Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Multiparametric Nonlinear MPC: A region free approach

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Abstract: Determination of active constraints forms an essential part of the multiparametric MPC approach for linear systems. An analysis of KKT conditions to identify active constraints provides piecewise affine control laws and their corresponding critical regions (CRs). However, an extension of multiparametric MPC for nonlinear systems requires overcoming significant challenges: predictions are nonlinear and so are constraints, in which case the MPC problems takes the form of a nonlinear program (NLP). Application of KKT conditions show that, in general, the MPC control law for nonlinear systems is piecewise, implicit, nonlinear function of the state. Moreover, the CRs have nonlinear boundaries. In this work, we propose an offline combinatorial approach to identify all active sets of constraints for the nonlinear MPC problem a priori. The offline approach uses implicit enumeration of the constraints based on feasibility of KKT conditions and a primal criteria. Thus, the offline step provides all the admissible CRs as well as the corresponding nonlinear system of KKT equations corresponding to each CR. The online MPC implementation uses a region-free approach, wherein the CR corresponding to the current state as well as the control action is determined by solving the nonlinear system of KKT equations online. The method is demonstrated using a numerical example from literature.

Keywords: Multiparametric MPC, Nonlinear systems, KKT conditions, Active sets

1. INTRODUCTION

Explicit model predictive control (e-MPC) was developed as an approach to obtain optimal, parameter (that is, state) dependent control law for linear systems. The control law takes the form of a piecewise affine (PWA) solution with each piece being defined over a polytopic region in the state space called the critical region (CR). The control law and the CRs are obtained via an offline analysis of the Karush-Kuhn-Tucker (KKT) conditions of the constrained optimal control problem. In e-MPC, the online MPC optimization problem is replaced with an online function evaluation problem (Bemporad et al., 2002). In context of nonlinear systems, the constrained optimal control problem takes the form of a Nonlinear programming (NLP). While a number of advances continue to be reported for linear systems (for example, Sun et al. (2019); Mönnigmann (2019)), solutions for nonlinear e-MPC (e-NMPC) are relatively scarce.

Methods for e-NMPC in (Dominguez and Pistikopoulos, 2011; Grancharova and Johansen, 2002; Johansen, 2002) use first order approximation of the KKT conditions or linear interpolation of the solution of the NLP obtained at different parametric realizations. These methods are approximate solutions of the multiparametric nonlinear program (mp-NLP) and, therefore, not exact. An attempt to provide an exact solution to the mp-NLP problem is presented in (Charitopoulos and Dua, 2016), wherein the parameter dependent KKT conditions are solved symbolically to yield an explicit e-NMPC control law. However, it is well-known that symbolic computations cannot be scaled to higher dimensional systems, particularly where inversion of symbolic matrices is involved. Robust tube based framework for e-NMPC, by decomposing the state-space into hyper-rectangles, has been reported in (Bayer et al., 2016).

In (Mönnigmann et al., 2015), the NLP is solved online and an implicit "regional control law" is determined corresponding to its "region of validity" characterized by the active set. If the active set indeed changes relative to the previous instant, then the original MPC problem is solved implicitly. However, in this approach the change in active set is "discovered" online and not known a priori. In this paper, we present an offline methodology to determine all optimal active sets of the NLP parameterized by the initial state, which also prescribes the regions of validity. The offline technique is an extension of the implicit enumeration approach given in (Gupta et al., 2011) for multiparametric NLP (mp-NLP).

Notation: Let $\mathbb{R}, \mathbb{Z}$, and $\mathbb{Z}_+$ denote the set of real numbers, integers and non-negative integers, respectively. $|\cdot|$ is cardinality of a set and $k \in \mathbb{Z}_+$. $\chi_N \triangleq (x(1), \dots, x(N))$ represents a sequence of predictions for the nonlinear system defined later. For an initial condition $x(0)$ at $k = 0$, $U_N \triangleq (u(0), \dots, u(N-1))$ represents a sequence of length

$N \in \mathbb{Z}_+$. U_N^* represents an optimal solution and $u^*(0)$ is its first element. $G_{\mathcal{A}}$ is the constraint matrix corresponding to an index set $\mathcal{A}$ and refers to a matrix comprising of specific rows of G as identified in $\mathcal{A}$.

2. PRELIMINARIES AND MULTIPARAMETRIC NONLINEAR MPC PROBLEM

Consider the discrete-time nonlinear system,

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

$x(k) \in \mathbb{X} \subset \mathbb{R}^n$ and $u(k) \in \mathbb{U} \subset \mathbb{R}^q$ represent states and inputs, respectively. Constraints $\mathbb{X}$ and $\mathbb{U}$ are non-empty, convex, polyhedral, compact and contain the origin in their interiors. $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ in (1) is assumed to be twice continuously differentiable and $f(0,0) = 0$. It is also assumed that all states are measurable.

Problem 1. **Nonlinear MPC Problem**: For an initial condition $x(0) \in \mathbb{X}$, solve the following optimization problem with horizon N,

$$J^0(x(0)) \triangleq \min_{U_N, \chi_N} \{V(x(N)) + \sum_{i=0}^{N-1} L(x(i), u(i))\} \tag{2a}$$

$$\text{Eq}(1), x(k) \in \mathbb{X}, u(k) \in \mathbb{U} x(N) \in \mathbb{X}_T \subset \mathbb{X} \tag{2b}$$

where $N \in \mathbb{Z}_{\geq 1}$,$V(x(N)) = x^T(N)P_T x(N)$ is the terminal cost, $L(x(i), u(i)) = x^T(i)Qx(i) + u^T(i)Ru(i)$ is the cost of i^{th} stage, $P_T \succ 0$, $Q \succ 0$, $R \succ 0$ are matrices of appropriate sizes, and $\mathbb{X}_T$ is a terminal set. The MPC Problem 1 is nonlinear in nature and the optimal solution (U_N^*, χ_N^*) is obtained for an initial condition $x(0)$. $u^*(0)$ is injected as control input into system Eq. (1) and MPC Problem 1 is solved again for the next sampling instant. Next, we discuss the nonlinear optimization problem resulting from Problem 1.

2a. Multiparametric Nonlinear Programming

It is readily seen that MPC Problem 1 takes the form of the following NLP problem.

Problem 2. **mp-NLP Problem**

$$J(x(0)) \triangleq \min_z \frac{1}{2} z^T H z \tag{3a}$$

$$\mathcal{F}(z, x(0)) = 0 \tag{3b}$$

$$Gz - b \leq 0 \tag{3c}$$

where $z \triangleq \begin{bmatrix} \chi_N \\ U_N \end{bmatrix}$, $z \in \mathbb{R}^{N(n+q)}$ and let the number of polyhedral (hence linear) constraints in (3c) be p. $G \in \mathbb{R}^{p \times N(n+q)}$, $b \in \mathbb{R}^p$, $H \in \mathbb{R}^{N(n+q) \times N(n+q)}$. Assume that the objective function is convex, that is, $H \succ 0$. The the $n \times N$ equality constraints $\mathcal{F} = 0$ in (3b), representing the system predictions, are assumed differentiable with their Jacobian being full rank $\forall x(0) \in \mathbb{X}$. The above parametric NLP represents point of departure from the conventional mp-QP treated in the literature, which has a simpler form,

$$\min_{U_N} \frac{1}{2} U_N^T H U_N + x^T(0) F U_N \tag{4a}$$

$$G U_N \leq S + E x(0) \tag{4b}$$

with parameter $x(0)$ and all matrices, defined analogously, are constant arising from the MPC formulation of LTI systems. Next, active sets and KKT conditions required for Problem 2 are introduced.

Definition 1. **Active and Inactive Sets**: Let $\mathcal{M} \triangleq \{1, 2, 3, ..., p\}$ be the set of indices of all constraints in (3c). For a given z and $x(0)$, the elements of $\mathcal{M}$ can be categorized into two sets as follows:

(i) an active set $\mathcal{A}(z, x(0))$, is one for which the equality in (3b)-(3c) holds, that is

$$\mathcal{A}(z, x(0)) \triangleq \{i \in \mathcal{M} | \mathcal{F}(z, x(0)) = 0, G_i z = b_i\} \tag{5}$$

(ii) the corresponding inactive set for active set in (i) is $\mathcal{J}(z, x(0))$, where strict inequality holds among constraints in (3c),

$$\mathcal{J}(z, x(0)) \triangleq \mathcal{M} \setminus \mathcal{A}(z, x(0)) \tag{6}$$

Note that indices of equality constraints (3b) are not included in $\mathcal{A}$ but implicitly assumed in the remainder of the paper for simplicity of notation. It is clear from (5) that the number of active sets $\mathcal{A}$ that could be constructed from $\mathcal{M}$ is related to the cardinality of the power set of $\mathcal{M}$ and hence p. Before going further we recall some definitions.

Definition 2. **Linear Independence Constraint Qualification (LICQ)** Nocedal and Wright (2006): Given the optimal point $z = z^*$ and corresponding active set $\mathcal{A}(z^*, x(0))$, we say that LICQ holds if the set of active constraint gradients $\begin{bmatrix} \nabla_z \mathcal{F}(z^*, x(0)) \\ \dots \\ G_{\mathcal{A}} \end{bmatrix}$ are linearly independent, that is, $\begin{bmatrix} \nabla_z \mathcal{F}(z^*, x(0)) \\ \dots \\ G_{\mathcal{A}} \end{bmatrix}$ has full row rank.

Remark 1. The case that some constraints in $G_{\mathcal{A}}$ and $\nabla_z \mathcal{F}(z^*, x(0))$ form an dependent set is excluded from this work. Thus, the assumption of Jacobian $\nabla_z \mathcal{F}(z^*, x(0))$ being full rank $\forall x(0) \in \mathbb{X}$ entails that the rank condition now depends on independence of the constraints $G_{\mathcal{A}}$.

KKT Conditions: Among the candidate active sets in (5), we restrict our attention to optimal active sets $\mathcal{A}(z^*, x(0))$, where the optimized decision variable z^* is obtained by solving the following KKT conditions corresponding to the mp-NLP Problem 2,

$$Hz + \nabla_z \mathcal{F}(z, x(0))^T \mu + G_{\mathcal{A}}^T \lambda_{\mathcal{A}} = 0 \tag{7a}$$

$$\mathcal{F}(z, x(0)) = 0 \tag{7b}$$

$$G_{\mathcal{A}} z - b_{\mathcal{A}} = 0 \tag{7c}$$

$$G_{\mathcal{J}} z - b_{\mathcal{J}} + s_{\mathcal{J}} = 0 \tag{7d}$$

$$\lambda_{\mathcal{A}} \geq 0, s_{\mathcal{J}} \geq 0 \tag{7e}$$

$\mu \in \mathbb{R}^{Nn}$ and $\lambda_{\mathcal{A}} \in \mathbb{R}^{|G_{\mathcal{A}}|}$ are column vectors of Lagrange multipliers corresponding to constraints in (3b) and $\{G_i z = b_i | i \in \mathcal{A}(z, x(0))\}$, respectively. Slack variables $s_{\mathcal{J}}$ correspond to inactive constraints $\mathcal{J}(z, x(0))$.

Definition 3. **Strict Complementarity Slackness** Nocedal and Wright (2006): Given the KKT pair (z^*, λ^*), SCS holds if exactly one of λ_i^* and $G_i z^* - b_i$ is zero for each $i \in \mathcal{M}$, that is $\lambda_i^* > 0$, for each $i \in \mathcal{M} \bigcap \mathcal{A}(z^*, x(0))$.

The exponential increase in the candidate active sets $\mathcal{A}$ due to p constraints in Problem 2 makes it computationally costly to verify satisfaction of KKT conditions for all candidate active sets, even for a small p. For conventional mp-QP, (Gupta et al., 2011) proposed an offline, implicit enumeration strategy based on pruning, in order to determine the optimal active sets $\mathcal{A}$ (and equivalently $\mathcal{J}$). In this work, the enumeration approach is presented to obtain solutions of the mp-NLP posed in Problem 2.

3. OFFLINE: IDENTIFYING OPTIMALLY ACTIVE AND INACTIVE CONSTRAINTS

To identify the set of optimally active constraints for the mp-NLP Problem 2 in (3a)-(3c), we extend the combinatorial method of (Gupta et al., 2011), which was limited only to conventional mp-QP in (4a), to mp-NLP Problem 2. Enumerate all candidate active sets for the mp-NLP Problem 2 in (3a)-(3c) from the power set $\mathcal{M}$ in the order of increasing cardinality from 0 to $N(n+q)$, the size of the decision variable z. The upper bound arises from the fact that only a maximum of $N(n+q)$ out of $Nn+p$ linearly independent constraint (Nn constraints correspond to (3b)) can be strongly active at the optimal solution (Nocedal and Wright, 2006). For each such candidate active set, a feasible solution to the KKT conditions in (7a)-(7e) is sought that satisfies strict primal and dual feasibility conditions ($s_{\mathcal{J}} > 0$, $\lambda_{\mathcal{A}} > 0$) for some value of the parameter $x(0) \in \mathbb{X}$. Obtaining such a parameter $x(0)$ implies that the candidate set $\mathcal{A}(z, x(0))$ is optimally active for some admissible value of the parameter, and this is equivalent to obtaining a critical region in the conventional mp-QP solution. This KKT analysis is achieved by solving the following NLP, which will be referred to as the "Dual Feasibility NLP".

Problem 3. **Dual Feasibility NLP**

$$\max_{t,z,x(0),\mu,\lambda_{\mathcal{A}},s_{\mathcal{J}}} \quad t \tag{8a}$$
$$te_1 \leq \lambda_{\mathcal{A}}, te_2 \leq s_{\mathcal{J}} \tag{8b}$$
$$Hz + \nabla_z \mathcal{F}(z, x(0))^T \mu + G_{\mathcal{A}}^T \lambda_{\mathcal{A}} = 0 \tag{8c}$$
$$\mathcal{F}(z, x(0)) = 0 \tag{8d}$$
$$G_{\mathcal{A}} z - b_{\mathcal{A}} = 0 \tag{8e}$$
$$G_{\mathcal{J}} z - b_{\mathcal{J}} + s_{\mathcal{J}} = 0 \tag{8f}$$
$$\lambda_{\mathcal{A}} \geq 0, s_{\mathcal{J}} \geq 0, t \geq 0 \tag{8g}$$
$$x(0) \in \mathbb{X} \tag{8h}$$

where t is an epigraphic scalar variable and e_1 and e_2 in (8b) are vectors of ones of appropriate dimensions corresponding to the vector of lagrange multipliers $\lambda_{\mathcal{A}}$ and slacks $s_{\mathcal{J}}$. The KKT conditions, active and inactive constraints are enforced through (8c), (8e), and (8f), respectively, while the primal, dual feasibility and positivity of t are enforced by (8g). The enumeration begins with the candidate active set of smallest cardinality namely, the empty set $\{\}$ as the root node and then active sets of successively increasing cardinality with a maximum cardinality of $N(n+q)$ as successive levels of the combinatorial tree are considered. Before presenting the pruning criterion to reduce the enumerations in fathoming the tree, a conjecture is made in the following remark.

Remark 2. Note that in view of Remark 1, LICQ failure can occur if $G_{\mathcal{A}}$ is not full row rank. In case of linear constraints in conventional mp-QP, it has been shown that provided a rank condition is satisfied, LICQ failure results in lower dimensional critical regions (Tøndel et al., 2003). We conjecture that for a candidate active set $\mathcal{A}$ in Problem 2, failure of LICQ due to rank deficiency in $G_{\mathcal{A}}$ will also result in lower dimensional CRs, whose solutions are subsumed by the full dimensional CRs. Thus, all supersets of $\mathcal{A}$ will also exhibit LICQ failure and will similarly result in lower dimensional CR.

Pruning Criterion: A two-fold pruning criteria is presented that speeds up the process of determining optimal active sets: (i) Pruning based on rank of $G_{\mathcal{A}}$: The matrix $G_{\mathcal{A}}$ is independent of $(z^*, x(0))$ and its row rank can be easily checked for LICQ failure. Now consider a candidate active set $\mathcal{A}(z, x(0))$, then from the conjecture in *Remark 2*, it is clear that Problem 3 corresponding to candidate active set, $\mathcal{A}(z, x(0))$, need not be solved.
(ii) Pruning based on primal infeasibility: Infeasibility of Problem 3 could arise from two situations 1) primal infeasibility: the candidate active (and corresponding inactive) constraints cannot be simultaneously satisfied for the range of parameters considered, that is, infeasibility of (8e)-(8h), or 2) dual infeasibility: constraints exhibit primal feasibility but KKT conditions in (8c) along with dual variable $\lambda_{\mathcal{A}} > 0$ cannot be simultaneously satisfied. To identify primal infeasibility, an associated NLP is solved that examines feasibility of only those constraints which are involved with the active and inactive constraints of Problem 2. This NLP is formed by excluding all constraints arising from the optimality condition (namely, all constraints that include the term $\lambda_{\mathcal{A}}$).

Problem 4. **Primal Infeasibility NLP**

$$\max_{t,z,x(0),s_{\mathcal{J}}} \quad t \tag{9a}$$
$$te_2 \leq s_{\mathcal{J}} \tag{9b}$$
$$\mathcal{F}(z, x(0)) = 0 \tag{9c}$$
$$G_{\mathcal{A}} z - b_{\mathcal{A}} = 0 \tag{9d}$$
$$G_{\mathcal{J}} z - b_{\mathcal{J}} + s_{\mathcal{J}} = 0 \tag{9e}$$
$$s_{\mathcal{J}} \geq 0, t \geq 0 \tag{9f}$$
$$x(0) \in \mathbb{X} \tag{9g}$$

If Problem 4 is found infeasible, it is clear that the same problem will be infeasible for all supersets of the candidate active set $\mathcal{A}$ (Gupta et al., 2011). Thus $\mathcal{A}$ and all its supersets can be pruned from the combinatorial tree from further examination. A graphical illustration of the combinatorial enumeration strategy and the involved pruning process is given in the form of a combinatorial tree diagram in Fig.1. The above is summarized in following proposition.

Proposition 1. Consider a candidate active set $\mathcal{A}(z, x(0))$ of the mp-NLP in Problem (2) and its corresponding inactive set $\mathcal{J}(z, x(0))$. Also assume that LICQ condition holds for Dual Feasibility NLP Problem 3 as well as primal Infeasibility NLP Problem 4, then the following statements, given as cases, considering all candidate active sets $\mathcal{A}'$ such that $\mathcal{A}' \supset \mathcal{A}$ are true:

(1.a) For a given candidate active set $\mathcal{A}(z, x(0))$, if the Primal Infeasibility NLP Problem 4 is infeasible with respect to $\mathcal{A}$, the Dual Feasibility NLP Problem 3 is infeasible with respect to all candidate active sets $\mathcal{A}^{'}$. Moreover the mp-NLP in Problem 2 will also be infeasible for all candidate active sets $\mathcal{A}^{'}, \forall x_0 \in \mathbb{X}$.

(1.b) For a given candidate active set $\mathcal{A}(z, x_0)$, if $t = 0$ is the optimal solution to Dual Feasibility NLP Problem 3, then the mp-NLP Problem 2 exhibits Strict Complimentary Slackness (SCS) failure with respect to active set $\mathcal{A}(z, x(0))$.

(1.c) For a given candidate active set $\mathcal{A}(z, x(0))$, existence of a local solution to Dual Feasibility NLP Problem 3 with $t > 0$ implies that $\mathcal{A}(z, x(0))$ is an optimal active set with respect to the mp-NLP Problem 2.

Proof. (1.a) This case is divided into three parts as given below. i) Infeasibility of Problem 4 with respect to candidate active set $\mathcal{A}(z, x(0))$ implies infeasible Problem 3 with respect to candidate active set $\mathcal{A}(z, x(0))$; ii) Infeasibility of Problem 4 with respect to candidate active set $\mathcal{A}(z, x(0))$ implies infeasibility of Problem 4 with respect to all candidate active sets $\mathcal{A}^{'} \supset \mathcal{A}$; and iii) Infeasibility of Problem 4 with respect to a candidate active set $\mathcal{A}(z, x(0))$ implies infeasibility of mp-NLP Problem 2 with respect to active set $\mathcal{A}(z, x(0))$. The proof for three part statement is given below.
i) Note that Problem 4 involves constraints (9b)-(9g) that distinctly appear in Problem 3. However, Problem 3, along with constraints $te_1 \leq \lambda_{\mathcal{A}}$ has additional equality constraints involving $\lambda_{\mathcal{A}}$ in (8c). Then, for a fixed $\lambda_{\mathcal{A}}^{'} \geq 0$, the feasible space of Problem 3 is a subset of Problem 4. Thus, for a candidate active set $\mathcal{A}(z, x(0))$ if Problem 4 is infeasible then, Problem 3 is also infeasible with respect to $\mathcal{A}(z, x(0))$. ii) Now consider all $\mathcal{A}^{'} \supset \mathcal{A}$ then, the feasible space of Problem 4 with respect to $\mathcal{A}$ will shrink further as additional constraints become active in $\mathcal{A}^{'}$ relative to $\mathcal{A}$. Thus, infeasibility of Problem 4 with respect to $\mathcal{A}$ also implies infeasibility of Problem 4 with respect to $\mathcal{A}^{'}$. The result given in part (i) of (1.a) therefore assures that Problem 3 is infeasible with respect to all active sets in $\mathcal{A}^{'}$.
iii) Since constraints of Primal Infeasibility NLP Problem 4 with respect to $\mathcal{A}^{'}$ are a part of the KKT conditions of mp-NLP Problem 2, infeasibility of the former problem implies infeasibility of mp-NLP Problem 2 with respect to $\mathcal{A}^{'}$.
(1.b) An optimal solution $t = \lambda^*_{\mathcal{A}_i} = 0$ of the Dual Feasibility NLP Problem 3 implies that for i^{th} constraint is weakly active weak active due to failure of SCS condition. Now construct another candidate active set $\mathcal{A}^{'}(z, x(0)) = \mathcal{A}(z, x(0)) \setminus i$ and its corresponding $\mathcal{J}^{'}(z, x(0)) = \mathcal{J}(z, x(0)) \bigcup i$. Then $s^*_{\mathcal{J}^{'}_i} = 0$ for the newly constructed candidate active set $\mathcal{A}^{'}(z, x(0))$ implies that $i \in \mathcal{J}^{'}(z, x(0))$ is a constraint that is weakly inactive. Thus, the feasibility Problem 3 with respect to constraint set $\mathcal{A}^{'}$ will also have an optimal solution $t = \min(s^*_{\mathcal{J}^{'}}) = 0$. Hence, it becomes clear from *Definition 3*, that both $\mathcal{A}$ and $\mathcal{A}^{'}$ will exhibit SCS failure.

(1.c) Let $t^* > 0$ be the optimal solution of Problem 3. The KKT conditions for Problem 2 are also the constraints

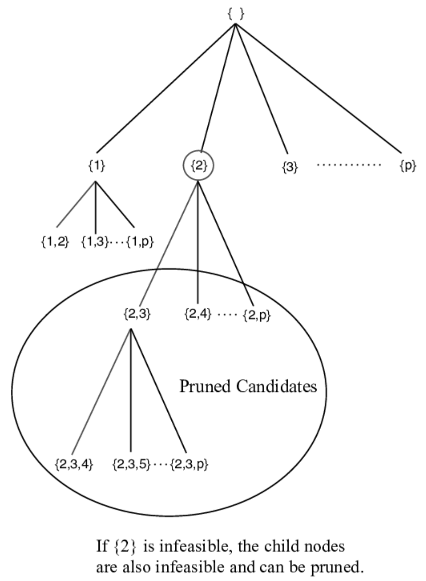

Fig. 1. Tree showing the active set based pruning criterion

(8c)-(8f) for Problem 3. Thus, an existence of a locally optimal solution to Problem 3 implies that $\exists\ \lambda^*_{\mathcal{A}} > 0$, $s^*_{\mathcal{J}} > 0$ for which (8e) holds with $\mathcal{A}(z, x(0))$ being the optimal active set.

Remark 3. Since Problem 3 is a non-convex NLP (due to Eq. (3b)), the outcome of the proposition is dependent on the choice of the solver: local or global. If a local solver is used, then case (1.c) may be misclassified as case (1.b). On the other hand, if a local solution results in case (1.c), it will not lead to misclassification.

Occurrence of cases (1.b) and (1.c) is characterized by feasibility of KKT conditions in (7a)-(7e), which ensures that there exist parameter values for which the set $\mathcal{A}(z, x(0))$ is optimally active. However, the nonlinearity of (7a) and (7b) in the KKT conditions restricts the possibility of having an explicit control law. In conventional mp-QP given by (4a)-(4b), the KKT matrix M is constant and can therefore be inverted in the offline step, while the parameter $x(0)$ appears linearly on the right side in $P(x(0))$. This inversion gives an explicit parametric solution as follows,

$$\begin{bmatrix} U_N(x(0)) \\ \lambda(x(0)) \end{bmatrix} = M^{-1} P(x(0)) \qquad (10)$$

However, in case of mp-NLP Problem 2, an attempt to find parameter-dependent solution $U_N(x(0))$ has two consequences: (1) the fully explicit solution for the MPC control law for nonlinear systems will take the form of piecewise, implicit, nonlinear function of $x(0)$ and obtaining it explicitly is, generally, not possible, (2) the critical region, that is, the largest set of the parameter $x(0)$ for which the optimally active set $\mathcal{A}^*(x(0))$ remains unchanged, is also nonlinear and hence not polyhedral. These consequences motivate use of approaches for which storage of critical regions and control laws is not required as discussed next.

4. ONLINE REGION-FREE SEARCH ALGORITHM

In the enumeration approach discussed in Section 3, all optimal active sets are determined independent of the critical region boundaries or the control law. In (Kvasnica et al., 2015), this feature is used in combination with region-free e-MPC for LTI systems (Borrelli et al., 2010), for an online solution of the conventional mp-QP in (4a), that, did not require construction and storage of critical regions offline. Corresponding to each optimal active set that is obtained in the offline step, the constant matrix M_i (or equivalently M_i^{-1}) and $P_i(x(0))$ needed in (10), is stored. The online step gives the optimal active set for a current value of parameter, say $x^{'}(0)$, by verifying if the primal and dual feasibility conditions in (7e) are satisfied. This approach has been termed as region free because explicit computation and storage of critical regions becomes redundant (Kvasnica et al., 2015). The region free approach can be used to overcome inability to obtain critical region boundaries and control law explicitly due to nonlinearity of the KKT conditions (see (7a) and (7b)). Note that in (Borrelli et al., 2010; Kvasnica et al., 2015), the motivation is to reduce memory footprint using region-free approach for e-MPC of LTI systems since only the constant factors $M_i^{-1}, P_i(x(0))$ for each optimally active set need to be stored. However, the region-free approach plays a decisive-role in enabling implementation of e-MPC for nonlinear systems. Next, we present the region-free approach in context of mp-NLP Problem 2.

Assume that a total of $N_{\mathcal{A}}$ optimally active sets $\bar{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{N_{\mathcal{A}}}\}$, and equivalently $N_{\mathcal{A}}$ inactive sets $\bar{\mathcal{J}} = \{\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_{N_{\mathcal{A}}}\}$, were obtained in the offline step by solving the Dual Feasibility NLP Problem 3 in (8a)-(8h) corresponding to mp-NLP Problem 2. In order to obtain a parameter dependent solution $z(x(0))$ online for a parameter say $x^{'}(0)$, (7a)-(7d) are solved simultaneously. Now, the dual and primal feasibility (namely, $\lambda_{\mathcal{A}_i} \geq 0$, $s_{\mathcal{J}_i} \geq 0$) is verified to identify the optimal set pair $(\mathcal{A}_i, \mathcal{J}_i)$ at $x^{'}(0)$. In this work, we implement a simple sequential search algorithm that evaluates the primal and dual variables $\lambda_{\mathcal{A}_i}$ and $s_{\mathcal{J}_i}$ for optimal active sets $\mathcal{A}_i^*(x^{'}(0))$ and corresponding inactive sets $\mathcal{J}_i(x^{'}(0)), i = 1, \ldots, R$ as identified in the offline step sequentially until feasibility of $\lambda_{\mathcal{A}_i}$, $s_{\mathcal{J}_i}$ is obtained.

Remark 4. Since the online step requires solution of nonlinear equations, this is potentially time consuming and may be afflicted with multiple admissible solutions. In terms of CRs, multiple solutions may manifest as overlapping regions. This issue has not been explored.

5. EXAMPLE

The proposed offline and online steps of explicit NMPC are demonstrated for the nonlinear plant in (Mönnigmann et al., 2015) given by,

$$x_1(k+1) = x_1(k) + 0.1x_2(k) + 0.1(0.5 + 0.5x_1(k))u(k) \quad (11a)$$

$$x_2(k+1) = x_2(k) + 0.1x_1(k) + 0.1(0.5 - 2.0x_2(k))u(k) \quad (11b)$$

$$-2 \leq (x_1(k), x_2(k)) \leq 2 \quad (11c)$$

$$-2 \leq u(k) \leq 2 \quad (11d)$$

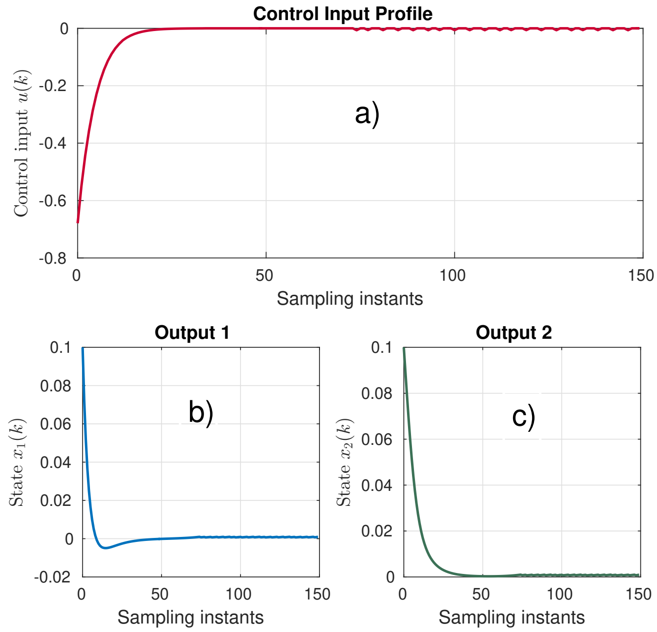

Fig. 2. Nonlinear system control input and output profile

Here $P = \begin{bmatrix} 5.9353 & 5.2774 \\ 5.2774 & 5.9353 \end{bmatrix}$, $Q = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}$ and $R = 0.1$ All offline and online computations were performed in MATLAB® (R2015b).

Table 1. Number of optimal active sets and constraint indices

$\mathcal{A}_i$	Constraints
1	{5}
2	{11}
3	{2,3}
4	{5,6}
5	{6,11}
6	{11,12}
7	{2,3,4}
8	{2,3,6}
9	{2,3,11}
10	{3,4,11}
11	{2,3,4,11}
12	{2,3,6,11}
13	{}

Table 2. Offline: Variation of number of NLPs P3 and P4 with horizon N and number of constraints p

N	p	Max NLPs	Pruned (P4 + LICQ) NLPs	P3: Solved NLPs	P4: Solved (Infeas, Feas)
2	12	2510	2413	97	(8,43)

Table 3. Offline: Information about infeasible candidates, SCS failures and optimal active sets

N	p	P4: t value (Infeas, $t = 0, t > 0$)
2	12	(80, 0, **13**)

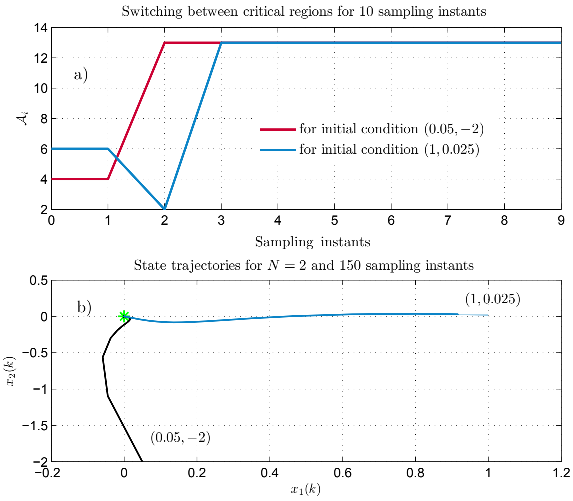

Fig. 3. (a) Switching between critical regions and (b)State trajectories for two different initial conditions

Determining number of optimal active sets(Offline): In this step, the NLP Problem 2 for $N = 2$ i.e for 12 constraints and 6 decision variables is solved. The theoretical maximum number of Dual Feasibility NLP problems, represented by (8a)-(8g), which would need solutions if the pruning criterion is not implemented is 2510 and corresponds to implicit enumeration of the active sets in (5). The number of Dual Feasibility NLPs to Problem 3 that resulted in case (1.c) of Proposition 1 corresponds to the total number of optimally active sets (or critical regions) and was found to be $N_{\mathcal{A}} = 13$ corresponding to $N = 2$. These optimal active sets include active constraints (7c) and equality constraints (7b) are always active. Table 1 shows the indices of active constraints pertaining to (3c) corresponding to each of the 13 optimal active sets. Table 3 gives information regarding infeasible candidate sets, the candidates corresponding to SCS failure and optimal active sets determined by solving Problem 3. The function *fmincon* was used for solving Problem 3 and Problem 4.

Multiparametric NMPC implementation(Online): The Problem 2 was solved for 150 sampling instants using sequential search for initial condition $x(0) = (0.1, 0.1)$. Fig.2a) shows the control input to the plant and Fig.2b)-2c) shows that the nonlinear system given by (11a)-(11b) reaches the origin. Fig.3a) shows the switching of the optimal active sets for two separate initial conditions. For the initial condition $(1, 0.025)$ the switching sequence for first two sampling instants corresponds to the active sets $\mathcal{A}_i, i = 6, 2$ (see Table 1) and subsequently the null set is persistently in the null set $\mathcal{A}_{13}$. Similarly for $(0.05, -2)$ the switching sequence for the first sampling instant is 4 and then the null set $\mathcal{A}_{13}$ applies for further instants. Function *fsolve* was used to implement the sequential search.

6. CONCLUSION

A novel technique for determining the optimal active and inactive constraints for nonlinear systems is presented. The possible optimal active sets are identified in the offline step by solving a Dual Feasibility NLP and a pruning criterion to overcome the exponential number of NLPs that would need solutions. Simulation results show that the number of NLPs that are actually solved is a tiny fraction of the theoretical maximum thereby achieving a practical method for problems of moderate size. The online step uses a simple region free sequential approach to obtain the implicit regional state dependent control law.

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