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On finite-time stability of sub-homogeneous differential inclusions ⋆

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Abstract: Sub-homogeneity property is introduced and is related to a differential inclusion (DI). It is shown that a nonlinear ordinary differential equation (ODE), which may not admit a homogeneous approximation, can be transformed into a sub-homogeneous DI (which is a homogeneous extension of the original ODE). Using this homogeneous extension, one can directly recover finite-time stability property for some particular classes of nonlinear systems. In the last section, such a sub homogeneity property is used to design a nonlinear finite-time observer.

Keywords: Homogeneity, Finite time stability, Nonlinear systems.

1. INTRODUCTION

DIs arise naturally in various branches of modern mathematics, such as mathematical economics, theory of games, convex analysis, etc. The theory of DIs was initiated in Marchaud (1934, 1936); Zaremba (1936), where authors showed that it plays a significant role in the description of processes in control theory. Hence, DIs appear as a model of dynamical systems which do not satisfy the classical assumptions of regularity Filippov (1988). Furthermore, they also apear as a homogeneous extension of non-homogeneous systems to study their finite-time stability (FTS) (see Braidiz et al. (2019a)).

Homogeneity was introduced for ordinary differential equations (in Zubov (1958); Khomenyuk (1961); Hermes (1986); Kawski (1995); Bhat and Bernstein (2005)), time delay systems Efimov et al. (2015), discrete-time systems Sanchez et al. (2019) and partial differential equations (in Polyakov et al. (2016, 2018)). The notion of homogeneity was also used to prove FTS of systems with disturbances in Braidiz et al. (2019) and weak homogeneous systems in Braidiz et al. (2019b). Nevertheless, requiring homogeneity for finite-time design is a restrictive constraint. However, many useful properties can be guaranteed for locally homogeneous systems, and the class of such dynamics is very large (see also Andrieu et al. (2008); Bernuau et al. (2014)), but still limited.

The goal of this note is to extend the homogeneity theory to the class of nonlinear dynamical systems, which do not admit a homogeneous approximation. To this end, a notion of homogeneous set-valued extension is defined, and its properties are investigated using the introduced concept of sub-homogeneity. Sub-homogeneity is a sort of symmetry, which appears as a general definition of homogeneity for some classes of DIs. It will be demonstrated that this concept, as the homogeneity Zubov (1958), is a useful tool for analysis of nonlinear dynamical systems. In particular, it allows us to investigate their FTS.

The main contribution of this work is proving the FTS of a class of DIs which are sub-homogeneous. The notion of sub-homogeneity which will be proposed in this paper can be used to study the FTS of globally asymptotically stable (GAS) systems when their homogeneous approximation does not exist. Hence, this allows us to enlarge a lot the class of considered systems.

The paper is organized as follows. After introducing notations and definitions about asymptotic/finite-time stability and homogeneity in Section 2, we investigate the FTS of sub-homogeneous DIs in Section 3. In the last section, to illustrate the obtained results, we present an example of finite-time observer design by using the sub-homogeneous DIs.

2. PRELIMINARIES

2.1 Notations

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ where $\mathbb{R}$ is the set of real numbers. $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\lceil x \rfloor^\alpha = |x|^\alpha \text{sign}(x)$ for $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$.
- $S = \{x \in \mathbb{R}^n : \ \|x\| = 1\}$ denotes the unit sphere in $\mathbb{R}^n$. $B(x_0, r) = \{x \in \mathbb{R}^n : \ \|x - x_0\| < r\}$ denotes the

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open ball of radius $r > 0$ centered at a point x_0. We define $B(K, r) = \left\{ y \in \mathbb{R}^n : \inf_{x \in K} \|y - x\| < r \right\}$ for all compact $K \subset \mathbb{R}^n$.

- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ (resp., $\mathcal{GKL}$) if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\lim_{t \to +\infty} \beta(s, t) = 0$ (resp., if for each fixed $t \in (0, T)$, β is a strictly decreasing function of its second argument $t \in \mathbb{R}_+$ for any fixed first argument $s \in \mathbb{R}_+ \setminus \{0\}$ and $\beta(s, T) = 0$ for some $0 \le T < +\infty$) for each fixed $s \in \mathbb{R}_+$.
- The notation $\nabla V(x) = \left(\frac{\partial V}{\partial x_1}(x), \cdots, \frac{\partial V}{\partial x_n}(x) \right)^\top$ stands for the first derivative of a continuously differentiable function V at point $x \in \mathbb{R}^n$. For $f : x \in \mathbb{R}^n \to f(x) \in \mathbb{R}^n$, the notation $\langle \nabla V(x), f(x) \rangle = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x) f_i(x)$ stands for the directional derivative of a continuously differentiable function V with respect to the vector field f evaluated at point $x \in \mathbb{R}^n$.
- $\mathcal{C}^p(\mathbb{R}^n, \mathbb{R})$ denotes the space of functions $f : \mathbb{R}^n \to \mathbb{R}$ which have p continuous derivatives and $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ denotes the space of functions $f : \mathbb{R}^n \to \mathbb{R}$ which are smooth. We denote by $\mathcal{CL}^0(\mathbb{R}^n, \mathbb{R})$ (respectively $\mathcal{CL}^k(\mathbb{R}^n, \mathbb{R})$) the set of continuous functions on $\mathbb{R}^n$, locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ with value in $\mathbb{R}^n$ (respectively the set of continuous functions on $\mathbb{R}^n$, $\mathcal{C}^k$ on $\mathbb{R}^n \setminus \{0\}$ with value in $\mathbb{R}$).
- For $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we denote by $\mathcal{D}(F)$ the domain of definition of F.
- $\overline{\text{co}} A$ denotes the closed convex hull of $A \subset \mathbb{R}^n$.

2.2 Stability properties and generalized homogeneity

In this subsection we give some definitions of homogeneity and different kinds of stability for a system:

$$\begin{cases} \dot{x}(t) = f(x(t)), \quad t \ge 0 \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n, f(0) = 0$ is considered such that (1) possesses a unique solution in forward time for all initial conditions $x_0 \in \mathbb{R}^n$ except the origin, denoted as $x(t, x_0)$.

Asymptotic, finite-time stability: The following definitions are inspired from Zubov (1964); Clarke et al. (1998); Bhat and Bernstein (2000) and Polyakov and Fridman (2014).

Definition 2.1. The origin of system (1) is

- **Lyapunov stable**: if there exists a nonempty open neighborhood of the origin $\mathcal{V} \subset \mathbb{R}^n$ and a function $\alpha \in \mathcal{K}_\infty$ such that for all $x_0 \in \mathcal{V}$ we have $\|x(t, x_0)\| \le \alpha(\|x_0\|), \forall t \ge 0$.
- **Asymptotically stable**: if it is Lyapunov stable and for all $x_0 \in \mathcal{V}$, $\lim_{t \to +\infty} x(t, x_0) = 0$.
- **FTS**: if it is Lyapunov stable and there exists a function $T : \mathcal{V} \to \mathbb{R}_+$ such that $\forall x_0 \in \mathcal{V}$, $x(t, x_0) = 0, \ \forall t \ge T(x_0)$. T is called a settling-time function of (1).

In addition, if $\mathcal{V} = \mathbb{R}^n$, then all these properties hold globally.

The following proposition shows how to investigate the FTS by using Lyapunov theory.

Proposition 2.1. Moulay and Perruquetti (2006) *Assume that* (1) *has a unique solution in forward time outside the origin. The origin of this system is FTS with a continuous settling-time function at the origin if and only if there exist a real number $c > 0, \alpha \in (0, 1)$ and a class $\mathcal{CL}^\infty$ Lyapunov function $V : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}_+, \ 0 \in \mathcal{V}$ satisfying $\langle \nabla V(x), f(x) \rangle \le -c[V(x)]^\alpha, \ \forall x \in \mathcal{V}$.*

Now we define the stability properties for DIs. Let us consider the following differential inclusion

$$\begin{cases} \dot{x}(t) \in F(x(t)), \ \forall t \ge 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$.

Definition 2.2. A set-valued map F is called upper semi-continuous at $\bar{x} \in \mathcal{D}(F)$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x \in B(\bar{x}, \delta)$, $F(x) \subset B(F(\bar{x}), \varepsilon)$.

For each $\Omega \subseteq \mathbb{R}^n$, we let $\mathbf{S}(\Omega) \subset \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ denote the set of solutions of (2) satisfying $x(0) \in \Omega$. If Ω is a singleton $\{x_0\}$, we will use the shorthand $\mathbf{S}(x_0)$. We denote $\mathbf{S} = \mathbf{S}(\mathbb{R}^n)$ as the set of all solutions. The domain of a solution $x(\cdot)$ will be denoted by $\mathbb{R}_+$ if it is forward complete. The next definition concerns GAS and FTS properties of DIs.

Definition 2.3. The origin of the system (2) is

- **strongly GAS** (resp., **FTS**) if all solutions $x(\cdot) \in \mathbf{S}$ of (2) are GAS (resp., FTS) (Definition 2.1).
- **weakly GAS/FTS** if there exists a subset $\hat{\mathbf{S}}$ of the set $\mathbf{S}$ of solutions of (2), such that all $x(\cdot) \in \hat{\mathbf{S}}$ are uniformly GAS/FTS.

By the definition of weak GAS (resp., FTS) $\exists \beta \in \mathcal{KL}$ (resp., $\beta \in \mathcal{GKL}$) such that $\|x(t)\| \le \beta(\|x_0\|, t)$, for all $x_0 \in \mathbb{R}^n, x(\cdot) \in \hat{\mathbf{S}}(x_0)$. Notice that, in the case of the weak uniform GAS or FTS, the set $\hat{\mathbf{S}}$ may depend on the function β:

$$\hat{\mathbf{S}} = \hat{\mathbf{S}}^\beta.$$

In this case, for $\sigma \in \mathcal{K}_\infty$ and $R > 0$ we denote

$$\overline{\mathcal{C}}_\sigma^\beta(R) = \left\{ y \in \mathcal{C}([0,1], \mathbb{R}^n); \begin{array}{c} y(1) = 0, \ y\left(\dfrac{\sigma(t)}{1 + \sigma(t)} \right) = x(t), \\ t \ge 0, x(\cdot) \in \hat{\mathbf{S}}^\beta(B(0, R)) \end{array} \right\}.$$

The following lemma proves that the set $\hat{\mathbf{S}}^\beta$ of weakly uniformly stable solutions of (2) is homeomorphic to a compact in $\mathcal{C}([0,1], \mathbb{R}^n)$.

Lemma 2.1. If F is upper semicontinuous and $\hat{\mathbf{S}}^\beta$ is the set of all GAS solutions of (2), *$R > 0$. Then, $\overline{\mathcal{C}}_\sigma(R)$ is a compactum in $(\mathcal{C}([0,1], \mathbb{R}^n), \|\cdot\|_{\mathcal{C}})$, with $\|f\|_{\mathcal{C}} = \sup_{x \in [0,1]} |f(x)|$ denotes the uniform norm.*

The proofs of several results are skipped or shortened due to space limitations.

Homogeneity: In control theory, homogeneity simplifies qualitative analysis of nonlinear dynamic systems. So that, it allows a local properties (e.g., local stability) to be extended globally using a property of the solutions (homogeneity is a Lie symmetry). In order to define this concept, let us introduce the notion of dilation group.

Definition 2.4. A map $\mathbf{d} : \mathbb{R} \to \mathbb{R}^{n\times n}$ is called a linear dilation in $\mathbb{R}^n$ iff it satisfies

- **Group property:** $\mathbf{d}(0) = I_n$, $\mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s)$, $t, s \in \mathbb{R}$.
- **Limit property:** $\lim\limits_{s\to-\infty} \|\mathbf{d}(s)x\| = 0$ and $\lim\limits_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty$ uniformly on the unit sphere.

Property 2.1. The matrix $G_{\boldsymbol{d}} \in \mathbb{R}^{n\times n}$ defined as $G_{\boldsymbol{d}} = \lim\limits_{s\to 0} \frac{\boldsymbol{d}(s)-I_n}{s}$ is known as the generator of the group $\boldsymbol{d}(s)$ and satisfies the following properties: $\frac{d}{ds}\boldsymbol{d}(s) = G_{\boldsymbol{d}}\boldsymbol{d}(s) = \boldsymbol{d}(s)G_{\boldsymbol{d}}$ and $\boldsymbol{d}(s) = e^{G_{\boldsymbol{d}}s} = \sum\limits_{i=0}^{+\infty} \frac{s^i G_{\boldsymbol{d}}^i}{i!}$.

Definition 2.5. The dilation $\mathbf{d}$ is monotone in $\mathbb{R}^n$, if $\|\mathbf{d}(s)\|_{\mathbb{R}^{n\times n}} < 1$, $\forall s < 0$, with $\|\mathbf{d}(s)\|_{\mathbb{R}^{n\times n}} = \sup\limits_{x\in S} \|\mathbf{d}(s)x\|$.

Now we will define the canonical homogeneous norm.

Definition 2.6. If the dilation $\mathbf{d}$ is monotone, the continuous function $\|\cdot\|_{\mathbf{d}} : x \in \mathbb{R}^n \to \|x\|_{\mathbf{d}} = e^{s_x} \in \mathbb{R}_+$, where $s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1$, $\|x\|_{\mathbf{d}} \to 0$ as $x \to 0$ and $\|\mathbf{d}(s)x\|_{\mathbf{d}} = e^s\|x\|_{\mathbf{d}} > 0$, is called the **canonical homogeneous norm**.

Note that in the sequel of this paper, we will use only monotone dilations. The existence of a homogeneous Lyapunov function for a GAS homogeneous system was provided by Zubov in 1957, Rosier (1992) and Polyakov (2018) by using weighted and generalized homogeneity.

3. SUB-HOMOGENEOUS DIFFERENTIAL INCLUSIONS (DI)

In this section, we define sub-homogeneous DI of (1) and then investigate its FTS, a way to obtain a sub-homogeneous DI for (1) is also indicated. Let us introduce the following assumptions for (2):

- $\mathcal{H}_{Fgas}$: (2) is strongly GAS,
- $\mathcal{H}_{Fsub}$: for all $x \in \mathbb{R}^n$, $e^{\nu s}\mathbf{d}(s)F(x) \subseteq F(\mathbf{d}(s)x)$, $\forall s \geq 0$, where $\mathbf{d}$ is a generalized linear dilation and $\nu \in \mathbb{R}$.
- $\mathcal{H}_{Fexi}$: $F(x)$ is nonempty, convex, compact and upper semicontinuous.

Definition 3.1. F is $\mathbf{d}$-sub-homogeneous with degree of homogeneity $\nu \in \mathbb{R}^n$, iff $\mathcal{H}_{Fsub}$ holds.

Defining the notion of sub-homogeneity comes from the fact that there are some DIs which satisfy this property (with inclusion) and not with the equality (when additionally $F(\mathbf{d}(s)x) \subseteq e^{\nu s}\mathbf{d}(s)F(x)$, $\forall s \leq 0$). This class of DIs appears sometimes as extension of dynamical systems, which do not have homogeneous approximation and which are not easy to be studied. As an example of this extension we have

$$F(x) = \overline{\mathrm{co}} \bigcup_{s\leq 0} \left\{e^{-\nu s}\mathbf{d}(-s)f(\mathbf{d}(s)x)\right\}; \tag{3}$$

with f is defined in (1). Such a set-valued map F is $\mathbf{d}$-sub-homogeneous and $f(x) \in F(x)$, $\forall x \in \mathbb{R}^n$. Hence, this extension will be used in the last section to design finite-time observer for non-homogenoeus systems. The following proposition ensures symmetry of solutions of (2) when it satisfies the assumption $\mathcal{H}_{Fsub}$.

Proposition 3.1. Let F be a $\boldsymbol{d}$-sub-homogeneous set-valued map of degree $\nu \in \mathbb{R}$. If for $x_0 \in \mathbb{R}^n$, the trajectory $x(\cdot, x_0)$ is a solution of (2) starting at x_0, then for all $s \geq 0$, the absolutely continuous curve $t \longmapsto \boldsymbol{d}(s)x(e^{\nu s}t, x_0)$ is a trajectory of (2) starting at $\boldsymbol{d}(s)x_0$, (i.e., if $x(\cdot, x_0) \in \mathbf{S}(x_0)$, then $\boldsymbol{d}(s)x(e^{\nu s}\cdot, x_0) \in \mathbf{S}(\boldsymbol{d}(s)x_0)$ for all $s \geq 0$, (and $\boldsymbol{d}(s)\mathbf{S}(x_0) \subseteq \mathbf{S}(\boldsymbol{d}(s)x_0)$) for all $s \geq 0$).

Proof. We consider a trajectory $x(\cdot, x_0)$ of (2) starting at x_0. The curve $t \longmapsto \mathbf{d}(s)x(e^{\nu s}t, x_0)$ is clearly a continuous curve for all $s \geq 0$. Moreover, for almost all $t \in \mathbb{R}$ we have: $\frac{d}{dt}[\mathbf{d}(s)x(e^{\nu s}t, x_0)] = e^{\nu s}\mathbf{d}(s)\dot{x}(e^{\nu s}t, x_0) \in e^{\nu s}\mathbf{d}(s)F(x(e^{\nu s}t, x_0))$.

Since F is $\mathbf{d}$-sub-homogeneous with degree ν, one gets

$$\frac{d}{dt}[\mathbf{d}(s)x(e^{\nu s}t, x_0)] \in F(\mathbf{d}(s)x(e^{\nu s}t, x_0)), \tag{4}$$

then, thus $t \longmapsto \mathbf{d}(s)x(e^{\nu s}t, x_0)$ is a solution of (2) for all $s \geq 0$.

Theorem 3.1. If the set-valued map F satisfies $\mathcal{H}_{Fsub}$, $\mathcal{H}_{Fgas}$ and $\mathcal{H}_{Fexi}$, then, for $k > \max\{-\nu, 0\}$, there exists a pair (V, W) of continuous functions and $p \in \mathbb{N}\setminus\{0\}$, such that all the eigenvalues of $(kI_n - pG_{\boldsymbol{d}})$ have a positive real parts, and

1) *$V \in \mathcal{C}^p(\mathbb{R}^n, \mathbb{R}_+)$, V is $\boldsymbol{d}$-homogeneous with degree of homogeneity $k > \max\{-\nu, 0\}$ and positive definite;*
2) *$W \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}_+)$, W is positive definite and $\boldsymbol{d}$-homogeneous with degree of homogeneity $k + \nu$;*
3) *$\max\limits_{h\in F(x)} \langle\nabla V(x), h\rangle \leq -W(x)$ for all $x \neq 0$.*

Proof. The sketch of the proof is as follows: under the introduced hypotheses and using the result from Rosier (1992) we construct a $\mathbf{d}$-homogeneous Lyapunov function V, which satisfies 1), 2) and 3).■

Theorem 3.1 asserts that a strongly GAS sub-homogeneous DI admits a homogeneous Lyapunov function. This result is a generalization of the theorem of Rosier (1992). The next theorem proves the FTS of the GAS sub-homogeneous DI (2) with negative degree of homogeneity.

Theorem 3.2. If the set-valued map F satisfies $\mathcal{H}_{Fexi}$, $\mathcal{H}_{Fgas}$, $\mathcal{H}_{Fsub}$ and its degree of homogeneity $\nu < 0$, then (2) is globally FTS.

Proof. According to Theorem 3.1, there exists a continuous pair (V, W) satisfying the conditions 1), 2) and 3) above. Let $\tau \geq 0$, using the homogeneity of V and the sub-homogeneity of F, we get

$$\begin{aligned} e^{(k+\nu)\tau} \max_{h\in F(x)} \langle\nabla V(x), h\rangle &\leq e^{k\tau} \max_{h\in e^{\nu\tau}\mathbf{d}(\tau)F(x)} \langle\nabla V(x), \mathbf{d}(-\tau)h\rangle \\ &\leq \max_{h\in F(\mathbf{d}(\tau)x)} \langle\nabla V(\mathbf{d}(\tau)x), h\rangle \\ &\leq -W(\mathbf{d}(\tau)x). \end{aligned} \tag{5}$$

Let $R > 0$, for $\tau = -\ln\left(\frac{\|x(t)\|_{\mathbf{d}}}{R}\right) \geq 0$, and from (5) we obtain

$$\begin{aligned} \dot{V}(x(t)) &= \max_{h \in F(x(t))} \langle \nabla V(x(t)), h \rangle \\ &\leq -e^{-(k+\nu)\tau} \min_{\|y\|_{\mathbf{d}}=R} W(y) \\ &\leq -\frac{\overline{W}}{R^{k+\nu}} \|x(t)\|_{\mathbf{d}}^{k+\nu} \\ &\leq -\frac{\overline{W}}{aR^{k+\nu}} V(x(t))^{\frac{k+\nu}{k}}, \end{aligned} \tag{6}$$

where $a = \left(\max_{\|y\|_{\mathbf{d}}=1} V(y)\right)^{\frac{k+\nu}{k}}$ and $\overline{W} = \min_{\|y\|_{\mathbf{d}}=R} W(y)$. If $\nu < 0$ and by using Proposition 2.1 we conclude that the origin of (2) is FTS (settling time function is estimated by $T(x_0) \leq \frac{[V(x_0)]^{1-\alpha}}{(1-\alpha)c}$) for all initial conditions $x_0 \in B(0,R)$ and for all $R > 0$. This proves the global FTS of (2).■

This result demonstrates how useful the notion of sub-homogeneity for analysis of FTS. In many papers, researchers study DIs which are strongly GAS. However, in this work, we will see that DIs which are weakly GAS appear as a sub-homogeneous extensions of non-homogeneous dynamical systems. Therefore, the next theorem will study the FTS of this class of DIs.

Theorem 3.3. If the set-valued map F satisfies $\mathcal{H}_{Fexi}$, $\mathcal{H}_{Fsub}$, there exists a set $\hat{\mathbf{S}}^\beta$ of GAS solutions of DI (2) ((2) is weakly GAS) and $\nu < 0$, then all the solutions in $\hat{\mathbf{S}}^\beta$ are globally FTS (origin of DI (2) is weakly globally FTS).

Proof. Let $\hat{\mathbf{S}}^\beta$ be the set of solutions of the DIs (2), which are GAS. Let $R > 0$, $x_0 \in B_{2R}$, $x(\cdot, x_0) \in \hat{\mathbf{S}}^\beta(x_0)$, $B_R = \{x \in \mathbb{R}^n : \ \|\mathbf{d}(-\ln(R))x\| \leq 1\}$ and let us define the time $\tau(x_0, B_R)$ for GAS solutions of (2) starting from x_0, to reach and stay in B_R by:

$$\tau(x_0, B_R) = \sup_{x(\cdot,x_0) \in \hat{\mathbf{S}}^\beta(x_0)} \inf \{T > 0 : \ x(t, x_0) \in B_R, \forall t \geq T\}.$$

Using Lemma 2.1, one can prove that τ is finite for every $x_0 \in B_R$. The fact that $\nu < 0$ implies the weak finite-time stability of the sub-homogeneous extension (2). This property holds for every $R > 0$, which implies the global weak FTS of (2).■

Theorem 3.3 generalizes the results from Bernuau et al. (2015) about homogeneous DI. Also, it shows that if a DI is weakly GAS and the degree of sub-homogeneity is negative, we can guarantee its weak FTS.

4. FINITE-TIME OBSERVER DESIGN USING SUB-HOMOGENEITY

The notion of sub-homogeneity is instrumental to introduce a new observer design method. Throughout this section we consider the following system

$$\begin{cases} \dot{x}(t) &= A(y)x(t) + \phi(y,u); \ x(t) \in \mathbb{R}^n, \\ y(t) &= Cx(t), \qquad\qquad\qquad t \in [0, +\infty). \end{cases} \tag{7}$$

where

$$A(y) = \begin{pmatrix} 0 & a_1(y) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & a_{n-1}(y) \\ 0 & \cdots & \cdots & & 0 \end{pmatrix}, C = (1, 0, \cdots, 0). \tag{8}$$

The functions $a_i : \mathbb{R} \to \mathbb{R}$ and $0 < \underline{a}_i \leq a(y) \leq \bar{a}_i < 1$, $\forall y \in \mathbb{R}, i = 1, \cdots, n-1$ are continuous and they do not have a homogeneous approximation. The goal of this section is to introduce an observer for the system (7). The observer is given by the system

$$\dot{\hat{x}}(t) = A(y)\hat{x}(t) + \phi(y,u) - L(E_1), \ t \in [0, +\infty),$$

where $\hat{x}(t) \in \mathbb{R}^n$, is the state estimate $L(E_1) = (k_1 \lceil E_1 \rfloor^{\alpha_1}, \cdots, k_n \lceil E_1 \rfloor^{\alpha_n})^\top$, $k_i > 0$, $\alpha_i = \frac{n-i}{n}$, $\forall i = 1, \cdots, n$; and $E = \hat{x} - x = (E_1, \cdots, E_n)^\top$ is the estimation error vector. Here is the observer error dynamics

$$\dot{E} = A(y)E - L(E_1) = f(E, y). \tag{9}$$

Let us consider the following Lyapunov function that has been proposed in Cruz-Zavala and Moreno (2018)

$$V(E) = \sum_{i=1}^{n-1} \beta_i g_i(E_i, E_{i+1}) + \frac{\beta_n}{p} |E_n|^p, \tag{10}$$

with $\beta_1 > 0, \cdots, \beta_n > 0$ and $g_i(E_i, E_{i+1}) = \frac{r_i}{p}|E_i|^{\frac{p}{r_i}} - E_i \lceil E_{i+1} \rfloor^{\frac{p-r_i}{r_{i+1}}} + \left(\frac{p-r_i}{p}\right)|E_{i+1}|^{\frac{p}{r_{i+1}}}$, where for $i = 1, \cdots, n$; we have $r_i = n + 1 - i$ and $p \geq r_1 + r_2 = 2n - 1 \geq r_i + r_{i+1} \geq 1$ for $n \geq 2$. For simplicity, we introduce the variables

$$\begin{aligned} \sigma_i &= \left(\lceil E_i \rfloor^{\frac{p-r_i}{r_i}} - \lceil E_{i+1} \rfloor^{\frac{p-r_i}{r_{i+1}}}\right), \\ \psi_i &= \left(\frac{p - r_i}{r_{i+1}}\right) |E_{i+1}|^{\frac{p-r_i-r_{i+1}}{r_{i+1}}} \left(\lceil E_{i+1} \rfloor^{\frac{r_i}{r_{i+1}}} - E_i\right). \end{aligned}$$

Note that g_i, σ_i and ψ_i vanish on the same set. The partial derivatives of g_i are

$$\frac{\partial g_i(E_i, E_{i+1})}{\partial E_i} = \sigma_i, \ \frac{\partial g_i(E_i, E_{i+1})}{\partial E_{i+1}} = \psi_i.$$

σ_i and ψ_i are continuous.

Theorem 4.1. If we choose carefully the gains $k_i > 0$, and $\beta_i > 0$ such that $\beta_n = 1$,

$$\begin{aligned} &0 < k_n < \frac{\frac{1}{p-1} - \frac{2\beta_{n-2}}{\beta_{n-1}(p-1)}}{\frac{1}{\beta_{n-1}} + \frac{2(p-2)^2}{p-1}}, \ 0 < k_{n-1} < \frac{1}{p-1} - \frac{k_n}{\frac{1}{\beta_{n-1}} + \frac{2(p-2)^2}{p-1}} - \frac{2\beta_{n-2}}{\beta_{n-1}(p-1)} \\ &0 < k_{n-2} < \frac{2}{p-1} - \frac{4\beta_{n-3}}{\beta_{n-2}(p-1)} - 2k_n \frac{\beta_{n-1}(p-2)}{\beta_{n-2}(p-1)} - (k_{n-1}+1)\left(\frac{\beta_{n-1}}{\beta_{n-2}} + \frac{2p-7}{p-1}\right) \end{aligned} \tag{11}$$

for $i = 3, \cdots, n-2$,

$$\begin{aligned} 0 < k_{i-1} < &\frac{r_{i-1} - 1}{p-1} - \frac{2\beta_{i-2} r_i}{\beta_{i-1}(p-1)} - (k_{i+1}+1)\frac{\beta_i r_i}{\beta_{i-1}(p-1)} \\ &-(k_i + 1)\left(\frac{\beta_i}{\beta_{i-1}} + \frac{2p - 2r_i - r_{i-1}}{p-1}\right) - \frac{2\beta_{i-2} r_i}{\beta_{i-1}(p-1)} \end{aligned} \tag{12}$$

and

$$\begin{cases} 0 < k_1 < \frac{r_1 - 1}{p-1} - (k_3+1)\frac{\beta_2 r_2}{\beta_1(p-1)} - (k_2+1)\left(\frac{\beta_2}{\beta_1} + \frac{2p - 2r_2 - r_1}{p-1}\right), \\ 1 + \frac{r_1(1+k_2)}{p-1} + \sum_{i=2}^{n-1} 2k_i \frac{r_{i+1}}{p-1} \frac{(\beta_i + \beta_{i-1})}{\beta_1} < k_1, \end{cases} \tag{13}$$

then (9) is uniformly GAS.

Proof. As it has been shown in Cruz-Zavala and Moreno (2018), since the Lyapunov function V is the sum of non negative terms, then it is non negative and the fact that $V(z) = 0 \Longrightarrow z = 0$, implies that V is positive definite. Due to homogeneity it is radially unbounded. Its derivative $\dot{V}$ is estimated as follows $\dot{V} \leq \sum_{i=1}^{n} \delta_i |E_i|^{\frac{p-1}{r_i}}$, where

$$\begin{aligned}
\delta_1 &= \left[\beta_1 \frac{p-r_1}{p-1}(1-k_1) + \beta_1 \frac{r_1(a_2(y)+k_2)}{p-1} + \sum_{i=2}^{n-1} 2k_i \frac{r_{i+1}}{p-1}(\beta_i + \beta_{i-1})\right], \\
\delta_2 &= \beta_1 \frac{p-r_1}{p-1}\left(k_1 - a_1(y)\left(\frac{p-1}{p-r_1} - (a_1(y))^{\frac{p-r_1}{r_2}}\right)\right) + \beta_2 \frac{p-r_2}{p-1}(a_2(y)+k_2) \\
&\quad + \beta_2 \frac{r_2(a_3(y)+k_3)}{p-1} + \beta_1(a_2(y)+k_2)\frac{2(p-r_2)-r_1}{p-1}.
\end{aligned}$$

For all $i = 3, \cdots, n-2$,

$$\begin{aligned}
\delta_i &= \beta_{i-1} \frac{p-r_{i-1}}{p-1}\left(k_{i-1} - a_{i-1}(y)\left(\frac{p-1}{p-r_{i-1}} - (a_{i-1}(y))^{\frac{p-r_{i-1}}{r_i}}\right)\right) \\
&\quad + \beta_i \frac{p-r_i}{p-1}(a_i(y)+k_i) + \beta_i \frac{r_i(a_{i+1}(y)+k_{i+1})}{p-1} \\
&\quad + \beta_{i-1}(a_i(y)+k_i)\frac{2(p-r_i)-r_{i-1}}{p-1} + \beta_{i-2}\frac{2r_i a_{i-1}(y)}{p-1}, \\
\delta_{n-1} &= \beta_{n-2} \frac{p-3}{p-1}\left(k_{n-2} - a_{n-2}(y)\left(\frac{p-1}{p-3} - (a_{n-2}(y))^{\frac{p-3}{2}}\right)\right) \\
&\quad + \beta_{n-1} \frac{p-2}{p-1}(a_{n-1}(y)+k_{n-1}) + \beta_{n-3}\frac{4a_{n-2}(y)}{p-1} \\
&\quad + \beta_{n-2}(a_{n-1}(y)+k_{n-1})\frac{2(p-2)-3}{p-1} + 2k_n\beta_{n-1}\frac{p-2}{p-1}, \\
\delta_n &= \beta_{n-1} \frac{p-2}{p-1}\left(k_{n-1} - a_{n-1}(y)\left(\frac{p-1}{p-2} - (a_{n-1}(y))^{p-2}\right)\right) + k_n\beta_n \\
&\quad + \beta_{n-2}\frac{2a_{n-1}(y)}{p-1} + 2k_n\beta_{n-1}\frac{(p-2)^2}{p-1}.
\end{aligned}$$

We choose the gains $k_i > 0$, and $\beta_i > 0$ such that (11), (12) and (13) hold. Then, one has $\delta_i < 0$, $\forall i \in \{1, \cdots, n\}$. Which implies that $\dot{V}$ is negative definite. This implies that the system (9) is uniformly GAS.■

Let F be a sub-homogeneous extension of f, which is given by

$$\begin{aligned}
F(E) &= \overline{\text{co}} \bigcup_{s\leq 0} \bigcup_{y\in\mathbb{R}} \{e^s \mathbf{d}(-s) f(\mathbf{d}(s)E, e^{r_1 s}y)\} \\
&= \overline{\text{co}} \bigcup_{s\leq 0} \bigcup_{y\in\mathbb{R}} \{A(e^{r_1 s}y)E - L(E_1)\}, \qquad (14)
\end{aligned}$$

By construction, the set valued $F(E)$ is nonempty ($f(E,y) \in F(E)$), compact, convex and upper semi-continuous (f is continuous, the proof is similar to the one of Lemma 3.1 in Braidiz et al. (2020)) and $\mathbf{d}$-sub-homogeneous with degree of homogeneity $\nu = -1$. Indeed,

$$F(\mathbf{d}(\tau)E) \subseteq e^{-\tau}\mathbf{d}(\tau)F(E), \; \forall \tau \leq 0,$$

and $\mathbf{d}(s) = \text{diag}\{e^{ns}, e^{(n-1)s}, \cdots, e^s\}$. The system (7) is GAS, this implies that there exists a subset $\hat{\mathbf{S}}^\beta$ of GAS solutions of the DI $\dot{x}(t) \in F(x(t))$. The degree of homogeneity of F is $\nu = -1 < 0$, then by using Theorem 3.3, $\dot{x}(t) \in F(x(t))$ is weakly globally FTS. Because the solutions of (7) are in $\hat{\mathbf{S}}^\beta$, then it is globally FTS. For $n = 2$, Fig. 1 shows the FTS of solution of the system (9) for $a : y \in \mathbb{R} \to (1 - \frac{1}{2}\cos(y)) \in \mathbb{R}$.

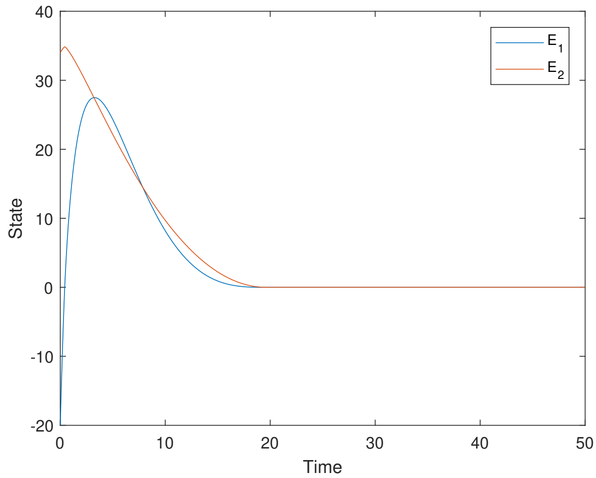

Fig. 1. The solutions of the system (9) with $a(y) = 1 - \frac{1}{100}\cos(y)$, and the initial condition $(x_0, y_0) = (3, 40)$.

5. CONCLUSION

The notions of sub-homogeneity were introduced. As it was shown, this property allows us to analyze systems which do not admit homogeneous approximations. FTS of sub-homogenoeus GAS DIs is investigated using Lyapunov function. The notion of weak FTS for DI was introduced and it was investigated for sub-homogeneous DI, which is weakly GAS. Example from Section 4 shows that the notion of sub-homogeneity can be used to design a finite-time observer.

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