Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Output feedback stochastic MPC with packet losses

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Abstract: The paper considers constrained linear systems with stochastic additive disturbances and noisy measurements transmitted over a lossy communication channel. We propose a model predictive control (MPC) law that minimizes a discounted cost subject to a discounted expectation constraint. Sensor data is assumed to be lost with known probability, and data losses are accounted for by expressing the predicted control policy as an affine function of future observations, which results in a convex optimal control problem. An online constraint-tightening technique ensures recursive feasibility of the online optimization and satisfaction of the expectation constraint without bounds on the distributions of the noise and disturbance inputs. The cost evaluated along trajectories of the closed loop system is shown to be bounded by the optimal predicted cost. A numerical example is given to illustrate these results.

Keywords: Model predictive control, output feedback, packet drops, chance constraints, convex optimization.

1. INTRODUCTION

Robust model predictive control often considers worst-case disturbance bounds, so that hard constraints on system states and control inputs are satisfied for all possible disturbances (Mayne et al., 2000; Mesbah, 2016; Kouvaritakis and Cannon, 2015). However, worst-case disturbance bounds can be extremely conservative or even non-existent, which motivates the development of stochastic MPC with chance constraints. In many applications of practical interest, system states cannot be measured directly and instead have to be estimated from output measurements. Existing stochastic MPC algorithms incorporating state estimation (e.g. Cannon et al., 2012; Dai et al., 2015) typically do not consider optimizing state feedback gains online, and the estimator gain is typically chosen as the steady state Kalman filter gain.

Control systems that rely on sensor signals transmitted over a network must tolerate communication delays and data losses. These pose additional challenges for estimation and control problems when constraints are present. From a control perspective, these features can be modelled as information losses by random processes, such as Bernoulli processes (Sinopoli et al., 2004) or Markov chains (Leong et al., 2017). In Sinopoli et al. (2004), the arrival of output observations is modelled as a Bernoulli process and fundamental results are derived, including bounds on the critical value for the arrival probability of the observation update and convergence properties of the algebraic Riccati equation for Kalman filters with intermittent observations. In Schenato et al. (2007), it is shown that the well-known separation principle holds with sensor packet losses, whereas this is not the case if constraints are present. Mishra et al. (2019) consider the problem of controlling linear systems with unbounded additive disturbances and measurement noise by using an affine policy, where both sensor measurements and control actions are lost with given probabilities. Alternatively, these problems can be modelled as jump linear systems (Mariton, 1990) switching between different states according to a transition probability matrix.

This paper designs an output-feedback MPC algorithm to minimize a discounted cost function subject to a discounted expectation constraint, assuming sensor measurements to be lost with a given probability. The discount setting is common to many control problems (e.g. Bertsekas, 1995; Van Parys et al., 2013; Kouvaritakis et al., 2003; Kamgarpour and Summers, 2017), and an appropriate discounting factor can provide stability guarantees (Postoyan et al., 2017). In this work, the discounting factor allows consideration of unbounded disturbances and measurement noise, and we derive bounds on the cost and constraints for the closed loop system using a constraint-tightening technique (Yan et al., 2018). Instead of choosing the future control policy as pre-stabilising feedback with perturbations (Cannon et al., 2011), we parameterise predicted control inputs as affine functions of future output measurements and show that the problem of optimizing the associated feedback gains is convex. This allows the distributions of future states to be controlled even when output measurements are lost.

This paper is organised as follows. We describe the control problem in Section 2, and introduce the controller parameterization and implementation in Section 3. We compute predicted state and control sequences via their first and second moments in Section 4. In Section 5, we derive the terminal conditions and give explicit expressions for the cost and constraints. Our main results, including a closed loop cost bound and constraint satisfaction, are in Section 6. Section 7 provides a numerical example and the paper is concluded in Section 8.

Notation: The $n\times n$ identity matrix is $I_{n\times n}$, and the $n\times m$ matrix with all elements equal to 1 is $\mathbf{1}_{n\times m}$. The vectorized form of a matrix $A=[a_1\ \cdots\ a_n]$ is $\text{vec}(A) := [a_1^\top\ \cdots\ a_n^\top]^\top$ and $A\otimes B$ is the Kronecker product. The Euclidean norm is $\|x\|$ and, for a matrix Q, $Q\succ 0$ ($Q\succeq 0$) indicates that Q is positive definite (semidefinite) and $\|x\|_Q^2 := x^\top Q x$.

2. PROBLEM DESCRIPTION

2.1 System model and feedback information

We assume a system with linear discrete time dynamics

$$x_{k+1} = Ax_k + Bu_k + Dw_k, \tag{1a}$$

$$y_k = Cx_k + v_k,\ z_k = \gamma_k y_k \tag{1b}$$

where $x\in\mathbb{R}^{n_x}$, $u\in\mathbb{R}^{n_u}$, $y\in\mathbb{R}^{n_y}$, $z\in\mathbb{R}^{n_y}$ are the state, control input, sensor measurement, and the measurement information received by the controller respectively. The disturbance, measurement noise and packet loss sequences, $\{w_k\}_{k=0}^\infty$, $\{v_k\}_{k=0}^\infty$ and $\{\gamma_k\}_{k=0}^\infty$, are assumed to have independent, identically distributed (i.i.d.) elements with

$$\begin{aligned} &\mathbb{E}\{w_k\}=0, && \mathbb{E}\{w_k w_k^\top\}=\Sigma_w\succeq 0,\\ &\mathbb{E}\{v_k\}=0, && \mathbb{E}\{v_k v_k^\top\}=\Sigma_v\succ 0,\\ &\mathbb{P}\{\gamma_k=0\}=1-\lambda, && \mathbb{P}\{\gamma_k=1\}=\lambda. \end{aligned}$$

The variable $\gamma_k\in\{0,1\}$ indicates whether sensor data at the kth sampling instant is received by the controller. The information available to the controller at time k consists of $\{u_i\}_{i=0}^{k-1}$, $\{(z_i,\gamma_i)\}_{i=0}^{k}$, the initial mean $\mathbb{E}\{x_0\}=\hat{x}_0$, and covariance $\mathbb{E}\{(x_0-\hat{x}_0)(x_0-\hat{x}_0)^\top\}=\Sigma_0$ of the model state.

We define the information sets

$$\mathcal{I}_k := \{\mathcal{I}_{k-1},(z_k,\gamma_k)\},\quad \mathcal{U}_k := \{\mathcal{U}_{k-1},u_k\},$$

for all $k\geq 0$, where $\mathcal{I}_{-1} := \{\hat{x}_0,\Sigma_0\}$, $\mathcal{U}_{-1} := \{\,\}$. Finally, we define conditional expectation operators as

$$\mathbb{E}_k\{\cdot\} := \mathbb{E}\{\cdot\,|\,\mathcal{U}_{k-1},\mathcal{I}_{k-1}\},\quad \mathbb{E}\{\cdot\} := \mathbb{E}_0\{\cdot\}.$$

Assumption 1. The pair (A,B) is stabilizable, and (A,C) is detectable.

2.2 Optimal control problem

We will employ a finite-horizon control policy with input at time k in the form

$$u_{i|k} = \kappa_i(\theta_k,\mathcal{U}_{k+i-1},\mathcal{I}_{k+i})$$

where $u_{i|k}$ for $i=0,1,\dots$ is the prediction of u_{k+i} at time k, and θ_k is a vector of controller parameters at time k. The dependence of $\kappa_i(\cdot)$ on the sets $\mathcal{U}_{k+i-1}$ and $\mathcal{I}_{k+i}$ ensures causality and the dependence on θ_k is chosen so that the optimal parameter vector, denoted θ_k^*, will be the solution of a convex problem.

Assumption 2. (i). The probability, λ, of successfully receiving sensor measurements is known. (ii). When θ_k^* is computed, (z_{k+i},γ_{k+i}) are unknown for all $i\geq 0$.

Assumption 2 requires θ_k^* to be a function of $\mathcal{U}_{k-1}$ and $\mathcal{I}_{k-1}$, and we therefore assume that θ_k^* is computed online prior to the kth sampling instant. However (z_k,γ_k) is known when the control law

$$u_k = \kappa_0(\theta_k^*,\mathcal{U}_{k-1},\mathcal{I}_k)$$

is applied to the plant.

We consider the problem of minimizing the discounted sum of expected future values of $\|x_k\|_Q^2+\|u_k\|_R^2$, where $Q\succeq 0$ and $R\succ 0$. This minimization is subject to a constraint on the discounted sum of second moments of an auxiliary output, defined for given matrix H by $\xi_k = Hx_k$, so that

$$\begin{aligned} \theta_k^* = \arg\min_{\theta_k}\ &\sum_{i=0}^\infty \beta^i \mathbb{E}_k\{\|x_{i|k}\|_Q^2+\|u_{i|k}\|_R^2\}\\ \text{s.t.}\ &\sum_{i=0}^\infty \beta^i\mathbb{E}_k\{\|Hx_{i|k}\|^2\}\leq\epsilon. \end{aligned} \tag{2}$$

Here $\beta\in(0,1)$ is a discounting factor and ϵ is a given bound on this infinite discounted sum of second moments. Instead of solving (2) directly, the control problem to be solved at time k is given by

$$\begin{aligned} \theta_k^* = \arg\min_{\theta_k}\ &\sum_{i=0}^\infty \beta^i \mathbb{E}_k\{\|x_{i|k}\|_Q^2+\|u_{i|k}\|_R^2\}\\ \text{s.t.}\ &\sum_{i=0}^\infty \beta^i\mathbb{E}_k\{\|Hx_{i|k}\|^2\}\leq\mu_k. \end{aligned} \tag{3}$$

Here $\mu_0=\epsilon$ and, for all $k>0$, μ_k is chosen as described in Section 6 to ensure that (3) is recursively feasible and that the constraint in (2) is satisfied with $k=0$ by the closed loop system.

3. CONTROLLER PARAMETERIZATION

Consider the output feedback control law defined by an observer and an affine feedback law:

$$\hat{x}_k = A\tilde{x}_{k-1}+Bu_{k-1},\ \tilde{x}_k=\hat{x}_k+\gamma_k M(y_k-C\hat{x}_k), \tag{4a}$$

$$u_k = K\tilde{x}_k + c_k. \tag{4b}$$

with $\hat{x}_0=\mathbb{E}\{x_0\}$, where $\hat{x}_k$ and $\tilde{x}_k$ are the a priori estimate and the posteriori estimate of x_k, respectively. A simplistic parameterization of the predicted control law $\kappa_i(\cdot)$ could be obtained if the observer gain M and feedback gain K were fixed and the optimization variables in problem (3) were defined as $\theta_k=\{c_{0|k},\dots,c_{N-1|k}\}$ for some fixed N, with the predicted control sequence defined as $u_{i|k}=K\tilde{x}_{i|k}+c_{i|k}$. Although this would require a number of optimization variables that grows only linearly with N, the parameters $\{c_{0|k},\dots,c_{N-1|k}\}$ constitute an open loop control sequence that does not vary with the future measurement noise and disturbance realizations. This is likely to provide poor performance and small sets of feasible initial conditions when the probability of packet loss is non-zero.

By using a parameterization that allows the dependence of the predicted control sequence on future realizations of model uncertainty to be optimized, the predicted probability distributions of states and control inputs can be controlled explicitly. This provides flexibility to balance conflicting requirements for performance and constraint satisfaction. However, similarly to the case of predicted control laws in which state feedback gains are decision variables (Löfberg, 2003; Goulart et al., 2006), the cost and constraints of problem (3) are nonconvex if time-varying gains M, K are considered as optimization variables. On the other hand, if predicted control inputs are parameterized in terms of affine functions of the future output measurements received by the controller, then the dependence of the first and second moments of predicted states and inputs on controller parameters is convex. Moreover, by incorporating affine terms in the future innovation sequence, a predicted control law with arbitrary linear dependence of

$\kappa_i(\cdot)$ on the received sensor measurements can be obtained. This approach allows the future control sequence to be optimized at every sampling instant, including those at which information from sensors is lost.

We therefore express the i steps ahead predicted control input $u_{i|k}$, for all $i = 0, 1, \ldots$, as

$$u_{i|k} = K\hat{x}_{i|k} + c_{i|k} + d_{i|k}, \tag{5a}$$

$$d_{i|k} = \gamma_{0|k}L_{i,0|k}(y_{0|k} - C\hat{x}_{0|k}) + \gamma_{1|k}L_{i,1|k}(y_{1|k} - C\hat{x}_{1|k}) + \cdots + \gamma_{i|k}L_{i,i|k}(y_{i|k} - C\hat{x}_{i|k}), \tag{5b}$$

$$\hat{x}_{i+1|k} = A\hat{x}_{i|k} + Bu_{i|k} + \gamma_{i|k}AM(y_{i|k} - C\hat{x}_{i|k}) \tag{5c}$$

where $c_{i|k} = 0$ and $L_{i,j|k} = 0$ for all $i \geq N$. Here $\gamma_{i|k}$ and $y_{i|k}$ are random variables, denoting the i-step-ahead predicted packet loss and sensor measurement at time k, respectively. Then, for all $i = 0, 1, \ldots$ the predicted state estimate satisfies

$$\hat{x}_{i+1|k} = \Phi\hat{x}_{i|k} + B(c_{i|k} + d_{i|k}) + \gamma_{i|k}AM(y_{i|k} - C\hat{x}_{i|k}) \tag{6}$$

where $\Phi := A + BK$. Since $x_{i+1|k} = Ax_{i|k} + Bu_{i|k} + Dw_{i|k}$ the predicted estimation error evolves according to

$$x_{i+1|k} - \hat{x}_{i+1|k} = \Psi_{i|k}(x_{i|k} - \hat{x}_{i|k}) - \gamma_{i|k}AMv_{i|k} + Dw_{i|k} \tag{7}$$

with $\Psi_{i|k} := A(I - \gamma_{i|k}MC)$. These relationships allow the first and second moments of $x_{i|k}$ to be determined in terms of the decision variable θ_k, which consists of the parameters $\{c_{0|k}, \ldots, c_{N-1|k}\}$ and feedback gains $L_{0,0|k}$, $\{L_{1,0|k}, L_{1,1|k}\}$, $\ldots$, $\{L_{N-1,0|k}, \ldots, L_{N-1,N-1|k}\}$.

The gains K and M in the predicted control law (5a-c) are chosen offline and satisfy the following assumption.

Assumption 3. $\xi_{i+1} = (A+BK)\xi_i$ is asymptotically stable and $\xi_{i+1} = A(I - \gamma_i MC)\xi_i$ is mean-square stable (Kushner, 1971).

Remark 1. Gains K and M exist satisfying Assumption 3 if Assumption 1 holds and if the probability, λ, of successfully receiving a sensor measurement is greater than some critical value (e.g. Sinopoli et al., 2004). Suitable choices for K, M are the optimal gains for (3) in the absence of constraints, or the certainty equivalent LQ feedback gain for a problem with state and control weighting matrices Q and R and the steady state Kalman filter gain (Sinopoli et al., 2004). We note also that time-varying gains K_k, M_k can be used within the framework of this paper, provided their dependence on γ_k is known in advance.

3.1 Controller implementation

The control law is implemented by the following procedure.

(i). Given $\mathcal{U}_{k-1}$ and $\mathcal{I}_{k-1}$, solve problem (3) for θ_k^*.

(ii). Given γ_k and $z_k = \gamma_k y_k$:
  - (a). apply the control input
  $$u_k = K\hat{x}_k + c^*_{0|k} + \gamma_k L^*_{0,0|k}(y_k - C\hat{x}_k),$$
  - (b). update the state estimate
  $$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + \gamma_k AM(y_k - C\hat{x}_k).$$

Note that this receding horizon control law includes (4) as a special case, since u_k and $\hat{x}_{k+1}$ in step (ii) would be equal to their counterparts in (4) if $(c^*_{0|k}, L^*_{0,0|k}) = (c_k, KM)$.

4. PREDICTED STATE AND CONTROL SEQUENCES

To simplify notation we express the predicted control law in terms of vectorized sequences, with $\mathbf{x}_k$ denoting the vectorized state sequence $\{x_{i|k}\}_{i=0}^{N-1}$, $\hat{\mathbf{x}}_k$ the estimate sequence $\{\hat{x}_{i|k}\}_{i=0}^{N-1}$, $\mathbf{u}_k$ the predicted control sequence $\{u_{i|k}\}_{i=0}^{N-1}$, $\mathbf{c}_k$ the predicted control perturbations $\{c_{i|k}\}_{i=0}^{N-1}$, $\mathbf{w}_k$ the disturbances $\{w_{i|k}\}_{i=0}^{N-1}$, $\mathbf{v}_k$ the sensor noise sequence $\{v_{i|k}\}_{i=0}^{N-1}$, and $\boldsymbol{\zeta}_k$ the future innovation sequence $\{\gamma_{i|k}(y_{i|k} - C\hat{x}_{i|k})\}_{i=0}^{N-1}$ at time k. For a given sequence of matrices $\{\Psi_{i|k}\}_{i=0}^{N-1}$ and matrix B let

$$\mathbf{S}_\Psi = \begin{bmatrix} I \\ \Psi_{0|k} \\ \vdots \\ \prod_{i=N-2}^{0} \Psi_{i|k} \end{bmatrix}, \ \mathbf{T}_{(\Psi,B)} = \begin{bmatrix} 0 & \cdots & 0 \\ B & & \\ \vdots & \ddots & \\ \prod_{i=N-2}^{1} \Psi_{i|k}B & \cdots & B \ \ 0 \end{bmatrix},$$

$$S_\Psi^N = \prod_{i=N-1}^{0} \Psi_{i|k}, \qquad T_{(\Psi,B)}^N = \begin{bmatrix} \prod_{i=N-1}^{1} \Psi_{i|k}B & \cdots & B \end{bmatrix}$$

where $\prod_{i=m}^{n} \Psi_{i|k} = \Psi_{m|k} \cdots \Psi_{n|k}$ for $m \geq n$, and define

$$\mathbf{L}_k = \begin{bmatrix} L_{0,0|k} & & & \\ L_{1,0|k} & L_{1,1|k} & & \\ \vdots & \vdots & \ddots & \\ L_{N-1,0|k} & L_{N-1,1|k} & \cdots & L_{N-1,N-1|k} \end{bmatrix},$$

$$\boldsymbol{\Gamma}_k = \mathrm{diag}\{\gamma_{0|k}, \ldots, \gamma_{N-1|k}\} \otimes I_{n_y \times n_y},$$

$\mathbf{K} = I_{N\times N} \otimes K$, $\mathbf{M} = I_{N\times N} \otimes M$ and $\mathbf{C} = I_{N\times N} \otimes C$. Then from (7) we have

$$\mathbf{x}_k - \hat{\mathbf{x}}_k = \mathbf{S}_\Psi(x_k - \hat{x}_k) - \mathbf{T}_{(\Psi,A)}\mathbf{M}\boldsymbol{\Gamma}_k\mathbf{v}_k + \mathbf{T}_{(\Psi,D)}\mathbf{w}_k \tag{8}$$

while (6) and (5b) give

$$\hat{\mathbf{x}}_k = \mathbf{S}_\Phi\hat{x}_k + \mathbf{T}_{(\Phi,B)}(\mathbf{c}_k + \mathbf{L}_k\boldsymbol{\zeta}_k) + \mathbf{T}_{(\Phi,A)}\mathbf{M}\boldsymbol{\zeta}_k.$$

Here $\boldsymbol{\zeta}_k = \boldsymbol{\Gamma}_k\mathbf{C}(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \boldsymbol{\Gamma}_k\mathbf{v}_k$ and $\mathbf{S}_\Phi$, $\mathbf{T}_{(\Phi,B)}$ are defined (analogously to $\mathbf{S}_\Psi$, $\mathbf{T}_{(\Psi,B)}$) in terms of Φ and B. Hence

$$\hat{\mathbf{x}}_k = \mathbf{S}_\Phi\hat{x}_k + \mathbf{T}_{(\Phi,B)}\mathbf{c}_k + (\mathbf{T}_{(\Phi,B)}\mathbf{L}_k + \mathbf{T}_{(\Phi,A)}\mathbf{M})\boldsymbol{\zeta}_k, \tag{9a}$$

$$\mathbf{u}_k = \mathbf{K}\hat{\mathbf{x}}_k + \mathbf{c}_k + \mathbf{L}_k\boldsymbol{\zeta}_k. \tag{9b}$$

Clearly the predicted estimation error, state and control sequences in (8) and (9a,b) depend linearly on the decision variables $\theta_k := (\mathbf{c}_k, \mathbf{L}_k)$.

4.1 First and second moments of predicted sequences

In order to express the cost and constraints of problem (3) in terms of the parameterization introduced in Section 3, we derive in this section expressions for the means and variances of predicted state and control sequences.

First consider the state x_k of the plant (1a) and the state estimate update $\hat{x}_k$ in step (ii) of the controller implementation in Section 3.1. By assumption we have $\mathbb{E}\{x_0\} = \hat{x}_0$ and $\mathbb{E}\{w_k\} = 0$, $\mathbb{E}\{v_k\} = 0$ for all $k \geq 0$, and hence the update of state estimates $\hat{x}_k$ in step (ii)(b) ensures that

$$\mathbb{E}_k\{x_k\} = \hat{x}_k \tag{10}$$

for all $k \geq 1$. Furthermore, from (1a) we have

$$x_k - \hat{x}_k = \Psi_{k-1}(x_{k-1} - \hat{x}_{k-1}) - \gamma_{k-1}AMv_{k-1} + Dw_{k-1}$$

for all $k \geq 1$. Let Σ_k denote the second moment of the state estimate error at time k:

$$\Sigma_k := \mathbb{E}_k\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^\top\}.$$

Then Σ_k evolves according to

$$\Sigma_k = \Psi_{k-1}\Sigma_{k-1}\Psi_{k-1}^\top + \gamma_{k-1}AM\Sigma_v M^\top A^\top + D\Sigma_w D^\top \tag{11}$$

for all $k \geq 1$, with initial condition Σ_0, and by Assumption 3 $\mathbb{E}\{\Sigma_k\}$ remains upper bounded $\forall k$.

We first derive the first and second moments of the predicted state sequence $\mathbf{x}_k$ and control sequence $\mathbf{u}_k$:

Proposition 2. Let $\boldsymbol{\pi}_k$, $\boldsymbol{\Pi}_k$, and $\boldsymbol{\Omega}_k$ be defined

$$\boldsymbol{\pi}_k = \mathbf{S}_\Phi \hat{x}_k + \mathbf{T}_{(\Phi,B)}\mathbf{c}_k, \quad \boldsymbol{\Pi}_k = \mathbf{T}_{(\Phi,B)}\mathbf{L}_k + \mathbf{T}_{(\Phi,A)}\mathbf{M},$$

$$\boldsymbol{\Omega}_k = \mathbb{E}_k\left\{ \begin{bmatrix} \mathbf{x}_k - \hat{\mathbf{x}}_k \\ \boldsymbol{\zeta}_k \end{bmatrix} \begin{bmatrix} \mathbf{x}_k - \hat{\mathbf{x}}_k \\ \boldsymbol{\zeta}_k \end{bmatrix}^\top \right\}.$$

Then

$$\mathbb{E}_k\{\mathbf{x}_k\} = \mathbb{E}_k\{\hat{\mathbf{x}}_k\} = \boldsymbol{\pi}_k, \tag{12a}$$

$$\mathbb{E}_k\{\mathbf{u}_k\} = \mathbf{K}\mathbb{E}_k\{\hat{\mathbf{x}}_k\} + \mathbf{c}_k = \mathbf{K}\boldsymbol{\pi}_k + \mathbf{c}_k, \tag{12b}$$

and

$$\mathbb{E}_k\{\mathbf{x}_k\mathbf{x}_k^\top\} = \boldsymbol{\pi}_k\boldsymbol{\pi}_k^\top + [I \ \ \boldsymbol{\Pi}_k]\,\boldsymbol{\Omega}_k \begin{bmatrix} I \\ \boldsymbol{\Pi}_k^\top \end{bmatrix}, \tag{13a}$$

$$\mathbb{E}_k\{\mathbf{u}_k\mathbf{u}_k^\top\} = (\mathbf{K}\boldsymbol{\pi}_k + \mathbf{c}_k)(\mathbf{K}\boldsymbol{\pi}_k + \mathbf{c}_k)^\top + [0 \ \ \mathbf{L}_k + \mathbf{K}\boldsymbol{\Pi}_k]\,\boldsymbol{\Omega}_k \begin{bmatrix} 0 \\ (\mathbf{L}_k + \mathbf{K}\boldsymbol{\Pi}_k)^\top \end{bmatrix}. \tag{13b}$$

Proof: From (8), (10) we have $\mathbb{E}_k\{\mathbf{x}_k - \hat{\mathbf{x}}_k\} = 0$. Therefore $\boldsymbol{\zeta}_k = \boldsymbol{\Gamma}_k\mathbf{C}(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \boldsymbol{\Gamma}_k\mathbf{v}_k$ implies $\mathbb{E}_k\{\boldsymbol{\zeta}_k\} = 0$ and (12a,b) follow from the expectations of (9a,b). To determine the second moments of $\mathbf{x}_k$ and $\mathbf{u}_k$, let

$$\mathbf{X}_k := \mathbb{E}_k\left\{ \begin{bmatrix} \mathbf{x}_k - \hat{\mathbf{x}}_k \\ \hat{\mathbf{x}}_k \end{bmatrix} \begin{bmatrix} \mathbf{x}_k - \hat{\mathbf{x}}_k \\ \hat{\mathbf{x}}_k \end{bmatrix}^\top \right\}.$$

Then from (8) and (9a) we have

$$\mathbf{X}_k = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{\pi}_k\boldsymbol{\pi}_k^\top \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \boldsymbol{\Pi}_k \end{bmatrix} \boldsymbol{\Omega}_k \begin{bmatrix} I & 0 \\ 0 & \boldsymbol{\Pi}_k \end{bmatrix}^\top, \tag{14}$$

and (13a,b) follow from $\mathbb{E}_k\{\mathbf{x}_k\mathbf{x}_k^\top\} = [I \ \ I]\,\mathbf{X}_k\,[I \ \ I]^\top$ and (9a,b), respectively. □

Since $\boldsymbol{\pi}_k$ and $\boldsymbol{\Pi}_k$ are linear in $(\mathbf{c}_k, \mathbf{L}_k)$ and $\boldsymbol{\Omega}_k$ is independent of $(\mathbf{c}_k, \mathbf{L}_k)$, it is clear from (12a,b) and (13a,b) that the first moments of the predicted state and input sequences are linear in $\theta_k = (\mathbf{c}_k, \mathbf{L}_k)$ while their second moments are quadratic functions of θ_k.

To determine $\boldsymbol{\Omega}_k$, note that $\mathbf{x}_k - \hat{\mathbf{x}}_k$ and $\boldsymbol{\zeta}_k$ can be written

$$\mathbf{x}_k - \hat{\mathbf{x}}_k = F(\boldsymbol{\Gamma}_k)q_k, \quad \boldsymbol{\zeta}_k = G(\boldsymbol{\Gamma}_k)q_k, \quad q_k = \begin{bmatrix} x_k - \hat{x}_k \\ \mathbf{v}_k \\ \mathbf{w}_k \end{bmatrix},$$

with $F(\boldsymbol{\Gamma}_k) = \begin{bmatrix} \mathbf{S}_\Psi & -\mathbf{T}_{(\Psi,A)}\mathbf{M}\boldsymbol{\Gamma}_k & \mathbf{T}_{(\Psi,D)} \end{bmatrix}$ and $G(\boldsymbol{\Gamma}_k) = \boldsymbol{\Gamma}_k\mathbf{C}F(\boldsymbol{\Gamma}_k) + [0 \ \ \boldsymbol{\Gamma}_k \ \ 0]$. So, by the law of total expectation,

$$\boldsymbol{\Omega}_k = \sum_j \begin{bmatrix} F(\boldsymbol{\Gamma}^{(j)}) \\ G(\boldsymbol{\Gamma}^{(j)}) \end{bmatrix} \mathbb{E}\{q_kq_k^\top\} \begin{bmatrix} F(\boldsymbol{\Gamma}^{(j)}) \\ G(\boldsymbol{\Gamma}^{(j)}) \end{bmatrix}^\top \mathbb{P}\{\boldsymbol{\Gamma}_k = \boldsymbol{\Gamma}^{(j)}\}, \tag{15}$$

where $\mathbb{E}\{q_kq_k^\top\}$ is the block-diagonal matrix:

$$\mathbb{E}\{q_kq_k^\top\} = \operatorname{diag}\{\Sigma_k, \bar{\Sigma}_v, \bar{\Sigma}_w\},$$

$$\bar{\Sigma}_v = I_{N\times N} \otimes \Sigma_v, \quad \bar{\Sigma}_w = I_{N\times N} \otimes \Sigma_w,$$

and where $\boldsymbol{\Gamma}^{(j)}$ for $j = 1, \ldots, 2^N$ enumerates the 2^N matrices with binary-valued diagonal elements defined by

$$\boldsymbol{\Gamma}^{(1)} = 0, \quad \boldsymbol{\Gamma}^{(2)} = \operatorname{diag}\{0, \ldots, 0, 1\} \otimes I_{n_y\times n_y} \quad \ldots$$

$$\ldots \quad \boldsymbol{\Gamma}^{(2^N-1)} = \operatorname{diag}\{1, \ldots, 1, 0\} \otimes I_{n_y\times n_y}, \quad \boldsymbol{\Gamma}^{(2^N)} = I.$$

Remark 3. $\boldsymbol{\Omega}_k$ in (15) can be computed conveniently via $\operatorname{vec}(\boldsymbol{\Omega}_k) =$

$$\left(\sum_j \begin{bmatrix} F(\boldsymbol{\Gamma}^{(j)}) \\ G(\boldsymbol{\Gamma}^{(j)}) \end{bmatrix} \otimes \begin{bmatrix} F(\boldsymbol{\Gamma}^{(j)}) \\ G(\boldsymbol{\Gamma}^{(j)}) \end{bmatrix} \mathbb{P}\{\boldsymbol{\Gamma}_k = \boldsymbol{\Gamma}^{(j)}\}\right) \operatorname{vec}\left(\begin{bmatrix} \Sigma_k & & \\ & \bar{\Sigma}_v & \\ & & \bar{\Sigma}_w \end{bmatrix}\right)$$

where the first term on the RHS can be determined offline given the probability distribution of γ_k. This allows $\boldsymbol{\Omega}_k$ to be computed online using the current value of Σ_k with a single matrix-vector multiplication.

Using the same arguments as the proof of Proposition 2, it can be verified that

$$X_{N|k} = \mathbb{E}_k\left\{ \begin{bmatrix} x_{N|k} - \hat{x}_{N|k} \\ \hat{x}_{N|k} \end{bmatrix} \begin{bmatrix} x_{N|k} - \hat{x}_{N|k} \\ \hat{x}_{N|k} \end{bmatrix}^\top \right\}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \pi_{N|k}\pi_{N|k}^\top \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \Pi_{N|k} \end{bmatrix} \Omega_{N|k} \begin{bmatrix} I & 0 \\ 0 & \Pi_{N|k} \end{bmatrix}^\top \tag{16}$$

where

$\Pi_{N|k} = T^N_{(\Phi,B)}\mathbf{L}_k + T^N_{(\Phi,A)}\mathbf{M}, \ \pi_{N|k} = S^N_\Phi \hat{x}_k + T^N_{(\Phi,B)}\mathbf{c}_k,$

$$\Omega_{N|k} = \sum_j \begin{bmatrix} F_N(\boldsymbol{\Gamma}^{(j)}) \\ G(\boldsymbol{\Gamma}^{(j)}) \end{bmatrix} \begin{bmatrix} \Sigma_k & & \\ & \bar{\Sigma}_v & \\ & & \bar{\Sigma}_w \end{bmatrix} \begin{bmatrix} F_N(\boldsymbol{\Gamma}^{(j)}) \\ G(\boldsymbol{\Gamma}^{(j)}) \end{bmatrix}^\top \mathbb{P}\{\boldsymbol{\Gamma}_k = \boldsymbol{\Gamma}^{(j)}\}$$

with $F_N(\boldsymbol{\Gamma}_k) = \begin{bmatrix} S^N_\Psi & -T^N_{(\Psi,A)}\mathbf{M}\boldsymbol{\Gamma}_k & T^N_{(\Psi,D)} \end{bmatrix}$.

5. COST AND CONSTRAINTS

We next show that the cost and constraints of (3) can be expressed as convex functions of $\theta_k = (\mathbf{c}_k, \mathbf{L}_k)$. First note that the objective in (3) can be written

$$\sum_{i=0}^{\infty} \beta^i \mathbb{E}_k\big\{\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big\}$$

$$= \operatorname{tr}(\mathbf{Q}_\beta\mathbf{X}_k) + \operatorname{tr}(\mathbf{R}_\beta\mathbf{U}_k) + f_N(\theta_k, \hat{x}_k, \Sigma_k) \tag{17}$$

where $\mathbf{X}_k$ is given by (14), and

$\mathbf{U}_k := \mathbb{E}_k\{\mathbf{u}_k\mathbf{u}_k^\top\}, \mathbf{Q}_\beta := \mathbf{1}_{2\times 2} \otimes \operatorname{diag}\{Q, \beta Q, \ldots, \beta^{N-1}Q\},$

$\mathbf{R}_\beta := \operatorname{diag}\{R, \beta R, \ldots, \beta^{N-1}R\},$

$$f_N(\theta_k, \hat{x}_k, \Sigma_k) := \sum_{i=N}^{\infty} \beta^i \mathbb{E}_k\big\{\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big\}.$$

Since $\mathbf{Q}_\beta \succeq 0$ and $\mathbf{R}_\beta \succ 0$, the term $\operatorname{tr}(\mathbf{Q}_\beta\mathbf{X}_k) + \operatorname{tr}(\mathbf{R}_\beta\mathbf{U}_k)$ in (17) can be expressed as a convex quadratic function of $\theta_k = (\mathbf{c}_k, \mathbf{L}_k)$ using (13b) and (14). To determine the terminal term, $f_N(\theta, \hat{x}_k, \Sigma_k)$, let $P_k = \sum_{i=N}^\infty \beta^i X_{i|k}$, where

$$X_{i|k} = \mathbb{E}_k\left\{ \begin{bmatrix} x_{i|k} - \hat{x}_{i|k} \\ \hat{x}_{i|k} \end{bmatrix} \begin{bmatrix} x_{i|k} - \hat{x}_{i|k} \\ \hat{x}_{i|k} \end{bmatrix}^\top \right\}.$$

Then for $i \geq N$ we have

$$X_{i+1|k} = \mathbb{E}\big\{\tilde{\Psi}(\gamma)X_{i|k}\tilde{\Psi}^\top(\gamma)\big\} + \mathbb{E}\big\{\tilde{D}(\gamma)\begin{bmatrix} \Sigma_v & \\ & \Sigma_w \end{bmatrix}\tilde{D}^\top(\gamma)\big\}$$

where

$$\tilde{\Psi}(\gamma) = \begin{bmatrix} A(I - \gamma MC) & 0 \\ \gamma AMC & \Phi \end{bmatrix}, \quad \tilde{D}(\gamma) = \begin{bmatrix} -\gamma AM & D \\ \gamma AM & 0 \end{bmatrix},$$

and γ is a random variable identically distributed as γ_k. Hence

$$\mathbb{E}\big\{\tilde{\Psi}(\gamma)P_k\tilde{\Psi}^\top(\gamma)\big\}$$

$$= \sum_{i=N}^{\infty} \beta^i \big(X_{i+1|k} - \mathbb{E}\{\tilde{D}(\gamma)\begin{bmatrix} \Sigma_v & \\ & \Sigma_w \end{bmatrix}\tilde{D}^\top(\gamma)\}\big)$$

$$= \beta^{-1}(P_k - \beta^N X_{N|k}) - \frac{\beta^N}{1-\beta}\mathbb{E}\{\tilde{D}(\gamma)\begin{bmatrix} \Sigma_v & \\ & \Sigma_w \end{bmatrix}\tilde{D}^\top(\gamma)\},$$

and the terminal term $f_N(\theta_k, \hat{x}_k, \Sigma_k)$ in (17) is equal to

$$\operatorname{tr}\left(\begin{bmatrix} Q & Q \\ Q & Q + K^\top RK \end{bmatrix} P_k\right)$$

with the additional constraint

$$P_k \succeq \beta\mathbb{E}\big\{\tilde{\Psi}(\gamma)P_k\tilde{\Psi}^\top(\gamma)\big\} + \beta^N X_{N|k} + \frac{\beta^{N+1}}{1-\beta}\mathbb{E}\big\{\tilde{D}(\gamma)\begin{bmatrix} \Sigma_v & \\ & \Sigma_w \end{bmatrix}\tilde{D}^\top(\gamma)\big\}. \tag{18}$$

Using (16) and Schur complements, (18) can be expressed as a linear matrix inequality in $\theta_k = (\mathbf{c}_k, \mathbf{L}_k)$ and P_k.

Re-writing the constraints of problem (3) using the matrix, $X_{i|k}$, of second moments yields the condition

$$\sum_{i=0}^{\infty} \beta^i \operatorname{tr}\big[(\mathbf{1}_{2\times 2} \otimes H^\top H) X_{i|k}\big] \le \mu_k,$$

which is equivalent to the constraint

$$\operatorname{tr}(\mathbf{H}_\beta \mathbf{X}_k) + \operatorname{tr}\big[(\mathbf{1}_{2\times 2} \otimes H^\top H) P_k\big] \le \mu_k \tag{19}$$

where $\mathbf{H}_\beta = \mathbf{1}_{2\times 2} \otimes \operatorname{diag}\{H^\top H, \beta H^\top H, \ldots, \beta^{N-1} H^\top H\}$.

The expressions for the cost and constraints in (17)-(19) allow the optimization (3) defining θ_k^* to be formulated as

$$\theta_k^* = \arg\min_{\theta_k, P_k} \operatorname{tr}(\mathbf{Q}_\beta \mathbf{X}_k) + \operatorname{tr}(\mathbf{R}_\beta \mathbf{U}_k) + \operatorname{tr}\Big(\begin{bmatrix} Q & Q \\ Q & Q+K^\top RK \end{bmatrix} P_k\Big)$$

$$\text{s.t. } (18), (19). \tag{20}$$

Remark 4. Problem (20) can be expressed as a semidefinite program in the variables $\theta_k = \{\mathbf{c}_k, \mathbf{L}_k\}$ and P_k using (13b), (14) and (16). Alternatively, we can eliminate P_k from (20) by writing the solution of the Lyapunov equation $P_k = \beta \mathbb{E}\{\tilde{\Psi}(\gamma) P_k \tilde{\Psi}^\top(\gamma)\} + \Xi$ for given $\Xi = \Xi^\top$ as $\operatorname{vec}(P_k) = \big[I - \beta(1-\lambda)\tilde{\Psi}(0)\otimes\tilde{\Psi}(0) - \beta\lambda\tilde{\Psi}(1)\otimes\tilde{\Psi}(1)\big]^{-1}\operatorname{vec}(\Xi)$ (where the matrix inverse can be computed offline), which allows problem (20) to be expressed, using standard matrix vectorization identities, as a convex quadratic program in $\theta_k = (\mathbf{c}_k, \mathbf{L}_k)$ with a single quadratic constraint.

6. CLOSED LOOP PROPERTIES

This section considers the performance of the closed loop system (1) with the control law of Section 3.1. We use the solution $\theta_k^* = \{\mathbf{c}_k^*, \mathbf{L}_k^*\}$ of (3) at time k to construct a feasible, but possibly suboptimal, solution for (3) at time $k+1$ (i.e. given $\mathcal{U}_k$, $\mathcal{I}_k$), which we denote $\theta_{k+1}^\circ = \{\mathbf{c}_{k+1}^\circ, \mathbf{L}_{k+1}^\circ\}$, where

$$\mathbf{c}_{k+1}^\circ := \begin{bmatrix} c_{1|k}^* \\ \vdots \\ c_{N-1|k}^* \\ 0 \end{bmatrix} + \begin{bmatrix} L_{1,0|k}^* \\ \vdots \\ L_{N-1,0|k}^* \\ 0 \end{bmatrix} \gamma_k (y_k - C\hat{x}_k), \tag{21a}$$

$$\mathbf{L}_{k+1}^\circ := \begin{bmatrix} L_{1,1|k}^* & & \\ \vdots & \ddots & \\ L_{N-1,1|k}^* & \cdots & L_{N-1,N-1|k}^* \\ 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{21b}$$

Following Yan et al. (2018), we define the constraint threshold μ_k in (3) for all $k > 0$ in terms of θ_k°. This ensures recursive feasibility of the MPC optimization without requiring bounds on the noise v_k and disturbance w_k. Thus

$$\mu_k := \begin{cases} \epsilon, & k = 0 \\ \operatorname{tr}(\mathbf{H}_\beta \mathbf{X}_k^\circ) + \operatorname{tr}\Big[(\mathbf{1}_{2\times 2} \otimes H^\top H) P_k^\circ\Big], & k > 0 \end{cases} \tag{22}$$

where

$$P_k^\circ = \beta \mathbb{E}\{\tilde{\Psi}(\gamma) P_k^\circ \tilde{\Psi}^\top(\gamma)\} + \beta^N X_{N|k}^\circ + \frac{\beta^{N+1}}{1-\beta} \mathbb{E}\{\tilde{D}(\gamma) \begin{bmatrix} \Sigma_v & \\ & \Sigma_w \end{bmatrix} \tilde{D}^\top(\gamma)\},$$

$$\mathbf{X}_k^\circ = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{\pi}_k^\circ \boldsymbol{\pi}_k^{\circ\top} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \boldsymbol{\Pi}_k^\circ \end{bmatrix} \boldsymbol{\Omega}_k \begin{bmatrix} I & 0 \\ 0 & \boldsymbol{\Pi}_k^\circ \end{bmatrix}^\top,$$

$$X_{N|k}^\circ = \begin{bmatrix} 0 & 0 \\ 0 & \pi_{N|k}^\circ \pi_{N|k}^{\circ\top} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \Pi_{N|k}^\circ \end{bmatrix} \Omega_{N|k} \begin{bmatrix} I & 0 \\ 0 & \Pi_{N|k}^\circ \end{bmatrix}^\top$$

with $\boldsymbol{\pi}_k^\circ = \mathbf{S}_\Phi \hat{x}_k + \mathbf{T}_{(\Phi,B)} \mathbf{c}_k^\circ$, $\boldsymbol{\Pi}_k^\circ = \mathbf{T}_{(\Phi,B)} \mathbf{L}_k^\circ + \mathbf{T}_{(\Phi,A)} \mathbf{M}$.

Theorem 5. If problem (3) is feasible at $k = 0$, then (3) remains feasible for all $k > 0$ and the state of (1) under the control law of Section 3.1 satisfies

$$\sum_{k=0}^{\infty} \beta^k \mathbb{E}\{\|H x_k\|^2\} \le \epsilon. \tag{23}$$

Proof: The definition (22) of μ_k trivially ensures feasibility for all $k > 0$. The definitions (21a,b) ensure that, at time k (given $\mathcal{U}_{k-1}$, $\mathcal{I}_{k-1}$), the distributions of the state and control sequences $\{x_{i|k+1}\}_{i=0}^\infty$ and $\{u_{i|k+1}\}_{i=0}^\infty$ are identical to the distributions of $\{x_{i+1|k}\}_{i=0}^\infty$ and $\{u_{i+1|k}\}_{i=0}^\infty$. Therefore

$$\sum_{i=0}^{\infty} \beta^i \mathbb{E}_k\{\|H x_{i|k}\|^2\} = \operatorname{tr}(\mathbf{H}_\beta \mathbf{X}_k) + \operatorname{tr}\big[(\mathbf{1}_{2\times 2} \otimes H^\top H) P_k\big]$$

implies

$$\beta \mathbb{E}_k\{\mu_{k+1}\} \le \mu_k - \mathbb{E}_k\{\|H x_{0|k}\|^2\} = \mu_k - \mathbb{E}_k\{\|H x_k\|^2\}.$$

Hence the trajectories of the closed loop system satisfy

$$\sum_{i=0}^{\infty} \beta^i \mathbb{E}_k\{\|H x_{k+i}\|^2\} \le \mu_k - \lim_{i\to\infty} \beta^i \mathbb{E}_k\{\mu_{k+i}\} \le \mu_k$$

for all $k \ge 0$. □

Corollary 6. Let $J_k := J(\theta_k^*, \hat{x}_k, \Sigma_k)$ denote the optimal value of the objective in (3). Then under the control law of Section 3.1, the trajectories of (1) satisfy

$$\sum_{k=0}^{\infty} \beta^k \mathbb{E}\{\|x_k\|_Q^2 + \|u_k\|_R^2\} \le J_0. \tag{24}$$

Proof: Applying the same argument used in the proof of Theorem 5 to the definition of the objective in (3) yields

$$\beta \mathbb{E}_k\{J(\theta_{k+1}^\circ, \hat{x}_{k+1}, \Sigma_{k+1})\} = J_k - \mathbb{E}_k\{\|x_k\|_Q^2 + \|u_k\|_R^2\},$$

and since $J_k \le J(\theta_k^\circ, \hat{x}_k, \Sigma_k)$ $\forall k$ by optimality, the bound in (24) follows. □

7. NUMERICAL EXAMPLES

This section gives a numerical example to demonstrate that the closed loop system satisfies (23) and (24) and to compare with the unconstrained optimal LQG controller. We consider a system obtained by discretising a linearised continuous time model of a double inverted pendulum with a sample time of 0.01 s as in (Kwakernaak and Westdyk, 1985). The system matrices are

$$A = \begin{bmatrix} 1.0005 & 0.01 & -0.0005 & 0 \\ 0.098 & 1.0005 & -0.0981 & -0.0005 \\ -0.0005 & 0 & 1.0015 & 0.01 \\ -0.0981 & -0.0005 & 0.2942 & 1.0015 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0001 & -0.0001 \\ 0.01 & -0.02 \\ -0.0001 & 0.0003 \\ -0.02 & 0.05 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = I,$$

and $\omega_k \sim \mathcal{N}(0, \Sigma_w)$, $v_k \sim \mathcal{N}(0, \Sigma_v)$, $\lambda = 0.6$. Here $\Sigma_w = \operatorname{diag}\{0.5, 0.2, 0.9, 0.3\}$ and $\Sigma_v = 1.1I$. Initial conditions are given by

$$x_0 = \begin{bmatrix} -0.8 \\ 0.4 \\ 0.55 \\ -0.5 \end{bmatrix}, \ \hat{x}_0 = \begin{bmatrix} 0.1 \\ 0.05 \\ 0.1 \\ 0.05 \end{bmatrix}, \ \Sigma_0 = \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}.$$

The constraint of (2) is defined by $\beta = 0.95$, $\epsilon = 111$ and

$$H = \begin{bmatrix} 0 & 0.1 & 0 & -0.1 \\ 0.1 & 0 & -0.1 & 0 \end{bmatrix}.$$

The weighting matrices in the cost function of (2) are given by $Q = \operatorname{diag}\{10, 0.1, 10, 0.1\}$, $R = 10^{-4}I$. We choose a prediction horizon $N = 5$, K as the unconstrained LQ-optimal, K_{LQ}, with respect to (A, B, Q, R) and $M =$

$\bar{\Sigma}C^\top(C\bar{\Sigma}C^\top + \Sigma_v)^{-1}$, where $\bar{\Sigma}$ is the solution of the algebraic Riccati equation

$$\bar{\Sigma} = A\bar{\Sigma}A^\top + \Sigma_w - \lambda A\bar{\Sigma}C^\top(C\bar{\Sigma}C^\top + \Sigma_v)^{-1}C\bar{\Sigma}A^\top.$$

Using the above information, we solve problem (20) and obtain $J_0 = 2.368 \times 10^4$.

Simulation A: To estimate empirically the LHS of (23) and (24), we consider their average values over 10^3 simulations, each of which has a length of 500 time steps. This gives $\sum_{k=0}^\infty \beta^k \mathbb{E}\{\|Hx_k\|^2\}$ and $\sum_{k=0}^\infty \beta^k \mathbb{E}\{\|x_k\|_Q^2 + \|u_k\|_R^2\}$ as 104.7 and 4.774×10^3 respectively. Therefore, these estimates agree with the bound (23) and (24). Moreover, $\beta^{500} = 7.3 \times 10^{-12}$, so a further increase in the horizon length has negligible effect on these estimates.

Simulation B: To compare with the above results, we run the same number of simulations with the same $\{\omega_k\}$, $\{v_k\}$,$\{\gamma_k\}$ sequences using the unconstrained optimal LQG controller, where $u_k = K_{LQ}\hat{x}_k$ and the estimator gain is time-varying and given by $M = \Sigma_k C^\top(C\Sigma_k C^\top + \Sigma_v)^{-1}$. Here Σ_k evolves as

$$\Sigma_{k+1} = A\Sigma_k A^\top + \Sigma_w - \gamma_k A\Sigma_k C^\top(C\Sigma_k C^\top + \Sigma_v)^{-1}C\Sigma_k A^\top.$$

This gives $\sum_{k=0}^\infty \beta^k \mathbb{E}\{\|Hx_k\|^2\}$ as 123.8, violating the bound (23), and $\sum_{k=0}^\infty \beta^k \mathbb{E}\{\|x_k\|_Q^2 + \|u_k\|_R^2\}$ as 3.626×10^3, which is smaller than that in Simulation A, as expected.

8. CONCLUSION

This paper describes an output feedback MPC algorithm for linear discrete time systems with additive disturbances and noisy sensor measurements transmitted over a packet-dropping communication channel. By designing a control policy with an affine dependence on future observations, we provide a convex formulation of a stochastic quadratic regulation problem subject to a discounted expectation constraint. Our controller parameterization ensures recursive feasibility of the MPC optimization problem and ensures a cost bound and constraint satisfaction in closed loop operation. Future work will explore interconnections between conditions for mean square stability of the MPC law and the values of the packet loss probability and the discount factor in the receding horizon optimization.

ACKNOWLEDGEMENTS

The authors would like to thank Prof. Subhrakanti Dey for helpful discussions about early versions of this paper.

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