Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Robust stabilization of an elementary logistic system with an input delay

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Abstract: In this paper, we are interested in the robustness analysis of an elementary logistic system having a fixed loss factor on the inventory level and uncertainties on the production delay. The problem is treated in control theory domain, where the model is considered as an input time delay system characterized by positivity and saturation constraints, and an external disturbance. Indeed, we apply a prediction state feedback control strategy using an affine control law, where the prediction of the future inventory level is based on a delay estimation of the delay uncertainty. Hence, the objective of the study is to quantify the impact of the uncertainty induced by this estimation on the performance of the controlled system. First, we use a frequency-domain technique to identify the robust stability condition in the set of parameters. In particular, we specify the range of the delay deviation such that the closed-loop system stability is guaranteed. Then, we move on to define the input-output flow variations that allows to check the system constraints, based on the invariance properties. Finally, a comparative simulation is provided to highlight the advantages of this study.

Keywords: Input-delay system, inventory control, predictor-feedback structure, delay uncertainty, robust study, system constraints, input-output flow variations.

1. INTRODUCTION

The dynamic behavior of many engineering processes, especially supply chains and production systems, contains time delays that are linked directly to the flow movements. In the past years, great attention has been paid to stability and robustness of time delay systems. Indeed, several studies have been done introducing the notions of input-output stability, as well as the stability by state estimation or state prediction for systems with uncertain time delay as Moon, Y.S. *et al.* (2001), Chiasson, J. and Loiseau J.J. (2007), Tarbouriech, S. *et al.* (2011), Sipahi, R. *et al.* (2011) and Wang, X. *et al.* (2012).

In this paper, we are interested on the inventory regulation problem for an elementary logistic system, that is composed of a production unit and a storage unit. The production system is characterized by the presence of a production delay that is defined with uncertainty. Moreover, the storage unit presents losses on the inventory level due to the manufacturing of perishable products. Furthermore, positivity and saturation constraints are imposed due to the specifications on the production and storage capacities. The main goal is to compromise between low storage level deterioration and high customer demands satisfaction. Such problems can be treated using different frameworks. In our study, we deal with a control theory approach where the system is considered as an input time-delayed system, with delay uncertainties. Simon, H.A. (1952) was the first to study the dynamics of logistics systems by a Servomechanism approach. Through the years, different studies were based on differential equations and feedback structures, to model and control production systems, see Ignaciuk P., and Bartoszewicz, A. (2011), Abbou *et al.* (2015) and Bou Farraa, B. *et al.* (2018).

The first contribution of this work provides a robust control law which guarantee the stability of the closed-loop system, using a feedback-predictor control structure. Different studies have treated the robustness with respect to delay uncertainty. Sufficient conditions for system stability with an input delay are obtained in Mondié, S. *et al.* (2001), and similar conditions are also obtained in Gu, K. *et al* (2005), for the stability crossing curves for systems with two delays. While in our study, necessary and sufficient conditions for the existence of a robust control are defined and expressed in terms of the delay deviation. The second contribution remains in specifying the input-output flow variations, in order to find the conditions that verify the system constraints. The main advantage in this study, is to apply the invariance principle introduced in Blanchini, F. (1990) and recently used in Bou Farraa, B. *et al.* (2018), on the exact prediction of the inventory level and in the presence of delay uncertainty.

The paper is organized as follows. Section 2 introduces the problem formulation. In Section 3, we recall some preliminary results. Then, the inventory control structure is described in Section 4. Section 5 is dedicated for the robust study in frequency domain. After, we identify the dynamics of the constrained input-output flow variables, in Section 6. Finally, we end the study by a simulation example in Section 7 and a brief summary in Section 8.

2. PROBLEM FORMULATION

2.1 Model description

In this study, we consider a logistic system composed of a production unit and a storage unit. The production unit is characterized by the incoming flow of production orders denoted by $u(t)$, and by a constant lead time θ which corresponds to a specified duration of the production task. Moreover, the function $\phi(t)$ corresponds to the production work in progress in the initial phase. The storage unit is characterized by the inventory level denoted by $y(t)$, whose outgoing flow satisfy the customers demand and sales made denoted by $d(t)$. The generic model for the inventory level dynamics is described by the following first order delayed differential equations:

$$\dot{y}(t) = \begin{cases} -\sigma y(t) + u(t-\theta) - d(t) & \text{, for } t \geqslant \theta, \\ -\sigma y(t) + \phi(t) - d(t) & \text{, for } 0 \leqslant t < \theta. \end{cases} \quad (1)$$

In our case of study, we are interested in the inventory dynamics of perishable products, so that the losses are modeled by a fixed expiration rate noted σ. This latter is a particular case of interest of the function $\sigma(t)$ with $0 \leqslant \sigma < 1$. Indeed, the elementary logistic system is considered as an input time-delay system, where $u(t)$ is the control input, $d(t)$ is the perturbation, and $y(t)$ is the system output.

2.2 Constraints and objective

Since the production unit and the storage unit are limited resources, the system is subject to positivity and saturation constraints that are defined as follows.

$$0 \leqslant y_{min} \leqslant y(t) \leqslant y_{max}, \quad (2)$$

and

$$0 \leqslant u_{min} \leqslant u(t) \leqslant u_{max}. \quad (3)$$

The customer demand $d(t)$ is supposed to be unknown but bounded by a minimum and a maximum demand rates, such that

$$0 \leqslant d_{min} \leqslant d(t) \leqslant d_{max}. \quad (4)$$

The problematic is to find a robust control strategy using a predictor state feedback structure, so that the operating constraints, (2) and (3), are satisfied for any customer demand verifying (4).

3. BACKGROUNDS AND PRELIMINARY RESULTS

3.1 Convolution systems

Given the input-output systems of the form $y(t) = (h * u)(t) = \int_0^t h(t-\tau)u(\tau)d\tau$, an important family of causal systems is characterized by convolution kernels of the form

$$h(t) = h_a(t) + \sum_{i=0}^{\infty} h_i.\delta(t - t_i), \text{ for } t \geqslant 0. \quad (5)$$

This set of kernels is denoted by $\mathcal{A}$ which forms a commutative Banach algebra with a norm verifying

$$||h||_{\mathcal{A}} = ||h_a||_{L_1} + \sum_{i=0}^{\infty} |h_i| < \infty \quad (6)$$

Such systems are known to belong to the Callier-Desoer class, introduced in Desoer, C.A. and Vidyasagar, M. (1975), which covers finite dimensional systems and infinite dimensional ones as well, such as time delayed systems. The authors showed that the following equality holds

$$\sup_{u \neq 0} \frac{||y||_\infty}{||u||_\infty} = ||h||_{\mathcal{A}}. \quad (7)$$

Hence, every system with kernel defined on $\mathcal{A}$ is said to be Bounded Input Bounded Output (BIBO)-stable. It means that for every bounded input to the system, results a bounded output for $t \geqslant t_0$.
Moreover, for a convolution system $(h_1 * h_2)(t)$ with only one positive kernel, the following property holds true:

$$||h_1 * h_2||_{\mathcal{A}} \leqslant ||h_1||_{\mathcal{A}}.||h_2||_{\mathcal{A}}. \quad (8)$$

These properties are powerful tools to identify the BIBO stability conditions for input-output systems, as presented in the sequel.

3.2 $\mathcal{D}$-invariance properties

In our work, we are interested in the invariance and $\mathcal{D}$-invariance principles, in the context of solving constrained control problems for logistic systems. Indeed, we formulate the explicit conditions for closed intervals $\mathcal{D}$-invariance in the case of single-variable systems studied in Blanchini, F. (1990), in the following theorem.

Theorem 1. We consider a system defined by $\dot{x}(t) = f(x(t)) - d(t)$ where $x(0) \in \Omega$ with $\Omega = [\alpha, \beta]$, and $d(t) \in \mathcal{D}$ with $\mathcal{D} = [\gamma, \delta]$. The interval Ω is said to be $\mathcal{D}$-invariant for this system if and only if the following conditions are fulfilled.

$$f(\alpha) \geqslant \delta, \text{ and } f(\beta) \leqslant \gamma. \quad (9)$$

This result is very basic in control theory. It answers as well to the issues of existence of feasible controllers for constrained systems. Hence, it is very useful in our study for reachable bounds identification and constraints meeting is such delayed systems.

4. INVENTORY CONTROL STRUCTURE

In this paper, the proposed approach to control the inventory dynamics for the logistic system (1) having an input delay, is based on a prediction state feedback structure that was first studied by Olbrot, A.W. (1978). The basic issue is to compensate the time delay by generating a control law that uses directly the corresponding delay-free system, as developed in Bou Farraa, B. *et al.* (2018) and Abbou *et al.* (2015). However, the system delay presents some uncertainties that are expressed by the following range:

$$\theta \in [\theta_{min}, \theta_{max}], \quad (10)$$

where θ_{min} and θ_{max} are positive values. Hence, the future state of the inventory level is predicted according to an estimated delay that we note θ_0. Thus, the predictive control structure is defined by an affine control law $u(t)$ and a prediction $z(t)$, based on the delay estimation θ_0 such that

$$u(t) = K(z_0 - z(t)), \quad (11)$$

and

$$z(t) = e^{-\sigma\theta_0} y(t) + \int_{t-\theta_0}^{t} e^{-\sigma(t-\tau)} u(\tau) d\tau \text{ , for } t \geqslant \theta_0. \quad (12)$$

The static gain K adjusts the production rate, and z_0 is the reference value for the estimated storage level. In addition, due to the uncertainty of the delay, $z(t)$ is no longer an exact prediction but an estimation of the future storage level. Furthermore, the dynamics of the prediction $z(t)$ can be expressed as follows:

$$\dot{z}(t) = -\sigma z(t) + u(t) - e^{-\sigma\theta_0} d(t) + e^{-\sigma\theta_0}(u(t-\theta) - u(t-\theta_0)). \quad (13)$$

The dynamics of the controlled system being defined by the above delayed differential equation, that depends on two non commensurable delays θ and θ_0, we can move to the robust stability analysis in the following section.

5. ROBUST STUDY IN FREQUENCY DOMAIN

The objective of the robustness study is to quantify the impact of the delay uncertainty on the performance of the inventory controlled system. The study consists of finding the conditions on the control parameters K and z_0 as well as θ_0, so that the closed loop system is stable. Indeed, we start by the closed-loop stability analysis.

5.1 Stability analysis of the closed loop transfer

In the frequency domain, the system output and the control law are described respectively by $(s+\sigma)\hat{y}(s) = e^{-s\theta}\hat{u}(s) - \hat{d}(s)$, and $\hat{u}(s) = \hat{C}(s)(Kz_0 - Ke^{-\sigma\theta_0}\hat{y}(s))$, where $\hat{C}(s) = \frac{1}{1+K(\frac{1-e^{-(s+\sigma)\theta_0}}{s+\sigma})}$.

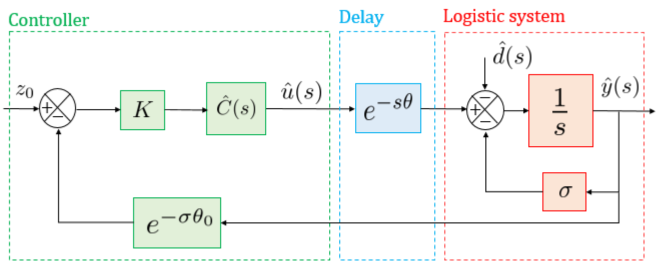

Fig. 1. Structure of the closed-loop system

We identify in the following proposition, the closed loop characteristic quasi-polynomial when the system delay θ is different from the design delay θ_0.

Proposition 1. Given the system (1) with a control design (11 - 12) based on a delay estimation θ_0, the closed loop characteristic quasi polynomial is equal to

$$\frac{1}{\hat{g}(s)} = s + \sigma + K - Ke^{-\sigma\theta_0}(e^{-s\theta_0} - e^{-s\theta}). \quad (14)$$

The proof is well-known in the control theory domain. Looking at the expression of $\hat{g}(s)$, we notice that the transfer depends explicitly on the gain parameter K and the two delays θ and θ_0. Indeed, a robust study in terms of the delay deviation between θ and θ_0 allows us to find the stability condition of the closed loop system. Thus, we express $\hat{g}(s)$ as a product of two transfers, in the following form

$$\hat{g}(s) = \hat{g}_1(s)\hat{g}_2(s). \quad (15)$$

The first transfer $\hat{g}_1(s) = \frac{1}{s+\sigma+K}$, corresponds in the time domain to the Kernel $g_1(t) = e^{-(\sigma+K)t}$, whose norm is equal to $||g_1||_{\mathcal{A}} = \frac{1}{\sigma+K}$. The second transfer is $\hat{g}_2(s) = \frac{1}{1-\hat{\varepsilon}(s)}$, with $\hat{\varepsilon}(s) = K\frac{e^{-\sigma\theta_0}(e^{-s\theta_0}-e^{-s\theta})}{s+\sigma+K}$. The function $\varepsilon(t)$ is equal to zero for $\theta = \theta_0$. Following this analysis, we will introduce the stability condition of the transfer $\hat{g}(s)$, in terms of the BIBO stability.

Proposition 2. Considering the following factorization,

$$\hat{g}(s) = \frac{1}{s+\sigma+K}\frac{1}{1-\hat{\varepsilon}(s)},$$

the transfer $\hat{g}(s)$ is stable if and only if $\frac{1}{1-\hat{\varepsilon}(s)}$ is stable.

Proof. We first remark that $\hat{g}_1(s)$ is BIBO-stable. As consequence, if $\hat{g}_2(s)$ is also stable, we obtain

$$||g||_{\mathcal{A}} \leqslant ||g_1||_{\mathcal{A}} . ||g_2||_{\mathcal{A}}.$$

This shows the sufficiency of Proposition 2. Reversely, we can write $\hat{g}_2(s) = 1 + \hat{g}_3(s).\hat{g}(s)$, with $\hat{g}_3(s) = Ke^{-\sigma\theta_0}(e^{-s\theta_0} - e^{-s\theta})$. Int the time domain, the kernel associated with the transfer $\hat{g}_3(s)$ is $g_3(t) = Ke^{-\sigma\theta_0}(\delta(t-\theta_0) - \delta(t-\theta))$, and that of $\hat{g}_2(s)$ is

$$g_2(t) = \delta(t) + Ke^{-\sigma\theta_0}(g(t-\theta_0) - g(t-\theta)).$$

The latter is integrable if $g(t)$ is also integrable, which completes the proof. ■

In the following, we will introduce a basic result that was introduced in Hille, E. and Phillips, R.S. (1957), and then developped in Desoer, C.A. and Vidyasagar, M. (1975) on the transfer stability, in order to identify the necessary and sufficient condition for the closed loop BIBO stability.

Lemma 1. Having an element α of $\mathcal{A}$, the complex function $\frac{1}{1-\hat{\alpha}}$ is an element of $\hat{\mathcal{A}}$ if and only if the following condition is satisfied

$$\inf_{Re(s)>0} \mid 1 - \hat{\alpha}(s) \mid > 0.$$

The application of this lemma allows to formulate the following condition on the robust stability of the transfer $\hat{g}(s)$.

Proposition 3. The transfer $\hat{g}(s)$ is stable if and only if

$$\sup_{w\in\mathbb{R}}\left(\frac{2Ke^{-\sigma\theta_0}}{\sqrt{w^2+(\sigma+K)^2}}\left|\sin\frac{w(\theta-\theta_0)}{2}\right|\right) < 1. \quad (16)$$

Proof. This result is deduced from Proposition 2 and Lemma 1. The condition in Lemma 1 reaches its maximum on the imaginary axis for $s = jw$, and is equivalent to

$$\sup_{w\in\mathbb{R}}\left|\frac{Ke^{-\sigma\theta_0}e^{-jw\theta_0}}{jw+(\sigma+K)}\right|\left|(1-e^{-jw(\theta_0-\theta)})\right| < 1.$$

The proof is achieved having that $\left|\frac{Ke^{-\sigma\theta_0}e^{-jw\theta_0}}{jw+(\sigma+K)}\right| = \frac{Ke^{-\sigma\theta_0}}{\sqrt{w^2+(\sigma+K)^2}}$, and $\left|(1-e^{-jw(\theta_0-\theta)})\right| = 2\left|\sin\frac{w(\theta-\theta_0)}{2}\right|$. ■

We notice that when the delay is known exactly, for $\theta = \theta_0$, $\sin\frac{w(\theta-\theta_0)}{2} = 0$ and the closed loop spectrum is equal to $\frac{1}{\hat{g}(s)} = s+\sigma+K$. Hence, the condition (16) is checked for every $w \in [0, \frac{\pi}{(\theta-\theta_0)}]$, and the closed-loop system is stable. Following the result of Proposition 3, we identify in the following theorem the conditions that guarantee the robustness of the controlled system against the delay uncertainty.

Theorem 2. Given a system of the form (1), the feedback control structure (11 - 12) is robust stabilizing with respect to the delay uncertainty, for

$$\begin{cases} \theta \geqslant 0 & , \text{ if } e^{-\sigma\theta_0} \leqslant \dfrac{1}{2}\dfrac{\sigma+K}{K}, \\ \theta \in]\max(0,\theta_0-\Delta),\theta_0+\Delta[& , \text{ if } e^{-\sigma\theta_0} > \dfrac{1}{2}\dfrac{\sigma+K}{K}, \end{cases} \quad (17)$$

where $\Delta = \frac{\pi}{2K\sqrt{(e^{-\sigma\theta_0})^2-(\frac{1}{2}\frac{\sigma+K}{K})^2}}$.

Proof. Using the condition (16) of Proposition 3, we notice that $\frac{2Ke^{-\sigma\theta_0}}{\sqrt{w^2+(\sigma+K)^2}}$ is a decreasing function, and that $\left|\sin\frac{w(\theta-\theta_0)}{2}\right| \leqslant \min(\frac{w(\theta-\theta_0)}{2}, 1)$ and it is satisfied at least if $w \in [0, \frac{\pi}{(\theta-\theta_0)}]$. We can therefore deduce the upper bound for the condition (16), which is equal to $\frac{2Ke^{-\sigma\theta_0}}{\sqrt{w_0^2+(\sigma+K)^2}}$. The condition (16) is therefore expressed by

$$\frac{2Ke^{-\sigma\theta_0}}{\sqrt{w_0^2+(\sigma+K)^2}} < 1.$$

It is checked for any positive value of w_0 if $e^{-\sigma\theta_0} \leqslant \frac{1}{2}\frac{\sigma+K}{K}$, which implies in particular that the gain K must be bigger enough. Hence, the system is stable for any value of $\theta \geqslant 0$. On the contrary, for $e^{-\sigma\theta_0} > \frac{1}{2}\frac{\sigma+K}{K}$, the upper bound of the condition (16) is obtained for $w_0 = \frac{\pi}{(\theta-\theta_0)}$. So, the transfer $\hat{g}(s)$ is stable if and only if

$$|\theta-\theta_0|\sqrt{4K^2e^{-2\sigma\theta_0}-(\sigma+K)^2} < \pi.$$

Indeed, the size of the delay deviation can be defined by Δ such that $|\theta-\theta_0| < \Delta$ with $\theta \geqslant 0$, which leads to the stability conditions of Theorem 2. ■

The necessary and sufficient conditions that are obtained in Theorem 2, allow to quantify the range of the delay deviation such that closed-loop system is stable. Indeed, we will move to specify in the following section the choice of the control parameters (K, θ_0), for which the control structure is robust and the condition (17) is verified.

5.2 Choice of the control parameters

The conditions of Theorem 2, depend on the set of parameters (K, σ, θ_0), where σ is an intrinsic parameter, and (K, θ_0) are the control parameters. The reference value z_0 does not appears in the stability analysis of the closed loop. In this section, we will identify the choice of the set of parameters (K, θ_0), for which the conditions of the robust stability are always satisfied. Indeed, the conditions on K are specified as follows.

Proposition 4. Given the closed-loop system of the form (1 - 11 - 12), the robust stability conditions (17) are verified for $\sigma > 0$, by choosing the control parameter K as follows:

$$\begin{cases} K \geqslant 0 & , \text{ for } e^{-\sigma\theta_0} \leqslant \dfrac{1}{2}, \\ K \in [0, \dfrac{\sigma}{2e^{-\sigma\theta_0}-1}] & , \text{ for } \dfrac{1}{2} < e^{-\sigma\theta_0} \leqslant \dfrac{1}{2}\dfrac{\sigma+K}{K}, \\ K \in]\dfrac{\sigma}{2e^{-\sigma\theta_0}-1}, \dfrac{\sigma}{2e^{-\sigma\theta_0}-1}\alpha[& , \text{ for } e^{-\sigma\theta_0} > \dfrac{1}{2}\dfrac{\sigma+K}{K}, \end{cases}$$

where $\alpha = \left(\frac{1+\sqrt{4e^{-2\sigma\theta_0}+(4e^{-2\sigma\theta_0}-1)\frac{\pi^2}{\sigma^2(\theta-\theta_0)^2}}}{2e^{-\sigma\theta_0}+1}\right)$.

Proof. Using the results of Theorem 2 to solve first the most complex case, the system is stable for $\theta \in]\max(0, \theta_0-\Delta), \theta_0+\Delta[$, which implies a choice of the control parameter K verifying

$$(4e^{-2\sigma\theta_0}-1)K^2 - 2\sigma K - (\sigma^2 + \frac{\pi^2}{(\theta-\theta_0)^2}) < 0.$$

This inequality is satisfied only if $e^{-\sigma\theta_0} > \frac{1}{2}$, for a choice of $K \in]\frac{\sigma}{2e^{-\sigma\theta_0}-1}, \frac{\sigma}{2e^{-\sigma\theta_0}-1}\alpha[$. In addition, for $\frac{1}{2} < e^{-\sigma\theta_0} \leqslant \frac{1}{2}\frac{\sigma+K}{K}$, the system is stable $\forall\theta \geqslant 0$ verifying $K \leqslant \frac{\sigma}{2e^{-\sigma\theta_0}-1}$. Following this analysis, we deduce one more case, for $e^{-\sigma\theta_0} \leqslant \frac{1}{2}$, where the system is stable $\forall\theta \geqslant 0$ and $K \geqslant 0$.■

We introduce in the following corollary the choice of the delay estimation θ_0, that allows to satisfy the conditions (17) given the interval of the delay variation (10).

Corollary 1. Given the system delay $\theta \in [\theta_{min}, \theta_{max}]$, the choice of the delay estimation θ_0 that satisfy the stability conditions (17) for Δ as defined in Theorem 2, is given by:

$$\begin{cases} \theta_0 \geqslant \dfrac{1}{\sigma}\log 2 & , \text{ for } K \geqslant 0, \\ \theta_0 < \dfrac{1}{\sigma}\log 2 & , \text{ for } K \leqslant \dfrac{\sigma}{2e^{-\sigma\theta_0}-1}, \\ \Delta > M & , \text{ for } K \in [\dfrac{\sigma}{2e^{-\sigma\theta_0}-1}, \dfrac{\sigma}{2e^{-\sigma\theta_0}-1}\beta[, \end{cases}$$

where $M = \max(\theta_{max}-\theta_0, \theta_0-\theta_{min})$, and $\beta = \left(\frac{1+\sqrt{1+(4e^{-2\sigma\theta_0}-1)(1+\frac{\pi^2}{\sigma^2M^2})}}{2e^{-\sigma\theta_0}+1}\right)$.

Proof. The first case on the choice of θ_0 is deduced from Proposition 4 for $e^{-\sigma\theta_0} \leqslant \frac{1}{2}$. Moreover, the second case is obtained from proposition 3, using that $\frac{1}{2} < e^{-\sigma\theta_0} \leqslant \frac{1}{2}\frac{\sigma+K}{K}$. Thus, the interval of variation of the system delay $[\theta_{min}, \theta_{max}]$, is a stabilizing solution in the two previous cases, for $e^{\sigma\theta_0} \leqslant \frac{1}{2}\frac{\sigma+K}{K}$. In the third case, for $e^{-\sigma\theta_0} > \frac{1}{2}\frac{\sigma+K}{K}$, the closed-loop is stable if and only if $\theta \in [\theta_{min}, \theta_{max}] \subset]\max(0, \theta_0-\Delta), \theta_0+\Delta[$, which implies that $\Delta > M$. Hence, this condition is guaranteed for $K \geqslant \frac{\sigma}{2e^{-\sigma\theta_0}-1}$ and bounded by $\frac{\sigma}{2e^{-\sigma\theta_0}-1}\beta$. ■

This section gives the conditions for which the control strategy is robustly stable in the presence of a delay uncertainty. However, the issue of the controller is to keep the production order and the inventory level, as far as possible within their limits, in order to forbid any overruns of the system constraints. So, the design of an admissible control law returns to define the conditions for which the closed-loop system would meet the system constraints in the presence of delay uncertainty. Hence, we will study the variation of the flow variables in the following section.

6. CONSTRAINED INPUT-OUTPUT SYSTEM

This section is dedicated for the study of the input-output variations using the predictive control structure, in order to satisfy the system constraints. As we have seen previously, the dynamics of the predictive control structure

(13) corresponds to a delayed differential equation that depends on two different delays θ and θ_0. So, let us consider in this section, the exact prediction $p(t)$ of the storage level, given by

$$p(t) = e^{-\sigma\theta}y(t) + \int_{t-\theta}^{t} e^{-\sigma(t-\tau)}u(\tau)d\tau, \text{ for } t \geqslant \theta. \quad (18)$$

Using the definitions of $p(t)$ and $z(t)$, and the properties of Artstein reduction as developped in Artstein, Z. (1982), we introduce the following proposition.

Proposition 5. For $z(t)$ and $p(t)$ being defined by (12) and (18) respectively, the following properties hold true.

$$\begin{aligned}
(i) \quad p(t) &= y(t+\theta) + \int_{t}^{t+\theta} e^{-\sigma(t+\theta-\tau)}d(\tau)d\tau,\\
(ii) \quad p(t) &= e^{-\sigma(\theta-\theta_0)}z(t) + e(t),\\
(iii) \quad \dot{p}(t) &= Kz_0 - (\sigma + Ke^{-\sigma(\theta_0-\theta)})p(t)\\
&\quad + Ke^{-\sigma(\theta_0-\theta)}e(t) - e^{-\sigma\theta}d(t),
\end{aligned}$$

where

$$e(t) = (1 - e^{-\sigma(\theta-\theta_0)})\int_{t-\theta_0}^{t} e^{-\sigma(t-\tau)}u(\tau)d\tau + \int_{t-\theta}^{t-\theta_0} e^{-\sigma(t-\tau)}u(\tau)d\tau.$$

Proof. The assertion (i) is obtained using the definition of $p(t)$ between t and $t+\theta$, and the system dynamics (1). The second assertion is deduced from the definitions (18) and (12) of $p(t)$ and $z(t)$ respectively. Finally, the assertion (iii) is obtained based on the Artstein reduction of (18), $\dot{p}(t) = -\sigma p(t) + u(t) - e^{-\sigma\theta}d(t)$, and using assertion (ii) and the definition of $u(t)$ (11). ■

Referring to the assertion (ii) of Proposition 5, the amount $e(t)$ allows to quantify the error on the prediction that is introduced by the delay uncertainty. In fact, when $\theta_0 = \theta$, we obtain $e(t) = 0$ and $z(t) = p(t)$ as consequence. Moreover, knowing that $u(t)$ is bounded by u_{min} and u_{max}, and that $\theta \in [\theta_{min}, \theta_{max}]$, we can find the exact bounds e_1 and e_2 of the variation of $e(t)$. Hence, the error of the prediction $e(t) \in [e_1, e_2]$, where

$$\begin{aligned}
e_1 &= \left(\frac{1 - e^{-\sigma\theta_{min}}}{\sigma} + e^{-\sigma\theta_{max}}\left(\frac{1 - e^{\sigma\theta_0}}{\sigma}\right)\right)u_{min},\\
e_2 &= \left(\frac{1 - e^{-\sigma\theta_{max}}}{\sigma} + e^{-\sigma\theta_{min}}\left(\frac{1 - e^{\sigma\theta_0}}{\sigma}\right)\right)u_{max}.
\end{aligned} \quad (19)$$

In the following, we use the $\mathcal{D}$-invariance properties in order to find the variation of the prediction $p(t)$, and then to define properly the interval of variations of $u(t)$ and $y(t)$.

Theorem 3. Given the system (1) with a disturbance $d(t)$ verifying (4), subject to a control strategy (11 - 12) with a delay uncertainty verifying (10), the following invariant sets of the input-output flow variables are satisfied.

$$u(t) \in [u_1, u_2], \; y(t) \in [y_1, y_2], \quad (20)$$

where u_1, u_2, y_1, and y_2 are defined by the following identities:

$$\begin{aligned}
u_1 &= \frac{K}{\sigma + Ke^{-\sigma(\theta_0-\theta_{max})}}(\sigma z_0 + e^{-\sigma\theta_{max}}d_{min} + \sigma e_1\\
&\quad - Ke^{-\sigma(\theta_0-\theta_{max})}(e_2 - e_1)),\\
u_2 &= \frac{K}{\sigma + Ke^{-\sigma(\theta_0-\theta_{min})}}(\sigma z_0 + e^{-\sigma\theta_{min}}d_{max} + \sigma e_2\\
&\quad - Ke^{-\sigma(\theta_0-\theta_{min})}(e_1 - e_2)),\\
y_1 &= \frac{1}{\sigma + Ke^{-\sigma(\theta_0-\theta_{min})}}(Kz_0 - e^{-\sigma\theta_{min}}d_{max}\\
&\quad + Ke^{-\sigma(\theta_0-\theta_{min})}e_1) - \frac{1 - e^{-\sigma\theta_{max}}}{\sigma}d_{max},\\
y_2 &= \frac{1}{\sigma + Ke^{-\sigma(\theta_0-\theta_{max})}}(Kz_0 - e^{-\sigma\theta_{max}}d_{min}\\
&\quad + Ke^{-\sigma(\theta_0-\theta_{max})}e_2) - \frac{1 - e^{-\sigma\theta_{min}}}{\sigma}d_{min}.
\end{aligned}$$

As consequence, the system constraints (2) and (3) are met if and only if u_1, u_2, y_1, and y_2 are so that

$$[u_1, u_2] \subset [u_{min}, u_{max}], \; [y_1, y_2] \subset [y_{min}, y_{max}]. \quad (21)$$

Proof. First, we apply the $\mathcal{D}$-invariance principle to the Artstein reduction expressed in assertion (iii) of Proposition 5. Using that $\dot{p}(d_{max}, \theta_{min}) \geqslant 0$ and that $\dot{p}(d_{min}, \theta_{max}) \leqslant 0$, we find the $\mathcal{D}$-invariant interval of the prediction $p(t)$, such that $p(t) \in [p_1, p_2]$, where the bounds are given by

$$\begin{aligned}
p_1 &= \frac{Kz_0 - e^{-\sigma\theta_{min}}d_{max}}{\sigma + Ke^{-\sigma(\theta_0-\theta_{min})}} + \frac{Ke^{-\sigma(\theta_0-\theta_{min})}e_1}{\sigma + Ke^{-\sigma(\theta_0-\theta_{min})}},\\
p_2 &= \frac{Kz_0 - e^{-\sigma\theta_{max}}d_{min}}{\sigma + Ke^{-\sigma(\theta_0-\theta_{max})}} + \frac{Ke^{-\sigma(\theta_0-\theta_{max})}e_2}{\sigma + Ke^{-\sigma(\theta_0-\theta_{max})}}.
\end{aligned}$$

Using the above bounds of $p(t)$ and the assertion (ii) of Proposition 5, we find the bounds of variation of $z(t)$. As consequence, we deduce the bounds u_1 and u_2 of $u(t)$ that is already defined by (11). In addition, one can check that for θ and $d(t)$ verifying (10) and (4) respectively, the integral $\int_t^{t+\theta} e^{-\sigma(t+\theta-\tau)}d(\tau)d\tau$ evolves between $\frac{1-e^{-\sigma\theta_{min}}}{\sigma}d_{min}$, and $\frac{1-e^{-\sigma\theta_{max}}}{\sigma}d_{max}$. Therefore, we can find easily the exact bounds y_1 and y_2 of $y(t)$ by replacing both p_1, p_2 and $\int_t^{t+\theta} e^{-\sigma(t+\theta-\tau)}d(\tau)d\tau$ by their expressions of evolution in the assertion (i) of Proposition 5. ■

In the end of this paper, we have found the exact and reachable bounds for both the input $u(t)$ and the output $y(t)$. These intervals allow to satisfy the system constraints, (2) and (3), given a customer demand verifying (4) and a delay uncertainty verifying (10).

7. SIMULATION EXAMPLE

This simulation illustrates the system response and highlights the effects of the delay uncertainties on the system stability and constraints verification. A co-design methodology is used to identify both system and control parameters. Indeed, for $d_{min} = u_{min} = y_{min} = 0$ and $\sigma = 0.02$, the control parameters are $K = 0.4468$ and $z_0 = 2800$. In addition, the system initialization is given by $y(0) = 2400$ and $\phi(t) = 70$. The results for either delay uncertainty or not, are given in the figures (2) and (3) for a rectangular signal of $d(t)$ limited by $d_{max} = 240$. We can notice that when $d(t) = 0$, the controlled structure makes it possible to replenish the storage level in order to reach the reference value z_0. Otherwise, the control input follows the variation of the demand, so that it is completely satisfied, and the storage level does not undergo a shortage. Hence, the system constraints are checked for both $y(t)$ and $u(t)$ verifying (20) and (21). In addition, the system responses when $\theta = \theta_0 = 6$ are very smooth, while they show some fluctuations and small variations when we introduce an delay uncertainty with $\theta_0 = 7$. Moreover, the robust stability conditions (17) are guaranteed for $\Delta = 5.06$, such that $\theta \in [\theta_0 - 5.06, \theta_0 + 5.06]$, and $K \in [0.0271, 2.2187[$.

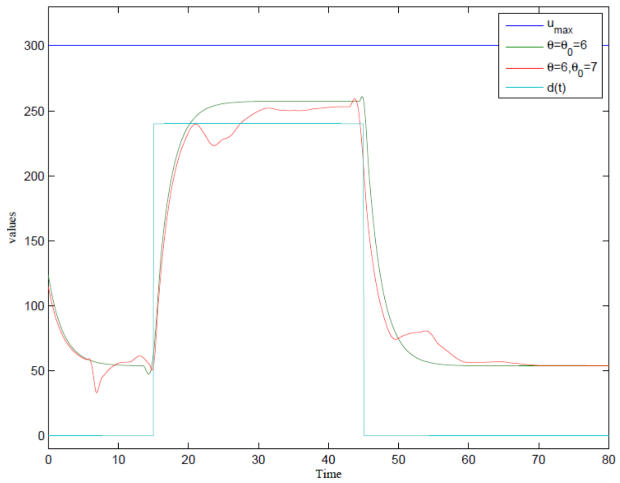

Fig. 2. Temporal evolution of $u(t)$ limited by $u_{max} = 300$

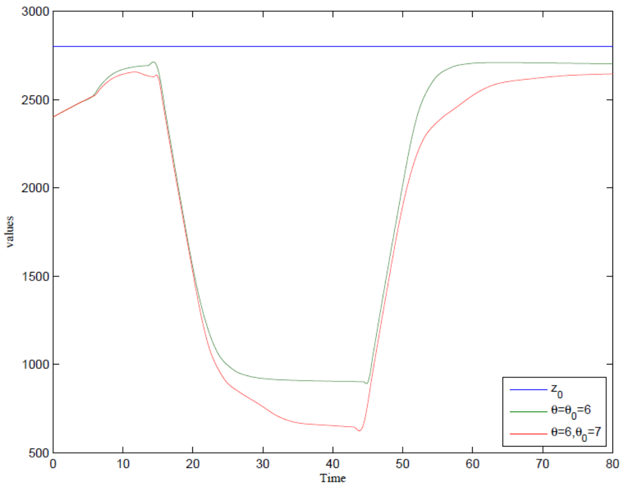

Fig. 3. Temporal evolution of $y(t)$, limited by $y_{max} = 3000$

8. CONCLUSION

Our work deals with the problem of inventory level management of an elementary logistic system, subject to a constant loss factor and a lead time uncertainty. The main advantage remains in the study of the robust stabilization using a predictive and affine feedback control structure. Indeed, necessary and sufficient conditions for which the robust stability of the controlled system is guaranteed, are given in terms of the size of delay deviation. In addition, we have identified the bounds of input-output variations, so that the system constraints are verified for any customer demand. In the continuity, different topics have worth future investigation. First, the robust analysis can be developed for variable input delays, and uncertain loss factor. Moreover, the losses can be modeled using a date of expiry instead of a preemption factor. Finally, it would be interesting to extend this study for distributed systems, that present real applications for logistic networks.

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