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Feedback synchronization in Persidskii systems ⋆

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Abstract: A general synchronization scheme for the common dynamics of Persidskii systems is presented in this paper. The conditions of output stability of the closed-loop systems and their synchronization are established in the form of linear matrix inequalities (LMI). An example of Chua's circuit is considered for examining the effectiveness of our proposed results. A designing method of feedback gains is also introduced.

Keywords: Synchronization, Persidskii systems, input-to-output stability, Chua's circuit, feedback gains.

1. INTRODUCTION

Synchronization is a complex phenomenon that is frequently observed in networked and interconnected systems. It has been extensively investigated in various fields, such as communication Akar and Shorten (2008), robotics Chung and Slotine (2007), cyber-physical systems Olfati-Saber et al. (2007). Mathematically, synchronization is a contraction property of the difference among the solutions of networked systems. The main methods for realizing synchronization of nonlinear systems include passivity theory Persis and Jayawardhana (2012); Hamadeh et al. (2012), output regulation Byrnes (2007); Persis and Jayawardhana (2014), incremental stability Angeli (2009), Lyapunov approach Polyak and Kvinto (2017), to mention a few recent results.

In this work, the research object is a family of Persidskii systems for which the conditions of their synchronization by a proper coupling are analyzed. A Persidskii system is an example of a nonlinear model for which there are known canonical forms of Lyapunov functions. For this kind of dynamics, the study in Barbashin (1961) first investigated a Lyapunov function as a linear combination of the integrals of the non-linearities, which had been extended by Persidskii (1969) to a Lyapunov function including the absolute values of the states. Further research works and extensions include Hsu et al. (2000); Kaszkurewicz and Bhaya (2005) and are surveyed in Kaszkurewicz and Bhaya (2000). Persidskii systems have been extensively studied in the context of neural networks Ferreira et al. (2005), power systems Hsu and Colvara (1987), stability analysis Kaszkurewicz and Hsu (1979). The present paper mainly focuses on the conditions of feedback synchronization of the common dynamics of Persidskii systems by utilizing a Lyapunov function recently proposed for the analysis of input-to-state stability in Efimov and Aleksandrov (2019). Taking into account the structure of the system and the form of the nonlinearities, a synchronization measure is introduced and used in the design of coupling gains. The obtained conditions and the guidelines for feedback tuning are given in the form of linear matrix inequalities (LMI). The model of Chua's circuit is considered as an application example for examining the efficiency of our proposed results. The method of devising the feedback gains is also studied as an auxiliary result.

The organization of this paper is as follow: in Section 2 the preliminaries are presented. The problem statement is described in Section 3, while in Section 4 the synchronization measure and the conditions of synchronization are given. An application of Chua's circuit is studied in Section 5. The designing method of feedback gains with the guarantee of synchronization is presented in Section 6.

Notation

- Let $\mathbb{R}$, $\mathbb{R}_+$ represent the set of real numbers and non-negative real numbers, respectively.
- The identity matrix of dimension n is denoted by I_n and the $n\times m$ zero matrix by $\mathbf{O}_{n\times m}$. Denote by $\operatorname{diag}(v)$ the $n\times n$ diagonal matrix with a vector $v\in\mathbb{R}^n$ on the diagonal. The notation $\overline{1,n}$ is used to represent the set of integers $\{1,\dots,n\}$.
- $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.
- Given a set $\mathscr{W}\subset\mathbb{R}^n$, the distance of a point $p\in\mathbb{R}^n$ to the set $\mathscr{W}$ is defined by $|p|_{\mathscr{W}}=\inf_{w\in\mathscr{W}}\|w-p\|$.
- For a Lebesgue measurable function $u\colon\mathbb{R}_+\to\mathbb{R}^m$, define the norm $\|u\|_{[t_1,t_2)}=\operatorname{ess\,sup}_{t\in[t_1,t_2)}\|u(t)\|$, for t_1, $t_2\in\mathbb{R}_+$. We denote by $\mathscr{L}_\infty^m$ the set of functions u with $\|u\|_\infty:=\|u\|_{[0,\infty)}<+\infty$.
- A continuous function $\sigma:\mathbb{R}_+\to\mathbb{R}_+$ belongs to class $\mathscr{K}$ if it is strictly increasing and $\sigma(0)=0$; it belongs to class $\mathscr{K}_\infty$ if it is also unbounded. A continuous function $\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ belongs to class $\mathscr{KL}$ if $\beta(\cdot,r)\in\mathscr{K}$ and $\beta(r,\cdot)$ is decreasing to zero for any fixed $r\in\mathbb{R}_+$.
- For a continuously differentiable function $V\colon\mathbb{R}^n\to\mathbb{R}$, denote by $\nabla V(x)v$ the gradient of V at $x\in\mathbb{R}^n$ in the direction of $v\in\mathbb{R}^n$.

2. PRELIMINARIES

Consider a class of nonlinear systems:

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$$\begin{aligned}&\dot{x}(t) = f(x(t), u(t)), t \geq 0, f(0,0) = 0, x(0) = x_0,\\ &y(t) = h(x(t)),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, for all $t \in \mathbb{R}_+$, and $f\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a locally Lipschitz continuous function. For an initial state $x_0 \in \mathbb{R}^n$ and an input $u \in \mathscr{L}_\infty^m$, we denote the corresponding solution of the system (1) by $x(t, x_0, u)$, then $y(t, x_0, u) = h(x(t, x_0, u))$.

Let us give some definitions that will be used in the sequel.

Definition 1. A forward complete system (1) is said to be *practical input-to-output stable* (pIOS) if there exist $\beta \in \mathscr{KL}, \gamma \in \mathscr{K}$ and $c \in \mathbb{R}_+$ such that

$$\|y(t, x_0, u)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_\infty) + c, \ \forall t \geq 0$$

for any $x_0 \in \mathbb{R}^n$ and $u \in \mathscr{L}_\infty^m$. The system is called IOS if $c = 0$. In the special case when $y = x$, the IOS property is reduced to the input-to-state stability (ISS).

Definition 2. A forward complete system (1) is *uniformly bounded input bounded state stable* (UBIBS) if there exists $\sigma \in \mathscr{K}$ such that

$$\|x(t, x_0, u)\| \leq \max\{\sigma(\|x_0\|), \sigma(\|u\|_\infty)\}, \ \forall t \geq 0$$

for all $x_0 \in \mathbb{R}^n$ and $u \in \mathscr{L}_\infty^m$.

Definition 3. A smooth function $V\colon \mathbb{R}^n \to \mathbb{R}_+$ is called an IOS-*Lyapunov function* for the system (1) if for some $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, $\chi \in \mathscr{K}$ and $\alpha_3 \in \mathscr{KL}$

$$\begin{aligned}\alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|x\|),\\ V(x) \geq \chi(\|u\|) \Rightarrow \nabla V(x) f(x,u) \leq -\alpha_3(V(x), \|x\|)\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Lemma 4. Sontag and Wang (2000) A UBIBS system (1) is IOS if and only if it admits an IOS-Lyapunov function.

3. PROBLEM STATEMENT

In this section, the synchronization problem of Persidskii systems is introduced. Our goal is to propose LMI-based sufficient conditions of the realization in this kind of dynamics.

Let N be a strictly positive integer. Consider a family of N systems of the following form:

$$\dot{x}_i(t) = A_{i,0}x_i(t) + \sum_{j=1}^{M} A_{i,j} f_j(x_i(t)) + B_i u_i(t), \forall i \in \overline{1,N}, \ t \in \mathbb{R}_+, \tag{2}$$

where $x_i(t) = [x_{i,1}(t) \ldots x_{i,n}(t)]^\top \in \mathbb{R}^n$ is the state vector of a subsystem, $A_{i,s} \in \mathbb{R}^{n\times n}$ ($s \in \overline{0,M}$), $B_i \in \mathbb{R}^{n\times m}$, $f_j(x_i(t)) = [f_j^1(x_{i,1}(t)), \ldots, f_j^n(x_{i,n}(t))]^\top$ ($j \in \overline{1,M}$) and $u_i(t) = [u_{i,1}(t) \ldots u_{i,m}(t)]^\top \in \mathbb{R}^m$ are the functions ensuring the existence of the solutions of the system (2) in the forward time at least locally.

The sector restrictions on f_j, $j \in \overline{1,M}$, are imposed as follows:

Assumption 5. Assume that for any $i \in \overline{1,n}$ and $j \in \overline{1,M}$,

$$\nu f_j^i(\nu) > 0, \ \forall \nu \in \mathbb{R}\backslash\{0\}.$$

Assume also there exists $r \in \overline{1,M}$ such that for all $i \in \overline{1,n}$, $k \in \overline{1,r}$

$$\lim_{\nu\to\pm\infty} f_k^i(\nu) = \pm\infty$$

and that there exists $p \in \overline{r,M}$ such that for all $i \in \overline{1,n}$, $k \in \overline{1,p}$

$$\lim_{\nu\to\pm\infty} \int_0^\nu f_k^i(r)dr = +\infty.$$

In this study, we consider the synchronization of the common dynamics of the system (2), *i.e.* a system in following form:

$$\dot{X}(t) = A_0 X(t) + \sum_{j=1}^{M} A_j F_j(X(t)) + BU(t), \ t \in \mathbb{R}_+, \tag{3}$$

where $X(t) = [x_1(t)^\top \ldots x_N(t)^\top]^\top \in \mathbb{R}^{Nn}$ is the state vector, $X(0) = X_0$, $A_s = \mathrm{diag}(A_{1,s}, \ldots, A_{N,s}) \in \mathbb{R}^{Nn\times Nn}$ ($s \in \overline{0,M}$), $B = \mathrm{diag}(B_1, \ldots, B_N) \in \mathbb{R}^{Nn\times Nm}$, $U(t) = [u_1(t)^\top \ldots u_N(t)^\top]^\top \in \mathbb{R}^{Nm}$, $F_j(X(t)) = [f_j(x_1(t))^\top, \ldots, f_j(x_N(t))^\top]^\top \in \mathbb{R}^{Nn}$ ($j \in \overline{1,M}$) are the functions ensuring the existence of the solution of (3) at least locally.

The corresponding solution of the system (3) at time t with an initial state X_0 is denoted by $X = X(t, X_0)$. We denote the *consensus set* of (2) as

$$\mathscr{W} := \left\{X \in \mathbb{R}^{Nn} \mid x_{i_1} = x_{i_2} \ \text{for} \ i_1, i_2 \in \overline{1,N}, i_1 \neq i_2\right\}.$$

In the sequel, to lighten the notation the time-dependency of functions might remain implicitly understood, for instance we might write x for $x(t)$.

4. SYNCHRONIZATION

In this section, the problem of realizing synchronization for the system (3) is connected with the stability analysis of the closed-loop system produced by a feedback controller. We first define the synchronization measure, then introduce a feedback controller with a specific form. Finally, the conditions ensuring the realization of synchronization are given.

The system (3) is in the *synchronous mode* if $X(t) \in \mathscr{W}$ for all $t \in \mathbb{R}_+$. To measure the closeness of the system to the synchronous regime we will use a synchronization measure: a continuously differentiable function $\rho : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ such that $\rho(X) = 0$ implies that $X \in \mathscr{W}$.

In this study, a controller with the feedback law $U = \psi(\rho(X))$ where $\psi\colon \mathbb{R}^{Nn} \to \mathbb{R}^{Nm}$ is a continuous function and $\psi(0) = 0$, is used to stabilize the system (3) and to realize synchronization for the resulting closed-loop system. In such a case the set $\mathscr{S} := \{X \in \mathscr{W} | \rho(X) = 0\}$ contains all synchronized solutions of the closed-loop system.

For example, we apply the following synchronization measure in this study:

$$\rho(X) = \Gamma X, \tag{4}$$

where

$$\Gamma = \begin{bmatrix} -I_n & I_n & 0 & \cdots & 0 \\ 0 & -I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I_n & I_n \\ I_n & 0 & \cdots & 0 & -I_n \end{bmatrix} \in \mathbb{R}^{Nn\times Nn},$$

also $\Gamma F_j(X) = 0$, for all $j \in \overline{1,M}$ and $X \in \mathscr{W}$ due to properties of F_j in the synchronization mode.

By considering

$$U = K_0 \Gamma X + \sum_{j=1}^{M} K_j \Gamma F_j(X) \tag{5}$$

to synchronize the system (3), we obtain the following closed-loop system

$$\dot{X}(t) = \tilde{A}_0 X(t) + \sum_{j=1}^{M} \tilde{A}_j F_j(X(t)), \tag{6}$$

where $\tilde{A}_s = A_s + BK_s\Gamma$ for $s \in \overline{0,M}$.

Proposition 6. If the system (6) admits a continuously differentiable Lyapunov function $V : \mathbb{R}^{Nn} \to \mathbb{R}_+$ such that for all $X \in \mathbb{R}^{Nn}$:

$$\begin{aligned} &\beta_1(\|\rho(X)\|) \le V(X) \le \beta_2(\|X\|), \\ &\dot{V}(X) \le -\beta_3(V(X)) \end{aligned} \tag{7}$$

or

$$\begin{aligned} &\beta_1(\|X\|) \le V(X) \le \beta_2(\|X\|), \\ &\dot{V}(X) \le -\beta_3(\|\rho(X)\|) \end{aligned} \tag{8}$$

for some $\beta_1, \beta_2 \in \mathscr{K}_\infty$ and $\beta_3 \in \mathscr{K}$, then the feedback control (5) ensures asymptotic attraction of the synchronous mode.

Proof. Reaching the synchronous mode is equivalent to stability and convergence of the system (6) with respect to the output $\rho(X)$. This is equivalent to IOS of (6) for a zero input.

The formulation (7) repeats exactly the conditions of Definition 3, and in this case there is no need in UBIBS property since $\beta_3 \in \mathscr{K}$. Hence, such a V is an IOS Lyapunov function with zero input and the required conclusion follows.

In the case of (8), since $\dot{V} \le 0$ and V is a positive definite function of the state X, all solutions are bounded. Then by LaSalle Invariance Principle Khalil (2002) all trajectories of the system converge to the set where $\rho(X) = 0$, as desired.

Theorem 7. Let Assumption 5 be satisfied. If there exist $P = P^\top \in \mathbb{R}^{Nn\times Nn}$, $\Xi_s = \mathrm{diag}\{\xi_s\} \in \mathbb{R}^{Nn\times Nn}$ with $\xi_s = [\xi_{s,1}, \dots, \xi_{s,nN}]^\top$ for $s \in \overline{0,M}$, $\Lambda_j = \mathrm{diag}\{\lambda_j\} \in \mathbb{R}^{Nn\times Nn}$ with $[\lambda_j^1, \dots, \lambda_j^{nN}]^\top$ for $j \in \overline{1,M}$, $\Upsilon_{s,\ell} = \mathrm{diag}\{\upsilon_{s,\ell}\} \in \mathbb{R}^{Nn\times Nn}$ with $\upsilon_{s,\ell} = [\upsilon_{s,\ell}^1, \dots, \upsilon_{s,\ell}^{nN}]^\top$ for $s \in \overline{0,M-1}$ and $\ell \in \overline{s+1,M}$ such that

$$\Lambda_j > 0,\ j \in \overline{1,M}, \quad \Xi_s \ge 0,\ \forall s \in \overline{0,M},$$

$$\Upsilon_{s,\ell} \ge 0,\ s \in \overline{0,M-1},\ \ell \in \overline{s+1,M}, \quad \sum_{j=1}^{p} \Lambda_j > 0,$$

$$\sum_{s=0}^{M} \Xi_s + 2\sum_{s=0}^{M}\sum_{\ell=s+1}^{M} \Upsilon_{s,\ell} > 0,$$

$$P > 0, \quad Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} & \cdots & Q_{1,M+1} \\ Q_{1,2}^\top & Q_{2,2} & \cdots & Q_{2,M+1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{1,M+1}^\top & Q_{2,M+1}^\top & \cdots & Q_{M+1,M+1} \end{bmatrix} \le 0,$$

where

$$\begin{aligned} Q_{1,1} &= (A_0^\top + \Gamma^\top K_0^\top B^\top)P + P(A_0 + BK_0\Gamma) + \Gamma^\top \Xi_0 \Gamma, \\ Q_{j+1,j+1} &= (A_j^\top + \Gamma^\top K_j^\top B^\top)\Lambda_j + \Lambda_j(A_j + BK_j\Gamma) + \Gamma^\top \Xi_j \Gamma, \\ Q_{1,j+1} &= P(A_j + BK_j\Gamma) + (A_0^\top + \Gamma^\top K_0^\top B^\top)\Lambda_j \\ &\quad + \Gamma^\top \Upsilon_{0,j}\Gamma,\ \forall j \in \overline{1,M}, \\ Q_{j+1,\ell+1} &= (A_j^\top + \Gamma^\top K_j^\top B^\top)\Lambda_\ell + \Lambda_j(A_j + BK_j\Gamma) + \Gamma^\top \Upsilon_{j,\ell}\Gamma, \\ &\quad j \in \overline{1,M},\ \ell \in \overline{j+1,M}, \end{aligned}$$

then the synchronous mode is reached under the controller (5).

Proof. Consider a candidate Lyapunov function:

$$V(X) = X^\top P X + 2\sum_{j=1}^{M}\sum_{i=1}^{nN}\left(\lambda_j^i \int_0^{X^i} F_j^i(r)dr\right).$$

Then, V is positive definite and radially unbounded due to the properties of the nonlinear functions F_j.

It can be shown that

$$\begin{aligned} \dot{V}(X) &= \dot{X}^\top P X + X^\top P\dot{X} + 2\sum_{j=1}^{M} \dot{X}^\top \Lambda_j F_j(X) \\ &= \begin{bmatrix} X \\ F_1(X) \\ \vdots \\ F_M(X) \end{bmatrix}^\top Q \begin{bmatrix} X \\ F_1(X) \\ \vdots \\ F_M(X) \end{bmatrix} - X^\top \tilde{\Xi}_0 X - \\ &\quad - \sum_{j=1}^{M} F_j(X)^\top \tilde{\Xi}_j F_j(X) - 2\sum_{j=1}^{M} X^\top \tilde{\Upsilon}_{0,j} F_j(X) \\ &\quad - 2\sum_{j=1}^{M}\sum_{\ell=j+1}^{M} F_j(X)^\top \tilde{\Upsilon}_{j,\ell} F_\ell(X) \\ &\le -X^\top \tilde{\Xi}_0 X - \sum_{j=1}^{M} F_j(X)^\top \tilde{\Xi}_j F_j(X) \\ &\quad - 2\sum_{j=1}^{M} X^\top \tilde{\Upsilon}_{0,j} F_j(X) \\ &\quad - 2\sum_{j=1}^{M}\sum_{\ell=j+1}^{M} F_j(X)^\top \tilde{\Upsilon}_{j,\ell} F_\ell(X), \end{aligned} \tag{9}$$

where $\tilde{\Xi}_s = \Gamma^\top \Xi_s \Gamma$ and $\tilde{\Upsilon}_{s,\ell} = \Gamma^\top \Upsilon_{s,\ell}\Gamma$ for $s \in \overline{0,M-1}$ and $\ell \in \overline{s+1,M}$. Thus, if all conditions in Theorem 7 hold true, then V satisfies (8), which implies that the controller (5) pushes the system (3) to the synchronous mode according to Proposition 6 as desired.

5. APPLICATION

The model of Chua's circuit is widely used to investigate chaotic behavior (for instance, in nonlinear control, in secure communication Yang and Chua (1997)). The general representation of Chua's circuit is:

$$\begin{aligned} \dot{a} &= \alpha\,(b - \phi(a)),\ \text{with } \phi(a) = a + g(a), \\ \dot{b} &= a - b + c, \\ \dot{c} &= -\beta b, \end{aligned} \tag{10}$$

where $a, b, c, \alpha, \beta \in \mathbb{R}$ and the function $g : \mathbb{R} \to \mathbb{R}$ in this application is defined as

$$g(a) = \begin{cases} m_1 a + m_1 - m_0, & a \le -1, \\ m_0 a, & -1 < a < 1, \\ m_1 a + m_0 - m_1, & a \ge 1, \end{cases}$$

where $m_0 = -\frac{8}{7}, m_1 = -\frac{5}{7}$. Let $a = x^1, b = x^2, c = x^3$ in the system (10), then it can be rewritten as

$$\dot{x} = \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \end{bmatrix} = \alpha_0 \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} + \alpha_1 f_1\left(\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}\right) \tag{11}$$

in the form of (3), where

$$\alpha_0 = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \alpha_1 = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$f_1(x) = \begin{bmatrix} g(x^1) \\ g(x^2) \\ g(x^3) \end{bmatrix}.$$

In this example, the values of the parameters α, β are chosen as $\alpha = 15.6, \beta = 31.5$ and the number of systems in the family N is set to 2. Therefore, the common dynamics of (11) is

$$\dot{X} = A_0 X + A_1 F_1(X) + Bu, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^6, \quad (12)$$

where $x_1, x_2 \in \mathbb{R}^3$ are solutions of (11). Consider a feedback controller in the form of (5) and $u \in \mathbb{R}^2$ a vector of scalar controls affecting the Chua's circuits to synchronize the system (12), then we obtain

$$\dot{X} = (A_0 + BK_0\Gamma)X + (A_1 + BK_1\Gamma)F_1(X).$$

The synchronization measure is selected as

$$\Gamma X = 0,$$

where

$$\Gamma = [I_3 \ -I_3],$$

by which we see that $e := x_1 - x_2 = \Gamma X$. Then,

$$\dot{e} = \alpha_0 e + \alpha_1 (f_1(x_1) - f_1(x_2)).$$

Let

$$b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}, K_0 = \mathbf{O}_{2\times 3}, K_1 = \begin{bmatrix} -0.1 & 0 & 0 \\ 0.1 & 0 & 0 \end{bmatrix},$$

we obtain that there exist $P, \Lambda_1, \Xi_0, \Xi_1, \Upsilon_{0,1}$ solving the proposed LMIs in Theorem 7. Moreover, the norm of the difference e and the state trajectory x_1 of the closed-loop system with distinct initial states $x_1(0), x_2(0)$ are shown in Fig. 1 and Fig. 2, respectively. The simulation results indicate that the two subsystems in system (12) are synchronized by the feedback controller.

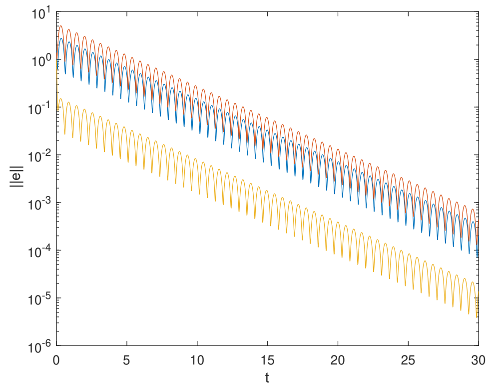

Fig. 1. The norm of the difference value e versus t

6. DESIGN OF FEEDBACK GAINS

In this section we propose a design method for the feedback gains $K_s\Gamma, s \in \overline{0,M}$ with the realization of synchronization (4). Since the terms with $K_s, s \in \overline{0,M}$ in inequality (9) only appear on Q, the design is concerned with the condition $Q \leq 0$. The main result is as follows:

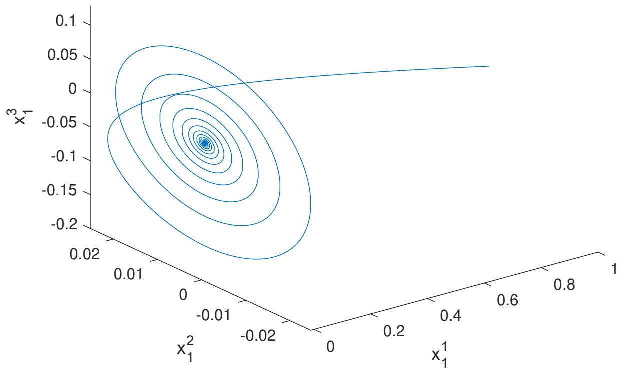

Fig. 2. The state trajectory of x_1

Theorem 8. Set

$$D = \begin{bmatrix} \Xi_0 & \Upsilon_{0,1} & \cdots & \Upsilon_{0,M} \\ \Upsilon_{0,1} & \Xi_1 & \cdots & \Upsilon_{1,M} \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{0,M} & \Upsilon_{1,M} & \cdots & \Xi_M \end{bmatrix},$$

and let all conditions of Theorem 7 be satisfied, except the condition $Q \leq 0$ is replaced by $D \leq 0$. If there exist matrices $Z_s \in \mathbb{R}^{Nm \times Nn}$, $s \in \overline{0,M}$ satisfying

$$L := \begin{bmatrix} L_{1,1} & L_{1,2} & \cdots & L_{1,M+1} \\ L_{1,2}^\top & L_{2,2} & \cdots & L_{2,M+1} \\ \vdots & \vdots & \ddots & \vdots \\ L_{1,M+1}^\top & L_{2,M+1}^\top & \cdots & L_{M+1,M+1} \end{bmatrix} \leq 0;$$

$$L_{1,1} := P^{-1}A_0^\top + Z_0^\top B^\top + A_0 P^{-1} + BZ_0;$$
$$L_{j+1,j+1} := \Lambda_j^{-1}A_j^\top + Z_j^\top B^\top + A_j\Lambda_j^{-1} + BZ_j, \ \forall j \in \overline{1,M};$$
$$L_{1,j+1} := P^{-1}A_0^\top + Z_0^\top B^\top + A_j\Lambda_j^{-1} + BZ_j; \ \forall j \in \overline{1,M};$$
$$L_{j+1,\ell+1} := \Lambda_j^{-1}A_j^\top + Z_j^\top B^\top + +A_\ell\Lambda_\ell^{-1} + BZ_\ell,$$
$$j \in \overline{1,M}, \ell \in \overline{j+1,M},$$

then the feedback gains

$$K_s\Gamma = \begin{cases} Z_0 P, & s = 0, \\ Z_s\Lambda_s, & s \in \overline{1,M} \end{cases}$$

form the controller (5) with the achievement of synchronization (4).

Proof. Consider the expression of $\dot{V}$ in (9). Let

$$Q = \check{Q} + \hat{Q},$$

where

$$\check{Q} := \begin{bmatrix} \Gamma^\top \Xi_0 \Gamma & \Gamma^\top \Upsilon_{0,1} \Gamma & \cdots & \Gamma^\top \Upsilon_{0,M} \Gamma \\ \Gamma^\top \Upsilon_{0,1} \Gamma & \Gamma^\top \Xi_1 \Gamma & \cdots & \Gamma^\top \Upsilon_{1,M} \Gamma \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma^\top \Upsilon_{0,M} \Gamma & \Gamma^\top \Upsilon_{1,M} \Gamma & \cdots & \Gamma^\top \Xi_M \Gamma \end{bmatrix};$$

$$\hat{Q} := \begin{bmatrix} \hat{Q}_{1,1} & \hat{Q}_{1,2} & \cdots & \hat{Q}_{1,M+1} \\ \hat{Q}_{1,2}^\top & \hat{Q}_{2,2} & \cdots & \hat{Q}_{2,M+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{Q}_{1,M+1}^\top & \hat{Q}_{2,M+1}^\top & \cdots & \hat{Q}_{M+1,M+1} \end{bmatrix};$$

$$\begin{aligned} \hat{Q}_{1,1} &:= (A_0^\top + \Gamma^\top K_0^\top B^\top) P + P(A_0 + BK_0\Gamma); \\ \hat{Q}_{j+1,j+1} &:= (A_j^\top + \Gamma^\top K_j^\top B^\top)\Lambda_j + \Lambda_j (A_j + BK_j\Gamma), \\ &\quad \forall j \in \overline{1,M}; \\ \hat{Q}_{1,j+1} &:= P(A_j + BK_j\Gamma) + (A_0^\top + \Gamma^\top K_0^\top B^\top)\Lambda_j, \\ &\quad \forall j \in \overline{1,M}; \\ \hat{Q}_{j+1,\ell+1} &:= (A_j^\top + \Gamma^\top K_j^\top B^\top)\Lambda_l + \Lambda_j (A_j + BK_j\Gamma), \\ &\quad j \in \overline{1,M}, \ell \in \overline{j+1,M}. \end{aligned}$$

Then, it can be shown that the condition $D \le 0$ is equivalent to

$$\begin{aligned} \check{Q} = &\Omega \operatorname{diag}(P^{-1}\Gamma^\top, \Lambda_1^{-1}\Gamma^\top, \ldots, \Lambda_M^{-1}\Gamma^\top) \\ &D \operatorname{diag}(\Gamma P^{-1}, \Gamma\Lambda_1^{-1}, \ldots, \Gamma\Lambda_M^{-1})\Omega \le 0, \end{aligned}$$

where

$$\Omega = \Omega^\top := \operatorname{diag}(P, \Lambda_1, \ldots, \Lambda_M).$$

Also, it holds that

$$\hat{Q} = \Omega\Omega^{-1}\hat{Q}\Omega^{-1}\Omega = \Omega L \Omega.$$

This implies that $\hat{Q} \le 0$ is equivalent to $L \le 0$.

Therefore, if the condition $D \le 0$ ($\check{Q} \le 0$) is satisfied and $L \le 0$ ($\hat{Q} \le 0$) holds true, then $Q \le 0$, which induces that synchronization is achieved since all conditions of Theorem 7 (except $Q \le 0$) are assumed to be satisfied.

7. CONCLUSION

In this paper, the synchronization measure of Persidskii systems was proposed and new conditions of realization of synchronization formed by LMI were presented. An application example of Chua's circuit was shown to validate the efficiency of the proposed results. Furthermore, the LMI-based design method of feedback gains with the achievement of synchronization was introduced. Reducing the conservativeness of the proposed LMI conditions is a potential future direction of interest, as well as considering the connection between input-to-output stability and synchronization.

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