On the Geometry of Consensus Algorithms with Application to Distributed Termination in Higher Dimension [★]

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Abstract: We present insights into the geometry of the ratio consensus algorithm that lead to finite time distributed stopping criteria for the algorithm in higher dimension. In particular we show that the polytopes of network states indexed by time form a nested sequence. This monotonicity allows the construction of a distributed algorithm that terminates in finite time when applied to consensus problems in any dimension and guarantees the convergence of the consensus algorithm in norm, within any given tolerance. The practical utility of the algorithm is illustrated through MATLAB simulations.

Keywords: Distributed Consensus; Multi-agent systems; Network-based computing systems; High-dimensional state algorithms; Convex hull;

1. INTRODUCTION

Monitoring and control of distributed physical environment warrants for the global information without the complete knowledge of the system. This task of getting the global information is mostly accomplished via a class of algorithms called consensus algorithms. In a consensus algorithm, agents iteratively and in a distributed manner agree on a common state. The ideas of distributed consensus algorithms can be traced back to the seminal works, see DeGroot (1974); Tsitsiklis (1984). Recent works on consensus algorithms are focused on designing protocols to drive agents to the average of their initial states, see Kempe et al. (2003); Hadjicostis and Charalambous (2013). These protocols were designed for cases where the state of each agent is a scalar value. However, the increasing storage and computation capabilities of the modern day sensor interfacing technologies have motivated large-scale applications, examples of which include distributed machine learning, see Predd et al. (2009), multi-agent control and co-ordination, see Fax and Murray (2002); Olfati-Saber et al. (2007), distributed optimization problems, see Nedić and Olshevsky (2014); Khatana et al. (2019), distributed sensor localization, see Khan et al. (2009). In order to meet the requirements of such applications there is need of distributed consensus algorithms that allow for vector states. Khan et al. (2010) presents such a higher dimensional consensus protocol. The framework in Khan et al. (2010) is based on a leader-follower architecture with the agents being partitioned into anchors and sensors. Anchors are agents with fixed states behaving as leaders in the algorithm, while the sensors change their state by taking

a convex combination of their state with the neighboring nodes' states. In such framework, algorithm converges to the state of the anchors. Cyber physical systems such as electrical power networks need to accommodate for large number of states for crucial applications such as state-estimations, optimal dispatch, demand response for ancillary services, etc. Patel et al. (2017) formulates the distributed apportioning problem using consensus protocols where only a single state is shared for each protocol. A similar situation involving higher dimensional states arises in distributed resource allocation problems where a fixed amount of resource, is required to be apportioned among all participating agents in a network and a convex cost associated with the each resource is to distributively minimized. The method used to solve the above resource allocation problem involves a higher dimensional consensus protocol, see Nedic et al. (2010).

Termination of consensus algorithms in finite time provides an advantage of getting an approximate consensus while saving the valuable computation and communication resources. For the scalar average consensus protocols discussed earlier, the authors in Yadav and Salapaka (2007); Prakash et al. (2018) have proposed a finite time stopping criteria utilizing two additional states namely the global maximum and global minimum over the network. This allows each agent to distributively detect the convergence to the (approximate) average and terminate further computations. The works in Saraswat et al. (2019); Prakash et al. (2019) generalize this result to the cases of dynamic interconnection topology and communication delays. The authors in Sundaram and Hadjicostis (2007) have also presented a method based on the minimal polynomial associated with the weight matrix in the state update iterations to achieve the consensus value in a finite number of iterations. However, to calculate the coefficients of the

minimal polynomial each node has to run n (total number of agents) different linear iterations each for at least $n+1$ time-steps.

In this article, we present a distributed stopping criteria for the higher dimensional consensus problem. We present the evolution of the convex hull of network states indexed by time, and show that they form a nested sequence. Based on this insight, we provide an algorithm which guarantees the convergence of consensus algorithm in norm, within any given tolerance.

The rest of the paper is organized as follows. In Section 2, the basic definitions needed for subsequent developments are presented. Further, we discuss the setup for the distributed average consensus in higher dimensions (called the vector consensus problem) using ratio consensus. Sections 3 presents an analysis on the polytopes of the network states generated in the ratio consensus algorithm. Section 4 establishes a norm-based finite-time termination criterion for the vector consensus problem. Theoretical findings are validated with simulations presented in Section 5 followed by conclusions in Section 6. Most proofs are omitted for space considerations and brevity of exposition. For a full discussion, the reader is directed to the journal version Melbourne et al. (2020).

2. DEFINITIONS, AND PROBLEM STATEMENT

2.1 Definitions and Notations

In this section we present basic notions of graph theory and linear algebra which are essential for the subsequent developments. Detailed description of graph theory and linear algebra notions are available in Diestel (2006); Horn and Johnson (2012) respectively.

Definition 1. (Cardinality of a set) Let A be a set. The cardinality of a set A denoted by $|A|$ is the number of elements of the set A.

Definition 2. (Directed Graph) A directed graph (denoted as digraph) $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set of vertices or nodes and $\mathcal{E}$ is a set of edges, which are ordered subsets of two distinct elements of $\mathcal{V}$. If an edge from $j \in \mathcal{V}$ to $i \in \mathcal{V}$ exists then it is denoted as $(i,j) \in \mathcal{E}$.

Definition 3. (Path) In a digraph, a directed path from node i to j exists if there is a sequence of distinct directed edges of $\mathcal{G}$ of the form $(k_1, i), (k_2, k_1), ..., (j, k_m)$. For the rest of the article, a path refers to a directed path.

Definition 4. (Strongly Connected Graph) A digraph is strongly connected if it has a path between each pair of distinct nodes i and j.

Definition 5. (In-Neighborhood) Set of in-neighbors of node $i \in \mathcal{V}$ is denoted by $N_i^- = \{j \mid (i,j) \in \mathcal{E}\}$. In this article, we assume $(i,i) \in N_i^-$ for all $i \in \mathcal{V}$.

Definition 6. (Diameter of a Graph) The diameter D of a graph is the longest shortest path between any two nodes in the network.

Definition 7. (Column Stochastic Matrix) A real $n \times n$ matrix $P = [p_{ij}]$ is called a column stochastic matrix if $1 \geq p_{ij} \geq 0$ for $1 \leq i, j \leq n$ and $\sum_{i=1}^{n} p_{ij} = 1$ for $1 \leq j \leq n$.

Definition 8. (Irreducible Matrix) A $N \times N$ matrix A is said to be irreducible if for any $i, j \in \{1, ..., N\}$, there exist

$m \in \mathbb{N}$ such that $(\mathbf{A}^m)(i,j) > 0$, that is, it is possible to reach any state from any other state in a finite number of hops.

Definition 9. (Primitive Matrix) A non negative matrix A is primitive if it is irreducible and has only one eigenvalue of maximum modulus.

Definition 10. (Convex hull) For a set $W \subseteq \mathbb{R}^d$, the convex hull of W is the smallest convex set containing E,

$$co(W) = \bigcap_{\{F \text{ convex } : A \subseteq F\}} F. \tag{1}$$

Definition 11. We define the usual dot product between two vectors $x, y \in \mathbb{R}^d$ by $x \cdot y = \sum_{i=1}^{d} x_i y_i$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots y_n)$.

2.2 Vector Consensus framework

Here, we extend a key result from Kempe et al. (2003); Dominguez-Garcia and Hadjicostis (2010) where a ratio of two states was maintained to reach average consensus. We consider the network topology to be represented by a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ containing n nodes and satisfies the following assumptions:

Assumption 1. The digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ representing the agent interconnections is strongly-connected.

Assumption 2. Let $P = [P_{ij}]$ be a primitive column stochastic matrix with digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with $p_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$

Each node $i \in \mathcal{V}$ maintains three state estimates at time k, denoted by $x^i(k) \in \mathbb{R}^d$ (referred as numerator state of node i), $y_i(k) \in \mathbb{R}$ (referred as denominator state of node i) and $r^i(k) \in \mathbb{R}^d$ (referred as ratio state of node i). Here d (≥ 1) is the dimension of each node's state. Node i updates its numerator and denominator states at the $(k+1)^{th}$ discrete iteration according to the following update law:

$$x^i(k+1) = \sum_{j \in N_i^-} p_{ji} x^j(k), \tag{2}$$

$$y_i(k+1) = \sum_{j \in N_i^-} p_{ji} y_j(k), \tag{3}$$

where, N_i^- is the set of in-neighbors of node i. The initial conditions for the numerator vector state and denominator state for any node $i \in \mathcal{V}$ are:

$$x^i(0) = [x_1^i(0) \ x_2^i(0) \ldots x_d^i(0)]^T, \ y_i(0) = 1. \tag{4}$$

Node i further updates its ratio state as:

$$r^i(k+1) = \frac{1}{y_i(k+1)} x^i(k+1). \tag{5}$$

Under Assumptions 1, 2 and the initialization in (4), ratio state in 5 is well defined. Next theorem establishes the convergence of the ratio state.

Theorem 2.1. Let $\{x^i(k)\}$, $\{y_i(k)\}$ and $\{r^i(k)\}$ be the sequences generated by (2), (3) and (5) respectively. Let the initial conditions for the network states be as defined in (4). Then, under Assumptions 1 and 2 the ratio state $r^i(k)$ asymptotically converges to $\bar{r} := \lim_{k \to \infty} \frac{1}{y_i(k)} x^i(k) = \frac{1}{n} \sum_{j=1}^{n} x^j(0)$ for all $i \in \{1, ..., n\}$.

Proof. Let $r_s^i(k)$ and $x_s^i(k)$ be the s-th elements of $r^i(k)$ and $x^i(k)$ respectively. Then the update equations (2) and (5) can be written as

$$x_s^i(k+1) = \sum_{j \in N_i^-} p_{ji} x_s^j(k)$$

$$r_s^i(k+1) = \frac{1}{y_i(k+1)} x_s^i(k+1)$$

Then from Kempe et al. (2003); Dominguez-Garcia and Hadjicostis (2010), for all $s \in \{1, 2, \ldots, d\}$ we have $\lim_{k \to \infty} r_s^i(k) = \frac{1}{n} \sum_{j=1}^n x_s^j(0)$. Therefore,

$$\lim_{k \to \infty} r^i(k) = \left[\lim_{k \to \infty} r_1^i(k), \lim_{k \to \infty} r_2^i(k), \ldots, \lim_{k \to \infty} r_d^i(k) \right]^T$$

$$= \left[\frac{1}{n} \sum_{j=1}^n x_1^j(0), \frac{1}{n} \sum_{j=1}^n x_2^j(0), \ldots, \frac{1}{n} \sum_{j=1}^n x_d^j(0) \right]^T$$

$$= \frac{1}{n} \sum_{j=1}^n [x_1^j(0), x_2^j(0), \ldots, x_d^j(0)]^T$$

$$= \frac{1}{n} \sum_{j=1}^n x^j(0) \qquad \square$$

Thus, we have convergence in the case when node states are vectors.

3. A MONOTONICITY RESULT FOR CERTAIN CONSENSUS PROTOCOLS

In this section we demonstrate a monotonicity property of the ratio consensus problem. We will show that under the update framework given by (5), the network states $\{r^i(k)\}_{i=1}^n$ at time k can be used to define a sequence of polytopes $\{r_k\}_{k=0}^\infty$ as

$$r_k := co(\{r^i(k)\}_{i=1}^n) \tag{6}$$

We then prove that these polytopes are descending in the sense that $r_k \supseteq r_{k+1}$. In the process we recover monotonicity results of the min-max protocols from the one dimensional case, see Prakash et al. (2019), as well as their generalization to the vector case, see Khatana et al. (2019).

This insight is used to develop a distributed stopping criteria, guaranteeing convergence of all nodes vector ratio state is within an ε-ball of the consensus for a general norm.

Lemma 3.1. Let $p_{ij} \geq 0$, $v \in \mathbb{R}^d$, $x^1, \ldots, x^n \in \mathbb{R}^d$, and $y_1, \ldots, y_n \in \mathbb{R}$ with $M_v \in \mathbb{R}$ such that $x_i \cdot v \leq M_v y_i$ for all i then

$$\sum_j p_{ij} x_j \cdot v \leq M_v \sum_j p_{ij} y_j.$$

holds for all i as well.

Proof. Computing directly,

$$\sum_j p_{ij} x_j \cdot v \leq \sum_j p_{ij} M_v y_j = M_v \sum_j p_{ij} y_j. \tag{7}$$

$\square$

Theorem 3.1. The vector ratio consensus algorithm is monotonically convex in the sense for $r^i(k)$ defined in (5) $r_k \subseteq r_{k+1}$, for r_k polytopes defined in (6).

Proof. If $r_k = co(\{r^i(k)\}_{i=1}^n)$ is contained in a half space $\{w : w \cdot v \leq M_v\}$ then in particular $(x^i(k)/y_i(k)) \cdot v \leq M_v$. Applying Lemma 3.1, $\sum_j p_{ij} x^j(k) \cdot v \leq M_v \sum_j p_{ij} y_j(k)$, so that $r^i(k+1) = \frac{x^i(k+1)}{y_i(k+1)}$ is an element of $\{x : x \cdot v \leq M_v\}$ as well. Since the half space is arbitrary, and r_{k+1} is the intersection of all closed halfspaces containing it, $r^i(k+1) \in r_k$, and hence $r_{k+1} \subseteq r_k$. $\square$

When $d = 1$, the convex hull is simply described, $co(r(k)) = [\min_j \{r^j(k)\}_{j=1}^n, \max_j \{r^j(k)\}_{j=1}^n]$, so that $co(r(k)) \supseteq co(r(k+1))$ gives the monotonicity results from Prakash et al. (2019), $\min_j \{r^j(k)\}_{j=1}^n, \leq \min_j \{r^j(k+1)\}_{j=1}^n$ and $\max_j \{r^j(k)\}_{j=1}^n, \geq \max_j \{r^j(k+1)\}_{j=1}^n$. In the d-dimensional case, this improves Theorem 4 of Khatana et al. (2019).

4. NORM BASED FINITE-TIME TERMINATION

Similar to the convex hull comprising all points (corresponding to each agent), radius of a minimal norm ball in d dimension enclosing all the points can also be used as a termination criterion. Once the radius is within some bound ε, it can be easily shown that every agent's state is within 2ε of the consensus value. Even in the $p-norm$ case calculation of minimum norm ball in a distributed manner is a difficult problem, see Fischer (1975). We provide an algorithm which distributedly finds an approximation of minimal ball at each agent. We show that the minimal ball is enclosed in this approximation, thus if the approximate ball's radius is within ε then the minimal ball's radius is within ε as well. This is established in next Lemma.

Lemma 4.1. Let $\{r^i(k)\}$ be the sequence generated by the consensus protocol of (5). For all $i \in \mathcal{V}$, let

$$R_i(k+1, k') := \max_{j \in N_i^-} \{\|r^i(k'+k+1) - r^j(k'+k)\|$$
$$+ R_j(k, k')\} \tag{8}$$

with $R_i(0, k') := 0$ and $k' \geq 0$. Then

$$r^j(k') \in B\{R_i(D, k'), r^i(k'+D)\} \tag{9}$$

where $B\{R_i(D, k'), r^i(k'+D)\}$ is a ball of radius $R_i(D, k')$ centered at $r^i(k'+D)$ and D is the diameter of the underlying graph topology.

Proof. Proof is omitted due to page constraints.

Lemma 4.1 provides an easy and distributed way to find a ball which encloses all the nodes. Only information needed by a node is the current radius of its neighbors (along with the states pertaining to ratio consensus) and it can calculate the final radius within D iteration. Further, since the ball $B\{R_i(D, k'), r^i(k'+D)\}$ encloses all the nodes, it also encloses the minimum ball, as mentioned earlier. We next present a framework which we use to prove that this radius converges to 0 and can be used as a distributed stopping criterion.

Let the element-wise maximum and minimum of the ratio states over all the agents at any time instant k be given by, $M(k) = [M_1(k) \ M_2(k) \ \ldots M_d(k)]^T$ and $m(k) = [m_1(k) \ m_2(k) \ \ldots m_d(k)]^T$ respectively. That is,

$$M_s(k) := \max_{i \in \mathcal{V}} r_s^i(k) \tag{10}$$

$$m_s(k) := \min_{i \in \mathcal{V}} r_s^i(k) \tag{11}$$

where $M_s(k) \in \mathbb{R}$, $m_s(k) \in \mathbb{R}$ for all $s \in \{1, 2, \dots, d\}$ and $r_s^i(k)$ is the s-th elements of $r^i(k)$. Then from Prakash et al. (2018), for all time instants $k' \geq k$ and for all $i \in \mathcal{V}$ and $s \in \{1, 2, \dots, d\}$,

$$m_s(k) \leq r_s^i(k') \leq M_s(k). \qquad (12)$$

Further from Prakash et al. (2018), for all $i \in \mathcal{V}$, $l \geq 0$ and $s \in \{1, 2, \dots, d\}$,

$$M_s((l+1)D) < M_s(lD) \qquad (13)$$
$$m_s((l+1)D) > m_s(lD). \qquad (14)$$

By using (12), (13) and (14), we can prove the following theorem.

Theorem 4.1. Consider the consensus protocol of (2), (3) and (5). Let Assumptions 1 and 2 hold. Then,

$$\lim_{l \to \infty} M(lD) = \frac{1}{n} \sum_{i=1}^{n} x^i(0), \ \lim_{l \to \infty} m(lD) = \frac{1}{n} \sum_{i=1}^{n} x^i(0)$$

where, $M(k)$ and $m(k)$ are as defined earlier.

Proof. Proof is omitted due to page constraints.

Corollary 4.1. Consider the consensus protocol of (2), (3) and (5). Let Assumptions 1 and 2 hold. Then,

$$\lim_{l \to \infty} \|M(lD) - m(lD)\| = 0.$$

Proof . The proof directly results from Theorem 4.1 and is left to the reader. □

It is clear from Lemma 4.1 that at any instant k, all agents ratio states are within $2R_i(D, k)$ of each other, that is,

$$\max_{i,j \in \mathcal{V}} \|r^i(k) - r^j(k)\| \leq 2R_i(D, k). \qquad (15)$$

Thus if $R_i(D, k)$ is within a tolerance $\varepsilon/2$, all the agents ratio state will be within ε of consensus. We next provide convergence result for $R_i(D, k)$ as $k \to \infty$.

Theorem 4.2. Consider the consensus protocol of (5) and update in (8). Let $\overline{R_i}(l) := R_i(D, lD)$ for $l = 0, 1, 2, \dots$ and all $i \in \mathcal{V}$. Then

$$\lim_{l \to \infty} \overline{R_i}(l) = 0 \qquad (16)$$

for all $i \in \mathcal{V}$.

Proof. Proof is omitted due to page constraints.

Notice that $R_i(l)$ can be different for different nodes and each node might detect ε-convergence ($R_i(l) < \varepsilon$) at different time instants. But according to Lemma 4.1, once $R_i(l) < \varepsilon$ for any $i \in \mathcal{V}$, $\|r^i(lD) - r^j(lD)\| < 2\varepsilon$, that is the ratio state is within 2ε of consensus value, and the consensus is achieved. Further, any node i which detects convergence can propagate a "converged flag" in the network. To take that into account, we run a separate 1-bit consensus algorithm (denoted as convergence consensus) for each node where each node maintains a convergence state $b_i(k)$ and shares it with neighbors. Each node initializes $b_i(k)$ at every lD iteration for $l \in \{0, 1, 2, \dots\}$ with 1 or 0 depending on the node has detected convergence or not, and updates its value on every iteration using,

$$b_i(k+1) = \bigcup_{j \in N_i^-} b_j(k), \qquad (17)$$

where $\bigcup$ denotes "OR" operation, $k \geq 0$ and $b_j(0) = 1$ if node j has detected convergence at initialization instant 0

Input:
 | $\varepsilon, x^i(0)$;   // Initial condition
Initialize:
 | $k := 0$; $R_i(0) := 0$; $y_i(0) = 1$; $b_i(0) = 0$; $l := 1$;
Repeat:
 | **Input:**
 | | $x^j(k), y_j(k), R_j(k), j \in N_i^-$
 | /* ratio consensus updates of node i given
 | by (2), (3) and (5)   */
 | $x^i(k+1) := \sum\limits_{j \in N_i^-} p_{ji}(k) x^j(k)$;
 | $y_i(k+1) := \sum\limits_{j \in N_i^-} p_{ji}(k) y_j(k)$;
 | $r^i(k+1) := \frac{1}{y_i(k+1)} x^i(k+1)$;
 | /* radius updates of node i given by (8)
 | */
 | $R_i(k+1) := \max\limits_{j \in N_i^-} \{\|r^i(k+1) - r^j(k)\| + R_j(k)\}$
 | $b_i(k+1) = \bigcup\limits_{j \in N_i^-} b_j(k,)$
 | **if** $k = lD$ **then**
 | | **if** $b_i(k+1) = 1$ **then**
 | | | **break**
 | | **else**
 | | | $\overline{R_i}(l) = R_i(k+1)$;
 | | | **if** $(\overline{R_i}(l) < \varepsilon)$ **then**
 | | | | $b_i(k+1) = 1$
 | | | **else**
 | | | | $R_i(k+1) = 0$; $b_i(k+1) = 0$;
 | | | | $l = l + 1$;
 | **end**
 | $k = k + 1$;

Algorithm 1: Finite-time termination of ratio consensus in higher-dimension d (at each node $i \in \mathcal{V}$)

and $b_j(0) = 0$ otherwise. Clearly, if $b_j(0) = 1$ for any $j \in \mathcal{V}$, then $b_i(D) = 1$ for all $i \in \mathcal{V}$ where D is the diameter. Thus each node can use $b_i(D)$ as a stopping criterion.

Using above discussion and Theorem 4.2, we present an algorithm (see Algorithm 1) which calculates the radius $\overline{R_i}(l)$ for $l = 0, 1, 2, \dots$ and all $i \in \mathcal{V}$ and provides a finite-time stopping criterion for vector consensus.

Theorem 4.3. Algorithm 1 converges in finite-time simultaneously at each node.

Proof . From Corollary 4.1, it follows that $\overline{R_i}(l) \to 0$ as $l \to \infty$. Thus, for any given $\varepsilon > 0$ and node $i \in \mathcal{V}$ there exists an integer $t(\varepsilon, i)$ such that for $l = t(\varepsilon, i)$, $\overline{R_i}(l) < \varepsilon$. As each node has access to $\overline{R_i}(l)$, convergence can be detected by each node and the convergence bit $b_i(lD + 1)$ will be set to 1. Thus $b_i(lD + D + 1) = 1$ for all $i \in \mathcal{V}$ and algorithm will stop simultaneously at each node. □

Remark 1. Notice that using the above protocol, each node detects convergence simultaneously. Further, the only global parameter needed for Algorithm 1 is the knowledge of diameter D. However, it should be noted that each node does not need to know the actual diameter D but some upper bound. In most applications, an upper bound on the diameter D is readily available.

Remark 2. It is to note here that for Algorithm 1, the only extra communication required between nodes is passing of

the current radius at each node, a scalar value, along with a single bit for convergence consensus. Therefore the total bandwidth required for each neighbor-neighbor interaction is $((d+1)B+1)$ where B is the bit length (usually 32) for floating point representation. Thus, the above protocol is suitable for *ad-hoc* communication networks where communication cost is high and bandwidth is limited.

A finite-time termination criterion for vector consensus was previously provided in Khatana et al. (2019). There, each element of ratio state required a maximum-minimum protocol (see (10) and (11)), with stopping criterion given by,

$$\| \max_{s\in\{1,2,...,d\}} \max_{i\in\mathcal{V}} r_s^i(k) - \min_{s\in\{1,2,...,d\}} \min_{i\in\mathcal{V}} r_s^i(k)\| < \varepsilon$$

This maximum-minimum is a special case for finding a minimum convex set in the form of a hyper rectangle (box) which encompasses all the points. Here, at each iteration, two extra states are shared by each node, namely, one state for element-wise maximum and the other for element-wise minimum. Thus the total communication bandwidth required for this algorithm is $3Bd$. An example case where $d=10, B=32$, requires bandwidth of 960 bits per interaction. For this example, Algorithm 1 only requires 353 bits per interaction, with a reduction of 63%. Thus for the applications with high dimensional vector consensus, algorithm reported here provides a reliable distributed stopping criterion with significantly less communication bandwidth.

5. RESULTS

In this section, we present simulation results to demonstrate finite-time stopping criterion for high-dimensional ratio consensus. A network of 25 nodes is considered which is represented by a randomly generated digraph (see Fig. 1 with diameter 6. Here the numerator state is chosen to be a 10-dimensional vector and selected randomly for every node. Equation (2), (3) and (5) are implemented in MATLAB and simulated. 2-norm of each node's ratio state is plotted in Figure 2 achieving convergence in 60 iterations.

Algorithm 1 is also implemented in MATLAB and the radius $\overline{R_i}(l)$ for all $i \in \mathcal{V}$ is plotted in Fig. 3. It can be clearly seen that radius comes under some pre-specified tolerance (0.0166, 1% of the norm of the consensus vector) within 60 iteration and is used as a stopping criterion by each node. Fig. 4 plots the two dimensional projection of the
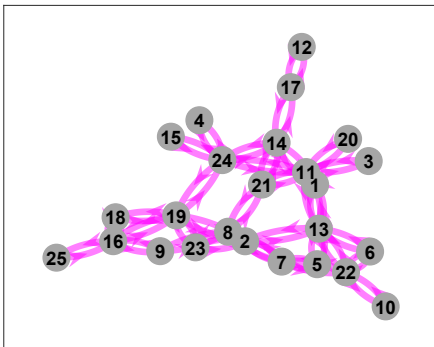

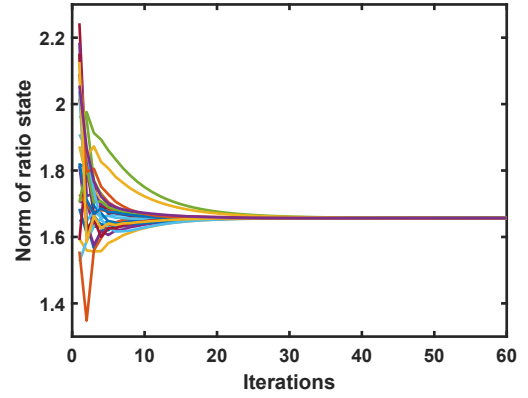Fig. 1. A communication network represented by a 25 node digraph.

Fig. 2. 2-norm of 10-dimensional ratio states of all the nodes (25) in the network.
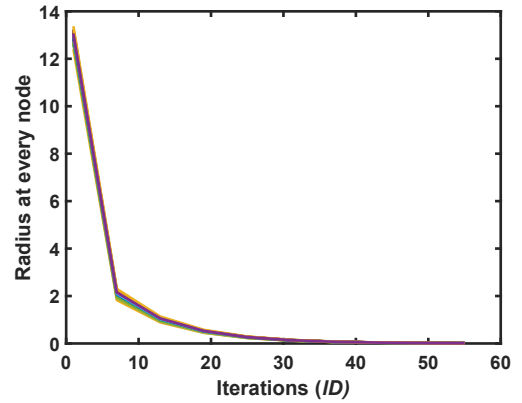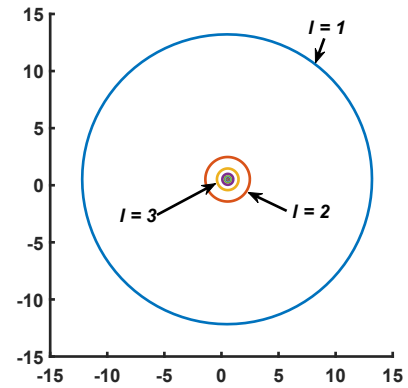

Fig. 3. Radius $\overline{R_i}(l)$ at each node.

Fig. 4. 2-dimensional projection of norm balls for node 1 with changing l.

norm ball $B\{\overline{R_i}(l), r^i(lD+D)\}$ for node 1 as l progresses over time. As expected, with increase in l, balls shrink in size. Similar observation is seen for all the other nodes as well. Thus we have established that Algorithm 1 can be successfully used as finite-time termination criterion for distributed ratio vector consensus.

6. CONCLUSION

In this article, we presented the monotonic property of convex hull of network states in vector consensus algorithms. We showed that this property can be used a finite-time stopping criterion and provided a distributed algorithm. We further provided an algorithm which calculates an approximation of minimum norm balls which contain all the network states at a given iteration. Radius of these balls was shown to converge to zero, and algorithm was presented to use that as a finite-time stopping criterion. This algorithm was shown to have much smaller communication requirement compared to existing methods. The effectiveness of our algorithm is validated by simulating a vector ($\in \mathbb{R}^{10}$) ratio consensus algorithm for a network graph of 25 nodes.

REFERENCES

DeGroot, M.H. (1974). Reaching a consensus. *Journal of the American Statistical Association*, 69(345), 118–121.

Diestel, R. (2006). *Graph Theory*. Springer-Verlag, Berlin, Germany.

Dominguez-Garcia, A.D. and Hadjicostis, C.N. (2010). Coordination and control of distributed energy resources for provision of ancillary services. In *Smart Grid Communications (SmartGridComm), 2010 First IEEE International Conference on*, 537–542. IEEE.

Fax, J.A. and Murray, R.M. (2002). Information flow and cooperative control of vehicle formations. *IFAC Proceedings Volumes*, 35(1), 115–120.

Fischer, K. (1975). *Smallest enclosing balls of balls*. Ph.D. thesis, ETH Zürich.

Hadjicostis, C.N. and Charalambous, T. (2013). Average consensus in the presence of delays in directed graph topologies. *IEEE Transactions on Automatic Control*, 59(3), 763–768.

Horn, R.A. and Johnson, C.R. (2012). *Matrix analysis*. Cambridge university press.

Kempe, D., Dobra, A., and Gehrke, J. (2003). Gossip-based computation of aggregate information. In *44th Annual IEEE Symposium on Foundations of Computer Science, 2003. Proceedings.*, 482–491. IEEE.

Khan, U.A., Kar, S., and Moura, J.M. (2009). Distributed sensor localization in random environments using minimal number of anchor nodes. *IEEE Transactions on Signal Processing*, 57(5), 2000–2016.

Khan, U.A., Kar, S., and Moura, J.M. (2010). Higher dimensional consensus: Learning in large-scale networks. *IEEE Transactions on Signal Processing*, 58(5), 2836–2849.

Khatana, V., Saraswat, G., Patel, S., and Salapaka, M.V. (2019). Gradient-consensus method for distributed optimization in directed multi-agent networks. *arXiv preprint arXiv:1909.10070*.

Melbourne, J., Saraswat, G., Khatana, V., Patel, S., and Salapaka, M.V. (2020). Monotonicity of consensus in high dimension and a peer to peer convex hull algorithm. *Preprint*.

Nedić, A. and Olshevsky, A. (2014). Distributed optimization over time-varying directed graphs. *IEEE Transactions on Automatic Control*, 60(3), 601–615.

Nedic, A., Ozdaglar, A., and Parrilo, P.A. (2010). Constrained consensus and optimization in multi-agent networks. *IEEE Transactions on Automatic Control*, 55(4), 922–938.

Olfati-Saber, R., Fax, J.A., and Murray, R.M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1), 215–233.

Patel, S., Attree, S., Talukdar, S., Prakash, M., and Salapaka, M.V. (2017). Distributed apportioning in a power network for providing demand response services. In *2017 IEEE International Conference on Smart Grid Communications (SmartGridComm)*, 38–44. IEEE.

Prakash, M., Talukdar, S., Attree, S., Patel, S., and Salapaka, M.V. (2018). Distributed Stopping Criterion for Ratio Consensus. In *2018 56th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, 131–135.

Prakash, M., Talukdar, S., Attree, S., Yadav, V., and Salapaka, M.V. (2019). Distributed stopping criterion for consensus in the presence of delays. *IEEE Transactions on Control of Network Systems*.

Predd, J.B., Kulkarni, S.R., and Poor, H.V. (2009). A collaborative training algorithm for distributed learning. *IEEE Transactions on Information Theory*, 55(4), 1856–1871.

Saraswat, G., Khatana, V., Patel, S., and Salapaka, M.V. (2019). Distributed finite-time termination for consensus algorithm in switching topologies. *arXiv preprint arXiv:1909.00059*.

Sundaram, S. and Hadjicostis, C.N. (2007). Finite-time distributed consensus in graphs with time-invariant topologies. In *2007 American Control Conference*, 711–716. IEEE.

Tsitsiklis, J.N. (1984). Problems in decentralized decision making and computation. Technical report, DTIC Document.

Yadav, V. and Salapaka, M.V. (2007). Distributed protocol for determining when averaging consensus is reached. In *45th Annual Allerton Conf*, 715–720.