Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Distributing Potential Games on Graphs Part I. Game formulation

Mohamed I. El-Hawwary *, Jonas Mårtensson ***

* *Division of Decision and Control Systems, School of Electrical Engineering & Computer Science, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden*
** *Department of Electrical Power Engineering, Faculty of Engineering, Cairo University, Giza 12613, Egypt*
(e-mails: melhawwary@eng.cu.edu.eg, jonas1@kth.se)

Abstract: The paper presents the problem of distributing potential games over communication graphs. Suppose a potential game can be designed for a group of agents (players) where each has access to all others' actions (strategies). The paper shows how to design a corresponding potential game for these agents if the full information assumption is replaced with communication over a network depicted by undirected graphs with certain properties. A state-based formulation for potential games is utilized. This provides degrees of freedom to handle the previous information limitation. Notions of Nash's equilibria for the developed game (called here distributed potential game) are presented, and relations between these equilibria and those of the full information game are studied. In part II of the paper learning Nash equilibria for the newly developed game is studied. The development focuses on providing a way to utilize available algorithms of the full information game. The motivation for the results comes from a platoon matching problem for heavy duty vehicles. Utilizing the newly developed distributed game, recent results based on potential games can be extended, providing a basis for an on-the-go strategy where platoon matching on road networks can be solved locally.

Keywords: Potential games, distributed optimization, multi-agents.

1. INTRODUCTION

Originally developed as a modeling method for describing interacting behaviours in societal systems Siegfried (2006), game theory has recently stood as a valuable tool in the study and control of distributed engineering systems. As in social systems, distributed engineering systems feature interactions between decision making elements. Ensuing collective behaviours of these interactions rely mainly on local decisions which are usually based on partial information. Game theory has been proven as a valuable tool in the study and control of different complex systems Marden and Shamma (2013).

Game theory has received particular interest in solving optimization problems of multiagent systems. The game theoretic approach has been used as a tool to solve global optimization problems of distributed systems Li and Marden (2013). Central solutions of optimization and control of multiagents face several limitations, e.g. those related to central computing. Game theoretic approaches are generally based on distributed adaptation rules which can mitigate those limitations. In that context, gaming methods can be used to solve central optimization problems if games can be designed such that their equilibria coincide with the desired global optimal states.

Conversely, game theoretic approaches can more naturally handle problems of competing agents, in situations where different agents are trying to optimize their individual objectives. The application presented in part II of this paper is an example of such situations. Those individual objectives in most cases do not depend only on individual actions, but also on joint variables. This is what elicits the game theoretic formulation.

Information sharing limitations in distributed engineering problems presents a challenge whether in the central or distributed applications of game theory. This paper provides a tool to address this challenge in situations where a potential game can be designed assuming full information access for all agents. The notion of Potential Games was coined in Monderer and Shapley (1996). These are non-cooperative games for which one can find auxiliary functions, denoted potential functions, with maxima corresponding to the Nash equilibria of the game. Potential games enjoy several useful properties González-Sánchez and Hernández-Lerma (2016). In a potential game a pure strategy Nash equilibrium is guaranteed to exist if certain conditions of the potential function are satisfied. Many systematic results for convergence of learning algorithms has been established for this class. Also, in potential games instances of coincidence of cooperative and Nash equilibria can be easily identified. Conversely however, (static) potential games suffer from limitations in handling multiagent systems challenges. For instance, often for multiagents actions must satisfy desired coupled constraints while achieving specified behaviours. Another is the derivation of local objective functions for multiagents coordination. It has been shown that there are theoretical obstacles in using potential games to handle these requirements Li and Marden (December, 2010), Li and Marden (2013). To overcome this, here we use the framework of state-based games Marden (2012). This framework endows the game with a state space structure which provides extra degrees of freedom that can help address such challenges.

In this paper we address the following problem. Consider a strategic game where all players have access to all others' strategies, called here the full information game. Suppose that individual utilities can be found such that the game can be formulated as an (exact) potential game, i.e. an (exact) potential function can be found for the previous utilities. This paper answers the following questions. If the same players don't have full access to others' strategies, but only a subset of those according to a communication graph, can a potential game be formulated for the new game? This new restricted information game shall be called distributed potential game. If yes, what is the relationship between the new game equilibria (Nash's) and that of the full information one. Moreover, in part II of the paper the following is addressed. Can a learning algorithm be devised for the new game that matches the outcome, i.e. the result converges to the same equilibria of the old one, without "much" increase in convergence time, and such that the algorithm works whether play is carried out player by player or some groups at once?

This problem is motivated from a platoon matching problem for heavy duty vehicles where, if a comprehensive algorithm is to be designed that would require vehicles applying a learning algorithm while moving, such algorithm can consequently face limited information access. According to the authors' knowledge no similar problem has been addressed in literature. Some results addressed games where utilities are based only on communicated agents information (local) as in Tekin et al. (2012). However, certain applications such as the platoon matching problem mentioned previously, and further studied in part II, would require individual utilities to be based on all agents actions, and hence the problem addressed here.

The paper is organized as follows. Section 2 presents preliminaries, assumptions and definitions. The notion of distributing a potential game over a communication graph is presented in Section 3, together with the motivating platooning problem. The distributed game is provided in Section 4, and equilibrium definitions and properties are presented in Section 5. Finally, conclusions are given in Section 6.

2. PRELIMINARIES

Consider a strategic form game with a set of agents $\mathcal{N} = \{1, \cdots, n\}$. Let for each $i \in \mathcal{N}$, S_i denote the set of strategies (actions) available to agent i. The set of joint strategies is denoted by $S := S_1 \times \cdots \times S_n$ with elements $s = (s_1, \cdots, s_n)$ the joint strategy profile. For a profile s, $s_{-i} = \{s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n\} \in S_{-i} := S_1 \times \cdots S_{i-1} \times S_{i+1} \cdots \times S_n$ denotes the strategies of all but i players. The functions $u_i : S \to \mathbb{R}$, $i \in \mathcal{N}$, denote the utility functions of the agents. This game will be denoted by $\langle \mathcal{N}, (S_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}} \rangle$. The interest in this paper is in finding solutions depicted by Nash equilibria.

Definition 1. A strategy profile $s^* \in S$ is a pure Nash equilibrium for the strategic game $\langle \mathcal{N}, (S_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}} \rangle$ if

$$(\forall i \in \mathcal{N})(\forall s_i \in S_i)\ u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

A Nash equilibrium is a stable state of the game as no player has incentive to deviate, i.e. cannot profitably do so, from that strategy given other players strategies.

2.1 Potential games

The focus here will be on potential games which is a class that enjoy useful properties especially when it comes to learning equilibria Lã et al. (2016).

Definition 2. A strategic game $\langle \mathcal{N}, (S_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}} \rangle$ is called an (exact) potential game if there exits a function $\phi : S \to \mathbb{R}$ such that for each $i \in \mathcal{N}$ and $s_{-i} \in S_{-i}$

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \phi(s_i, s_{-i}) - \phi(s_i', s_{-i}), \forall s_i, s_i' \in S_i.$$

In this case the function ϕ is called a potential function of the game.

Potential functions resemble energy functions of physical systems. This function assigns a value for every $s \in S$, and for every $i \in \mathcal{N}$, $\phi(s_i, s_{-i})$ gives information about $u_i(s_i, s_{-i})$. Among the useful properties of potential games is that a pure strategy Nash equilibrium is guaranteed to exist if the strategy sets are finite or compact. In this case, a best reply dynamics Nisan et al. (2008) converges to a Nash equilibrium limiting strategy. In addition to best reply dynamics, several learning algorithms with convergence guarantees exist. Also, learning Nash equilibrium for potential games is inherently robust. These properties make potential games a useful tool for solving engineering optimization problems.

2.2 State-based Games

A main tool used here to address strategic games with limited information is state-based games. A simplified version of stochastic games Shapley (1953), state-based games are an extension of strategic form games where a state space structure is introduced in the gaming environment.

Consider the set of $\mathcal{N}$ agents and a state space $\mathcal{X}$. Here the state space will encompass the set of strategies and their estimates. Define state dependent actions sets $\mathcal{A}_i(x)$ for $i \in \mathcal{N}$ and $x \in \mathcal{X}$. The joint action profile is $a \in \mathcal{A}(x)$ where $\mathcal{A}(x) := \mathcal{A}_1(x) \times \cdots \times \mathcal{A}_n(x)$. The agents utilities are defined by functions $v_i : \mathcal{X} \times \mathcal{A}(x) \to \mathbb{R}$. This game will be denoted by $\langle \mathcal{N}, \mathcal{X}, \mathcal{A}, (v_i)_{i\in\mathcal{N}}, f \rangle$ where $\mathcal{A} := \cup_{x\in\mathcal{X}} \mathcal{A}(x)$, and $f : \mathcal{X} \times \mathcal{A} \to \mathcal{X}$ is a deterministic function which defines the state transition (evolution) as influenced in part by actions. At (learning) time $\mathtt{t} \geq 0$, each agent action is selected such that $a_i(\mathtt{t}) \in \mathcal{A}_i(x(\mathtt{t}))$, i.e. it depends on the current state. The current state $x(\mathtt{t})$, and the joint action $a(\mathtt{t}) \in \mathcal{A}(x(\mathtt{t}))$ determine the ensuing state $x(\mathtt{t} + 1) = f(x(\mathtt{t}), a(\mathtt{t}))$, and the current utilities $v_i(x(\mathtt{t}), a(\mathtt{t}))$.

Definition 3. For the state-based game $\langle \mathcal{N}, \mathcal{X}, \mathcal{A}, (v_i)_{i\in\mathcal{N}}, f \rangle$ an action profile $\mathbf{0} \in \mathcal{A}(x)$ is called a null action if $x = f(x, \mathbf{0})$.

The definition (corresponding to Definition 2) of (exact) potential games for state-based games was given in Li and Marden (2013) as follows.

Definition 4. A (deterministic) state-based game $\langle \mathcal{N}, \mathcal{X}, \mathcal{A}, (v_i)_{i\in\mathcal{N}}, f \rangle$ with null action $\mathbf{0}$ is called a (deterministic) state-based potential game if there exists a function $\Phi : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$ such that for every $x \in \mathcal{X}$, $i \in \mathcal{N}$, $a_{-i} \in \mathcal{A}_{-i}(x)$ [1],

$$v_i(x, a_i, a_{-i}) - v_i(x, a'_i, a_{-i}) = \Phi(x, a_i, a_{-i}) - \Phi(x, a'_i, a_{-i}),$$

for all $a_i, a'_i \in \mathcal{A}_i(x)$, and for every $a \in \mathcal{A}(x)$ the potential function satisfies

$$\Phi(x, a) = \Phi(\tilde{x}, \mathbf{0})$$

where $\tilde{x} = f(x, a)$ denotes the ensuing state.

Remark 5. Here too Nash equilibria are the solutions goals. Two equilibrium definitions, pertinent to the development, for state-based games are presented in the following section.

3. DISTRIBUTING POTENTIAL GAMES

Consider a strategic form game $\langle \mathcal{N}, (S_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}} \rangle$ for which one can find an (exact) potential function $\phi : S \to \mathbb{R}$. Now, suppose that each agent $i \in \mathcal{N}$ can access the strategies of only a subset $\mathcal{N}_i$ of the agents, according to a connection graph $\mathcal{G}$.

Assumption 1. $\mathcal{G}$ is an undirected graph. If agent i sees (communicates with) agent j according to $\mathcal{G}$ then agent i has access to the full strategy profile as seen by j. It is not assumed though that agent i has access to $\mathcal{N}_j$, i.e. it doesn't know which of the strategies seen by j are true and which are estimates except for s_j.

Assumption 2. For all $i \in \mathcal{N}$, agent $j \in \mathcal{N}_i$ passes on to agent i all the values it receives for s_i from the other agents in $\mathcal{N}_j$.

By distributing the previous potential game on $\mathcal{G}$, it is meant here to solve the following.

- Find new utility functions and $\Phi(S, \cdot)$ with codomain $\mathbb{R}$, satisfying Assumptions 1 and 2, and utilizing if possible the full information utility functions and the potential function ϕ, such that in some sense (determine what) the game is a potential game with potential function Φ.
- Determine the Nash equilibria for the new game.

In part II of the paper the following is also addressed.

- Find a learning algorithm, utilizing if possible the available ones for the full information game, such that the strategies of the new game converges to an equilibrium of the latter. Such learning algorithm should not require significantly longer time, and if possible, works when players update their strategies one by one or in groups.

3.1 Motivating application

In part II of the paper the distributed potential game developed here will be used to solve a platoon matching problem for heavy duty vehicles. Truck platooning has acquired significant attention in recent years due to demonstrated potential for fuel savings and other beneficial aspects Tsugawa et al. (2016), Bishop et al. (2017), Alam et al. (2015).

[1] a_{-i} and $\mathcal{A}_{-i}(x)$ are congruent to s_{-i} and S_{-i}.

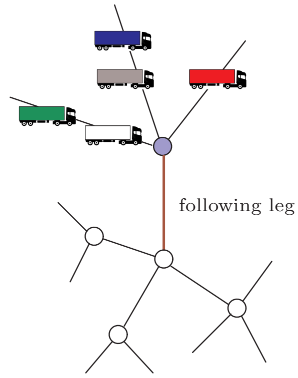

Fig. 1. Platoon matching problem

On of the main challenges of platooning is the matching problem, i.e. how to decide when, where and with whom to platoon. Several methods have been used to solve this problem. One can identify two main classes: centralized optimization and game theoretic approaches. Centralized strategies rely on formulating the problem as optimization problems with a goal of minimizing/maximizing global objective functions Larsson et al. (2015), Liang et al. (2016), van de Hoef et al. (2018). These methods faces several challenges related to the size of the problem: platooning on road networks involves a large number of agents with several solution variables; compatibility of goals where objectives for trucks belonging to different fleets or operators can be difficult to consolidated in a common goal; and information sharing due to communication limitations or privacy restrictions.

A game theoretic approach to platoon matching would be more suitable in situations of conflicting goals such as multi-fleet truck platooning. Game theoretic solutions, where agents are modeled as competing agents seeking to optimize individual profit functions received recent attention Farokhi and Johansson (2013), Johansson et al. (2018).

In Johansson et al. (2018) a potential game was developed to solve platoon matching. The standard assumption was that a group of trucks start their journeys from a common point (e.g. parking lot) in a road network, sharing in their itineraries at least the outgoing road segment from that point. Each truck is assumed to have preferred departure time, but can deviate from those to achieve platooning. Individual utility functions were formulated. If trucks platoon this would mean a saving in their utility functions, and conversely, a deviation from their preferred departure times would translate into a cost.

The algorithm presented in Johansson et al. (2018) for platoon matching requires the following.

- All trucks need to start from the same point.
- Trucks need to share their complete journey plans with all other trucks.

This limits the applicability of the algorithms on actual road networks where trucks can start their journeys from different places, the times of arrival and departure at junctions are affected by external factors such as traffic and weather, and where exchange of itineraries can be limited due to privacy or communication restrictions. Here, it is suggested to apply the gaming algorithm for platooning on-the-go (i.e. whenever vehicles are approaching a junction where platooning is possible), optimizing utilities over only the outgoing link (following leg), refer to Figure 1. This can be used to develop dynamic strategies where platoon matching on large road networks can be handled by applying matching algorithms locally, at junctions where platooning is possible, where trucks repeat the algorithms at every such junction in their journeys. This would offer the advantage of handling a global problem by solving multiple relatively simpler local ones.

In this situation one would have:

- An already developed potential game, which depends on strategies of all vehicles approaching the junction, and that share the same following leg in their itineraries
- As the algorithm is to be applied while the vehicles are moving, i.e. not all are at a common point, it could happen that some vehicles are not able to acquire all others' strategies.

This motivates the restricted information potential game presented next.

4. GAME DESIGN

In this section a strategic game $g = \langle \mathcal{N}, (S_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}} \rangle$ with potential function ϕ is distributed over the graph $\mathcal{G}$ utilizing state-based formulation.

4.1 Distributed potential game on $\mathcal{G}$

Let $e = (e^1, \cdots, e^n) \in (\mathbb{R}^n)^n$ $((\mathbb{R}^n)^n := \mathbb{R}^n \times \cdots \times \mathbb{R}^n)$ define the estimation profile, where $e^i = (e^i_1, \cdots, e^i_n)$ is agent's i estimate of the joint strategy profile s.

Assumption 3. For $j \in \mathcal{N}_i$, $e^i_j = s_j$, i.e. agent i only estimates the strategies of the agents it cannot see.

Definition 6. A state-based version of the strategic game g distributed on $\mathcal{G}$ is given as $G = \langle \mathcal{N}, \mathcal{X}, \mathcal{A}, (v_i)_{i\in\mathcal{N}}, f \rangle$ where

- The state of the game is defined as $x = (s, e)$, the state space is given by $\mathcal{X} = S \times (\mathbb{R}^n)^n$, and the individual state is $x_i := (s_i, e^i)$.
- Each agent is assigned an action set $a_i = (\hat{s}_i, \hat{e}^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i})$ where $\hat{s}_i$ indicates some change in the agent i strategy s_i, and $\hat{e}^i_j$ indicates some change in the agent's estimation strategy e^i_j.
- The joint action profile is defined as $a = (\hat{s}, \hat{e})$ where by Assumption 3, $\hat{e}^i_j = \hat{s}_j$, for all $j \in \mathcal{N}_i$, $i \in \mathcal{N}$.
- The state transition function is defined as $f = (f_s, f_e)$ where the ensuing state $\tilde{x} = (\tilde{s}, \tilde{e})$ is given by
$$\tilde{s} = f_s(s, \hat{s}), \;\; \tilde{e} = f_e(e, \hat{e})$$
with $\tilde{e}^i_j = \tilde{s}_j$, for all $j \in \mathcal{N}_i$.
- The utility function of agent i is defined as follows

$$\begin{aligned} v_i(x,a) = &\sum_{k\in\mathcal{N}_i} u_i(\tilde{s}_j|_{j\in\mathcal{N}_k}, \tilde{e}^k_j|_{j\in\mathcal{N}\setminus\mathcal{N}_k}) \\ &- \alpha \sum_{k\in\mathcal{N}_i}\sum_{l\in\mathcal{N}_k} \mathfrak{p}\left(\tilde{s}_i - \tilde{e}^k_l\right) \\ &- \alpha \sum_{k\in\mathcal{N}_i}\sum_{j\in\mathcal{N}\setminus\mathcal{N}_i} \mathfrak{p}\left(\tilde{e}^i_j - \tilde{e}^k_j\right) \end{aligned} \tag{1}$$

where $\alpha > 0$ and $\mathfrak{p} : \mathbb{R} \to [0, \infty)$.

Remark 7. Note that (1) is equivalent to

$$\begin{aligned} v_i(x,a) = &\sum_{k\in\mathcal{N}_i} u_i(\tilde{s}_j|_{j\in\mathcal{N}_k}, \tilde{e}^k_j|_{j\in\mathcal{N}\setminus\mathcal{N}_k}) \\ - \alpha &\sum_{k\in\mathcal{N}_i}\sum_{l\in\mathcal{N}_k} \mathfrak{p}\left(\tilde{s}_i - \tilde{e}^k_l\right) - \alpha \sum_{k\in\mathcal{N}_i}\sum_{j\in\mathcal{N}\setminus(\mathcal{N}_k\cup\mathcal{N}_i)} \mathfrak{p}\left(\tilde{e}^i_j - \tilde{e}^k_j\right) \\ - \alpha &\sum_{k\in\mathcal{N}_i}\sum_{j\in\mathcal{N}_k\setminus(\mathcal{N}_k\cap\mathcal{N}_i)} \mathfrak{p}\left(\tilde{e}^i_j - \tilde{e}^k_j\right) \end{aligned}$$

However, this form cannot be used as agent i does not have access to $\mathcal{N}_j$ for $j \in \mathcal{N}_i$.

Example 1. Consider 4 agents where agent i can communicate with agents $i-1$ and $i+1$ ($i \bmod 4 \in \mathcal{N}$). The utility functions are given as follows

$$\begin{aligned} v_1 =&u_1(s_1, e^4_2, s_3, s_4) + u_1(s_1, s_2, e^1_3, s_4) + u_1(s_1, s_2, s_3, e^2_4) \\ &- 2\alpha\mathfrak{p}(s_1 - e^3_1) - 2\alpha\mathfrak{p}(e^1_3 - s_3) \\ v_2 =&u_2(s_1, s_2, e^1_3, s_4) + u_2(s_1, s_2, s_3, e^2_4) + u_2(e^3_1, s_2, s_3, s_4) \\ &- 2\alpha\mathfrak{p}(s_2 - e^4_2) - 2\alpha\mathfrak{p}(e^2_4 - s_4) \\ v_3 =&u_3(s_1, s_2, s_3, e^2_4) + u_3(e^3_1, s_2, s_3, s_4) + u_3(s_1, e^4_2, s_3, s_4) \\ &- 2\alpha\mathfrak{p}(s_3 - e^1_3) - 2\alpha\mathfrak{p}(e^3_1 - s_1) \\ v_4 =&u_4(e^3_1, s_2, s_3, s_4) + u_4(s_1, e^4_2, s_3, s_4) + u_4(s_1, s_2, e^1_3, s_4) \\ &- 2\alpha\mathfrak{p}(s_4 - e^2_4) - 2\alpha\mathfrak{p}(e^4_2 - s_2) \end{aligned}$$

The next result shows how the previously defined state-based game is a potential game.

Proposition 8. If the strategic game g is a potential game with potential function ϕ, then the state-based game G in Definition 6 is a potential game according to Definition 4 with potential function

$$\begin{aligned} \Phi(x,a) = &\sum_{i\in\mathcal{N}} \phi\left(\tilde{s}_j|_{j\in\mathcal{N}_i}, \tilde{e}^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i}\right) \\ &- \frac{\alpha}{2}\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{N}_i}\sum_{j\in\mathcal{N}\setminus(\mathcal{N}_k\cup\mathcal{N}_i)} \mathfrak{p}\left(\tilde{e}^i_j - \tilde{e}^k_j\right) \\ &- \alpha\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{N}_i}\sum_{j\in\mathcal{N}_k\setminus(\mathcal{N}_k\cap\mathcal{N}_i)} \mathfrak{p}\left(\tilde{e}^i_j - \tilde{s}_j\right) \end{aligned} \tag{2}$$

where $\alpha > 0$, and $\mathfrak{p} : \mathbb{R} \to [0, \infty)$.

Proof.

Consider two actions for agent i,

$$a_i = (\hat{s}_i, \hat{e}^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i}) \text{ and } a'_i = (\hat{s}'_i, \hat{e}'^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i})$$

The individual utility functions can be written as

$$\begin{aligned} v_i(x,a) = &-\alpha \sum_{k\in\mathcal{N}_i}\left\{\sum_{l\in\mathcal{N}_k} \mathfrak{p}\left(\tilde{s}_i - \tilde{e}^k_l\right) + \sum_{j\in\mathcal{N}\setminus\mathcal{N}_i} \mathfrak{p}\left(\tilde{e}^i_j - \tilde{e}^k_j\right)\right\} \\ &+ u_i(\tilde{s}_j|_{j\in\mathcal{N}_i}, \tilde{e}^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i}) \\ &+ \sum_{k\in\mathcal{N}_i\setminus\{i\}} u_i(\tilde{s}_j|_{j\in\mathcal{N}_k}, \tilde{e}^k_j|_{j\in\mathcal{N}\setminus\mathcal{N}_k}) \end{aligned}$$

Using this, and the fact that the full information game is a potential game with potential function ϕ, $v_i(x, a_i, a_{-i}) - v_i(x, a_i', a_{-i})$ can be written as in (3).

Using Lemma 16 (Appendix A), the fact that $\tilde{s}_i$ and $\tilde{e}^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i}$ do not appear in

$$\sum_{k\in\mathcal{N}\setminus\mathcal{N}_i} \phi\left(\tilde{s}_j|_{j\in\mathcal{N}_k}, \tilde{e}^k_j|_{j\in\mathcal{N}\setminus\mathcal{N}_k}\right),$$

and that

$$\sum_{j\in\mathcal{N}\setminus\mathcal{N}_i}\sum_{k\in\mathcal{N}_i\setminus\{i\}} \mathfrak{p}\left(\tilde{e}^i_j - \tilde{e}^k_j\right), \sum_{k\in\mathcal{N}_i\setminus\{i\}}\sum_{l\in\mathcal{N}_k\setminus\{k\}} \mathfrak{p}\left(\tilde{s}_i - \tilde{e}^k_l\right)$$

are the only terms containing $\tilde{e}^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i}$ and $\tilde{s}_i$ in

$$\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{N}_i}\sum_{j\in\mathcal{N}\setminus(\mathcal{N}_k\cup\mathcal{N}_i)} \mathfrak{p}\left(\tilde{e}^i_j - \tilde{e}^k_j\right)$$

and

$$\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{N}_i}\sum_{j\in\mathcal{N}_k\setminus(\mathcal{N}_k\cap\mathcal{N}_i)} \mathfrak{p}\left(\tilde{e}^i_j - \tilde{s}_j\right),$$

the result follows. □

5. EQUILIBRIA

A definition of Nash equilibria for the the state-based game G can be given as follows.

Definition 9. A state action pair (x^*, a^*) is a stationary state Nash equilibrium for the game G if

- $(\forall i \in \mathcal{N})(\forall a_i \in \mathcal{A}_i(x^*))\ v_i(x^*, a^*) \geq v_i(x^*, a_i, a^*_{-i})$
- $x^* = f(x^*, a^*)$

This is a restatement of Definition 3 in Li and Marden (2013). An alternative definition is given as follows.

Let $a_i = (a_{i1}, \cdots, a_{in_i})$, where $n_i = 1 + \bar{n}_i$, $\bar{n}_i = |\mathcal{N}\setminus\mathcal{N}_i|$, and $a_{i\_j} = a_i\setminus\{a_{ij}\}$. Also, let $x_i = (x_{i1}, \cdots, x_{in_i})$, where $x_{i\_j} = x_i\setminus\{x_{ij}\}$.

Definition 10. A state action pair (x^*, a^*) is a stationary state Nash equilibrium for the game G if

- $(\forall a_i \in \mathcal{A}_i(x^*))\ v_i(x^*, a^*) \geq v_i(x^*, a_{ij}, a^*_{i\_j}, a^*_{-i})$, for all $i \in \mathcal{N}$, $j \in (1, \cdots, n_i)$
- $x^* = f(x^*, a^*)$

Note that the equilibria according to Definition 9 are a subset of those according to Definition 10.

A characterization of the previously defined equilibria for the state-based potential game can be given as follows.

Lemma 11. For the state-based game G with potential function Φ and null action $\mathbf{0}$, if x^* satisfies either

$$(\forall i \in \mathcal{N})\ \Phi(x^*, \mathbf{0}) \geq \Phi(x_i, x^*_{-i}, \mathbf{0}) \text{ or} \tag{4}$$

$$(\forall i \in \mathcal{N})(\forall j \in (1, \cdots, n_i))\ \Phi(x^*, \mathbf{0}) \geq \Phi(x_{ij}, x^*_{i\_j}, x^*_{-i}, \mathbf{0}) \tag{5}$$

then $(x^*, \mathbf{0})$ is a stationary state Nash equilibrium according to Definitions 9 or 10 respectively, and for any a' such that $x^* = f(x^*, a')$, (x^*, a') is an equilibrium.

The proof of this result, and the following ones, is omitted here for space limitations and will be provided elsewhere.

The following lemma gives a sufficient condition for when Definition 10 implies Definition 9.

Lemma 12. If (x, a) is a Nash equilibrium for the potential game G according to Definition 10, then it is an equilibrium according to Definition 9 if for all $i \in \mathcal{N}$, and (x, a')

$$\begin{aligned} &\phi\left(\tilde{s}_i', \tilde{s}_j|_{j\in\mathcal{N}_i\setminus\{i\}}, \tilde{e}^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i}\right) \\ &\qquad - \phi\left(\tilde{s}_i', \tilde{s}_j|_{j\in\mathcal{N}_i\setminus\{i\}}, \tilde{e}'^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i}\right) \\ &\geq \phi\left(\tilde{s}, \tilde{e}^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i}\right) - \phi\left(\tilde{s}, \tilde{e}'^i_1, \cdots, \tilde{e}^i_{n_i}\right) + \cdots + \\ &\quad \phi\left(\tilde{s}, \tilde{e}^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i}\right) - \phi\left(\tilde{s}, \tilde{e}^i_1, \cdots, \tilde{e}'^i_{n_i}\right) \end{aligned}$$

The following result establishes a relation between equilibria of the potential game g and that of the state-based potential game G.

Proposition 13. If the fixed points of G lie in the set $\mathcal{S} = \{x : e^i_j = s_j,\ i, j \in \mathcal{N}\}$, then

a. (x, a), where $x = (s, e)$, is a Nash equilibrium of G according to Definition 10 if s is a Nash equilibrium of g.
b. (x, a), where $x = (s, e)$, is a Nash equilibrium of G according to Definition 9 if s is a Nash equilibrium of g, and if for all $i \in \mathcal{N}$, $s' \in S$

$$\alpha \sum_{j\in\mathcal{N}\setminus\mathcal{N}_i}\sum_{k\in\mathcal{N}_i} \mathfrak{p}\left(s_j' - s_j\right) \geq |\phi(s_i', s_l|_{l\in\mathcal{N}\setminus\{i\}}) - \phi(s_i', s_l|_{l\in\mathcal{N}_i\setminus\{i\}}, s_j'|_{j\in\mathcal{N}\setminus\mathcal{N}_i})|.$$

The following result gives a characterization of G when Φ is differentiable.

Proposition 14. Consider the state based potential game G with continuous S, $\phi, \mathfrak{p} \in C^1$, where $\mathfrak{p}'(\cdot)$ is odd, and $\mathcal{A}_i(x)$ is open for any equilibrium. If (x, a), where $x = (s, e)$, is a Nash equilibrium of G according to either Definition 9 or 10, then

$$\sum_{i\in\mathcal{N}} \nabla\phi(s_j|_{j\in\mathcal{N}_i}, e^i_j|_{j\in\mathcal{N}\setminus\mathcal{N}_i}) = 0. \tag{6}$$

Corollary 15. If s is an isolated (strict) Nash equilibrium of the potential game g, then in a neighbourhood of $\bar{s} := (s, e) \cap \mathcal{S}$, $\mathcal{S} = \{x : e^i_j = s_j,\ i, j \in \mathcal{N}\}$, of the state based potential game G, (6) is satisfied only at $\bar{s}$, and this point is a Nash equilibrium of G according to Definition 10.

Proof. If s is an isolated Nash equilibrium, then there exists a neighbourhood of s, $U \subset S$ where $\nabla\phi(s') \neq 0$ for all $s' \in U\setminus\{s\}$. Consequently, the first part of the Proposition applies for the neighbourhood $U \times U^n$. From this it follows that $\bar{s}$ is a fixed point of G, and the second part follows directly from Proposition 13. □

The result in the previous corollary motivates the learning algorithm presented in part II.

6. CONCLUSION

The paper formulated the problem of distributing potential games over communication graphs. A restricted information potential game was developed for undirected graphs, using a state-based formulation. Equilibria properties were studied. In part II of the paper a learning algorithm with prescribed properties will be developed. In addition, the results will be used to developed an on-the-go method for platoon matching of heavy duty vehicles.

$$
\begin{aligned}
&v_i(x,a_i,a_{-i}) - v_i(x,a_i',a_{-i}) = \\
&u_i(\tilde{s}_i, \tilde{s}_j|_{j\in\mathcal{N}_i\setminus\{i\}}, \tilde{e}_j^i|_{j\in\mathcal{N}\setminus\mathcal{N}_i}) - u_i(\tilde{s}_i', \tilde{s}_j|_{j\in\mathcal{N}_i\setminus\{i\}}, \tilde{e}_j'^i|_{j\in\mathcal{N}\setminus\mathcal{N}_i}) - \alpha \sum_{k\in\mathcal{N}_i\setminus\{i\}} \sum_{l\in\mathcal{N}_k\setminus\{k\}} \mathfrak{p}\left(\tilde{s}_i - \tilde{e}_l^k\right) - \mathfrak{p}\left(\tilde{s}_i' - \tilde{e}_l^k\right) \\
&+ \sum_{k\in\mathcal{N}_i\setminus\{i\}} u_i(\tilde{s}_i, \tilde{s}_j|_{j\in\mathcal{N}_k\setminus\{i\}}, \tilde{e}_j^k|_{j\in\mathcal{N}\setminus\mathcal{N}_k}) - u_i(\tilde{s}_i', \tilde{s}_j|_{j\in\mathcal{N}_k\setminus\{i\}}, \tilde{e}_j^k|_{j\in\mathcal{N}\setminus\mathcal{N}_k}) - \alpha \sum_{j\in\mathcal{N}\setminus\mathcal{N}_i} \sum_{k\in\mathcal{N}_i\setminus\{i\}} \mathfrak{p}\left(\tilde{e}_j^i - \tilde{e}_j^k\right) - \mathfrak{p}\left(\tilde{e}_j'^i - \tilde{e}_j^k\right) \\
&= \phi\left(\tilde{s}_i, \tilde{s}_j|_{j\in\mathcal{N}_i\setminus\{i\}}, \tilde{e}_j^i|_{j\in\mathcal{N}\setminus\mathcal{N}_i}\right) - \phi\left(\tilde{s}_i', \tilde{s}_j|_{j\in\mathcal{N}_i\setminus\{i\}}, \tilde{e}_j'^i|_{j\in\mathcal{N}\setminus\mathcal{N}_i}\right) - \alpha \sum_{k\in\mathcal{N}_i\setminus\{i\}} \sum_{l\in\mathcal{N}_k\setminus\{k\}} \mathfrak{p}\left(\tilde{s}_i - \tilde{e}_l^k\right) - \mathfrak{p}\left(\tilde{s}_i' - \tilde{e}_l^k\right) \\
&+ \sum_{k\in\mathcal{N}_i\setminus\{i\}} \phi(\tilde{s}_i, \tilde{s}_j|_{j\in\mathcal{N}_k\setminus\{i\}}, \tilde{e}_j^k|_{j\in\mathcal{N}\setminus\mathcal{N}_k}) - \phi(\tilde{s}_i', \tilde{s}_j|_{j\in\mathcal{N}_k\setminus\{i\}}, \tilde{e}_j^k|_{j\in\mathcal{N}\setminus\mathcal{N}_k}) - \alpha \sum_{j\in\mathcal{N}\setminus\mathcal{N}_i} \sum_{k\in\mathcal{N}_i\setminus\{i\}} \mathfrak{p}\left(\tilde{e}_j^i - \tilde{e}_j^k\right) - \mathfrak{p}\left(\tilde{e}_j'^i - \tilde{e}_j^k\right)
\end{aligned}
\tag{3}
$$

Appendix A

Lemma 16. If the game $\langle \mathcal{N}, (S_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}}\rangle$ is a potential game with potential function ϕ, then for any $\mathcal{N}' \subset \mathcal{N}$, with $n' = |\mathcal{N}'|$, and $s_{-\mathcal{N}'} \in S_{-\mathcal{N}'}$ where $s_{-\mathcal{N}'} = \{s_i | i \in \mathcal{N}\setminus\mathcal{N}'\}$ and $S_{-\mathcal{N}'} := S^n \setminus S_{\mathcal{N}'}$, $S_{\mathcal{N}'} = S_{1'} \times \cdots \times S_{n'}$, and i' denotes the i-th element of $\mathcal{N}'$,

$$
\begin{aligned}
&u_i(s_{1'}', \cdots, s_{n'}', s_{-n'}) - u_i(s) = \phi(s_{1'}', \cdots, s_{n'}', s_{-n'}) - \phi(s), \\
&\qquad \forall (s_{1'}', \cdots, s_{n'}'), (s_{1'}, \cdots, s_{n'}) \in S_{\mathcal{N}'}
\end{aligned}
$$

Proof. $u_i(s_{1'}', \cdots, s_{n'}', s_{-n'}) - u_i(s)$ can be rewritten as

$$
\begin{aligned}
&u_i(s_{1'}', \cdots, s_{n'}', s_{-n'}) - u_i(s_{1'}', \cdots, s_{n'-1}', s_{n'}, s_{-n'}) \\
&+ u_i(s_{1'}', \cdots, s_{n'-1}', s_{n'}, s_{-n'}) \\
&- u_i(s_{1'}', \cdots, s_{n'-2}', s_{n'-1}, s_{n'}, s_{-n'}) + \cdots \\
&+ u_i(s_{1'}', s_{2'}, \cdots, s_{n'}, s_{-n'}) - u_i(s)
\end{aligned}
$$

By applying the property of Definition 2 the result follows. □

REFERENCES

Alam, A., Besselink, B., Turri, V., Mårtensson, J., and Johansson, K.H. (2015). Heavy-duty vehicle platooning for sustainable freight transportation: A cooperative method to enhance safety and efficiency. *IEEE Control Systems Magazine*, 35(6), 34–56.

Bishop, R., Bevly, D., Humphreys, L., Boyd, S., and Murray, D. (2017). Evaluation and testing of driver-assistive truck platooning: Phase 2 final results. *Transportation Research Record*, 2615(1), 11–18.

Farokhi, F. and Johansson, K.H. (2013). A game-theoretic framework for studying truck platooning incentives. In *Proceedings of the 16^{th} International IEEE Conference on Intelligent Transportation Systems (ITSC 2013)*. The Hague, Netherlands.

González-Sánchez, D. and Hernández-Lerma, O. (2016). A survey of static and dynamic potential games. *Science China Mathematics*, 59(11), 2075–2102.

Johansson, A., Nekouei, E., Johansson, K.H., and Mårtensson, J. (2018). Multi-fleet platoon matching: A game-theoretic approach. In *Proceedings of the 21^{st} International Conference on Intelligent Transportation Systems (ITSC 2018)*. Maui, Hawaii, USA.

Lã, Q.D., Chew, Y.H., and Soong, B.H. (2016). *Potential Game Theory*. Springer.

Larsson, E., Sennton, G., and Larson, J. (2015). The vehicle platooning problem: Computational complexity and heuristics. *Transportation Research Part C: Emerging Technologies*, 60, 258–277.

Li, N. and Marden, J.R. (December, 2010). Designing games to handle coupled constraints. In *Proceeding of the 49^{th} IEEE Conference on Decision and Control*, 250 – 255. Atlanta, GA, USA.

Li, N. and Marden, J.R. (2013). Designing games for distributed optimization. *IEEE Journal of Selected Topics in Signal Processing*, 7(2), 230–242.

Liang, K., Mårtensson, J., and Johansson, K.H. (2016). Heavy-duty vehicle platoon formation for fuel efficiency. *IEEE Transactions on Intelligent Transportation Systems*, 17(4), 1051–1061.

Marden, J.R. (2012). State based potential games. *Automatica*, 48, 3075–3088.

Marden, J.R. and Shamma, J.S. (2013). Game theory and distributed control. In H. Young and S. Zamir (eds.), *Handbook of Game Theory*, volume 4. Elsevier Science, Amsterdam.

Monderer, D. and Shapley, L.S. (1996). Potential games. *Games and Economic Behavior*, 14(1), 124–143.

Nisan, N., Schapira, M., and Zohar, A. (2008). Asynchronous best-reply dynamics. In C. Papadimitriou and S. Zhang (eds.), *Internet and Network Economics*, volume 5385 of *Lecture Notes in Computer Science*. Springer, Berlin, Heidelberg.

Shapley, L.S. (1953). Stochastic games. *Proc. Nat. Acad. Sci. USA*, 39(10), 1095–1953.

Siegfried, T. (2006). *A Beautiful Math: John Nash, Game Theory, and the Modern Quest for a Code of Nature*. Joseph Henry Press, Washington, DC.

Tekin, C., Liu, M., Southwell, R., Huang, J., and Ahmad, S.H.A. (2012). Atomic congestion games on graphsand their applications in networking. *IEEE/ACM Transactions on Networking*, 20(5), 1541–1552.

Tsugawa, S., Jeschke, S., and Shladover, S.E. (2016). A review of truck platooning projects for energy savings. *IEEE Transactions on Intelligent Vehicles*, 1(1), 68–77.

van de Hoef, S., Johansson, K.H., and Dimarogonas, D.V. (2018). Fuel efficient en route formation of truck platoons. *IEEE Transactions on Intelligent Transportation Systems*, 19(1), 102–112.