $\sigma_e^2 = 0.64$, which corresponds to the *Signal-to-Noise Ratio* (SNR),

$$\text{SNR} = 10 \log \frac{\sum_{k=1}^N y_\text{o}^2(k)}{\sum_{k=1}^N e_\text{o}^2(k)} = 11.7 \text{ dB}.$$

The training dataset consist of $N = 5000$ input/output samples gathered from the system (16).

For the identification, we consider the PWA-OE model structure defined in (2) with $s = 3$ modes, and model orders $n_a = 1$, $n_b = 1$. The tuning hyper-parameter λ is set to the inverse of the noise variance, *i.e.*, $\lambda = \sigma_e^{-2} = 1.56$. The model parameters and the unknown mode sequence are estimated by running Algorithm 1 for $M = 20$ iterations with a randomly generated initial guess $\{\sigma^0(k)\}_{k=1}^N$ of the mode sequence. The computation time to run Algorithm 1 is 2.6 sec.

The estimated affine submodel parameters obtained with the proposed bias-correction approach are reported in Table 1, along with the ones obtained via standard least squares (i.e., using the LS estimate (5) in Algorithm 1). It can be seen that the LS estimates are biased while the bias-corrected estimates match closely with the true system parameters. This is further highlighted by the obtained norms of the parameter estimation errors reported in Table 2, for BC and LS estimates.

Table 1. True (θ_i^o) and estimated model parameters: LS (θ_i^LS) *vs* BC (θ_i^BC) estimates

Mode	θ^o	θ^BC	θ^LS
$s=1$	-0.4000	-0.3933	-0.1889
	1.0000	1.0007	0.9691
	1.5000	1.4816	2.2521
$s=2$	0.5000	0.5041	0.3341
	-1.0000	-1.0003	-1.0022
	-0.5000	-0.4943	-0.6273
$s=3$	-0.3000	-0.3132	-0.2489
	0.5000	0.5056	0.5122
	-1.7000	-1.6765	-1.9091

The accuracy of the estimated mode sequence is expressed by the *Mode-Fit* (MF) index

$$\text{MF} = \left(\frac{1}{N} \sum_{k=1}^{N} \mathbb{I}(\sigma^M(k) = \sigma^\star(k)) \right) \times 100\%, \quad (17)$$

where $\mathbb{I}(\cdot)$ is the indicator function, $\sigma^M(k)$ and $\sigma^\star(k)$ are the estimated and the true [2] modes at time k, respectively. The MF index achieved over the training data is 98.76% with the bias-corrected estimates while it is 98.50% with the LS estimates.

Based on the clusters estimated through Algorithm 1, the polyhedral partition of the regressor space is computed by solving the linear multicategory discrimination problem (15), with regularization parameter $\kappa = 10^{-5}$. The true and the estimated polyhedral partitions of the regressor space are shown in Fig. 1.

The estimated PWA-OE model is validated on a new dataset of length $N_\text{val} = 500$. Based on the estimated submodel parameters and the polyhedral partitions, the output is simulated in open-loop as shown in Fig. 2 with bias-corrected and LS estimates. Specifically, in validation, using the estimated parameters $\{\omega^i, \gamma^i\}_{i=1}^s$ defining the partition of the regressor space, the active mode is selected based on the value of the separator function (14) and corresponding model parameter θ^BC is used to simulate the output. For the sake of better visualization, only a subset of validation data is plotted. The MF index achieved over the validation dataset are 88.40% and 77.80% with bias-corrected and LS estimates, respectively.

The obtained results indicate that the affine submodel parameters, the unknown mode sequence as well as the partition of the regressor space have been accurately estimated with the proposed approach.

6. CONCLUSION

We have presented a method for the identification of PWA output-error models. A bias-correction scheme is combined with recursive clustering to estimate the model parameters

[2] The true mode sequence $\sigma^\star(k)$ is considered only for evaluating the MF index and it is not used in the estimation algorithm.

Table 2. Norm of the parameter estimation error. LS *vs* BC estimates.

Mode	$\|\theta^{\circ} - \theta^{\mathrm{BC}}\|$	$\|\theta^{\circ} - \theta^{\mathrm{LS}}\|$
$s = 1$	0.0196	0.7818
$s = 2$	0.0070	0.2091
$s = 3$	0.0275	0.2156

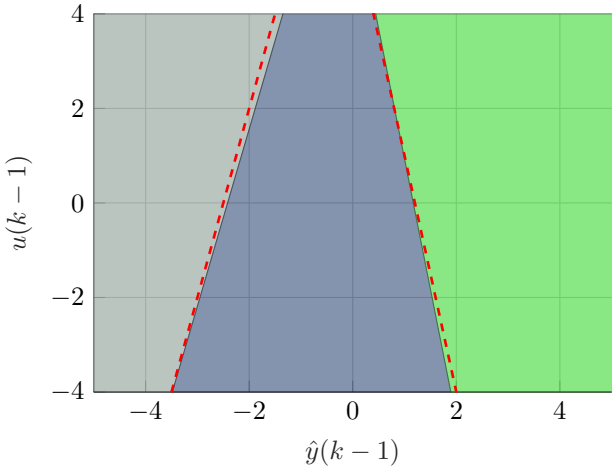

Fig. 1. True partition (dashed red lines) *vs* estimated polyhedral partition of the regressor space.

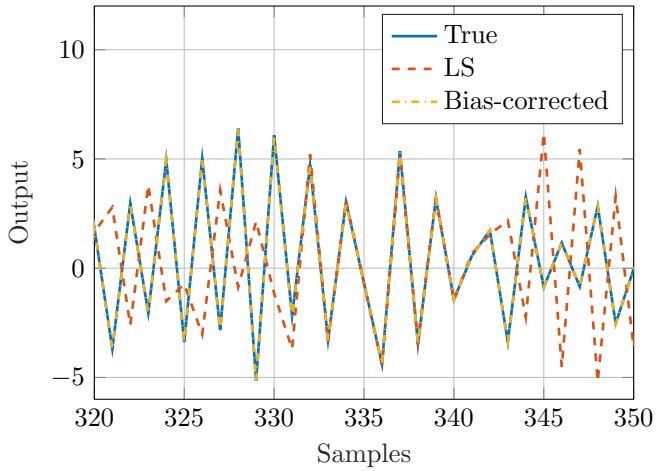

Fig. 2. Validation dataset: true output, simulated output of the LS model and simulated output of the BC model.

as well as the unknown mode sequence. The estimates are proven to be consistent under suitable assumptions. The reported example shows that the bias-correction scheme outperforms standard least squares in terms of achieving a consistent estimate. Furthermore, the presented algorithm estimates the partition of the regressor space with high accuracy. Future research activities include the problem of estimation of model orders and number of modes directly from data. This requires to define a proper criterion to be optimized, taking into account the output-error model structure in the fitting cost.

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