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An explicit model predictive controller for constrained stochastic linear systems

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Abstract: In this paper we introduce an explicit Model Predictive Controller (eMPC) for a linear system subject to an additive stochastic disturbance with bounded support. The finite horizon control problem that is solved to determine the eMPC consists in minimizing an average quadratic cost subject to robust linear constraints involving state and input. By resorting to a control law parametrization that is affine in the disturbance, the finite horizon control problem is reformulated as a convex quadratic optimization program and solved via multiparametric quadratic programming. The resulting eMPC is piecewise affine as a function of the state. The proposed approach is compared with an alternative min-max approach from the literature on a numerical example.

Keywords: Constrained control; explicit model predictive control; stochastic linear systems.

1. INTRODUCTION

Model predictive control is a model-based control paradigm that is able to cope with constrained multivariable systems. The control input to be applied at each sampling time is determined by solving a finite horizon constrained optimization problem that exploits the prediction of the system behavior starting from the current state, thus resulting in a state feedback control law. Online computations have to be performed in a sampling time interval, and may become prohibitive for systems with fast dynamics. This led to the study of methods for determining a solution to the finite horizon optimization problem that is parametric in the state, so as to reduce the online effort to a function evaluation. The resulting model predictive control strategy is then called explicit.

Explicit model predictive control has been extensively studied in the literature, Alessio and Bemporad (2009). The considered classes of systems include linear and hybrid linear systems, possibly affected by disturbances. In the explicit Model Predictive Controller (eMPC) computation, both nominal and robust formulations have been considered with different costs, and using either an open-loop or a closed-loop control policy, Bemporad et al. (2002a), Bemporad et al. (2002b), Bemporad et al. (2003).

Concerning nominal eMPC design, in Munoz de la Pena et al. (2004) a dynamic programming approach is presented for linear systems with a quadratic cost. Robust eMPCs are typically designed by minimizing the worst-case performance while enforcing state and input constraints. In Munoz de la Pena et al. (2005), a worst-case quadratic cost is minimized by parametrizing the input as the sum of an open-loop term and a linear feedback term whose gain is a-priori fixed. In Bemporad et al. (2003), a dynamic programming approach has been instead applied for worst-case costs based on 1-norm and ∞-norm. Robust eMPC design has been tackled also in Pistikopoulos et al. (2009), and, more recently, in Kouramas et al. (2013), where uncertainty enters the system through its matrices, characterized by the sum of a nominal term and an error term. The proposed method uses dynamic programming and multi-parametric programming to minimize a quadratic cost and satisfy state and input constraints, robustly with respect to uncertainty.

In this paper we consider linear systems subject to a stochastic additive disturbance with bounded support. We minimize an expected quadratic cost subject to robust constraints on state and input, with the objective of designing a control law that is feasible for any uncertainty instance since constraints are satisfied robustly, while imposing high performance only on average, for most of the uncertainty realizations, accepting a performance degradation but only for uncertainty instances that are unlikely to occur. A different philosophy is adopted in the worst-case approach where guarantees of a certain (optimized) performance level are enforced for all uncertainty instances, irrespectively of their likelihood to occur, and in the nominal approach where only the nominal system performance is optimized. Choice of the approach depends on the problem at hand and is also affected by the complexity of the resulting solution.

Inspired by Goulart et al. (2006), we parametrize the finite horizon control law as an affine function of the past disturbance values, which allows us to reformulate the finite horizon optimization problem as a convex quadratic program. Furthermore, such a parametrization is equivalent to a state feedback policy since disturbances can be reconstructed from state measurements (see Goulart et al. (2006)).

Robust eMPC design has been addressed in a stochastic framework also in Sakizlis et al. (2004) and Grancharova and Johansen (2010). In Sakizlis et al. (2004), constrained linear systems subject to additive uncertainty

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are considered. An expected quadratic cost function is introduced and both open-loop and closed-loop robust parametric controllers are computed. Differently from our setup, where we express the expectation as a quadratic function in the control law parameters, in Sakizlis et al. (2004) it is proposed an approximate solution based on discretization of the expectation to a set of uncertainty scenarios. In Grancharova and Johansen (2010), instead, an approximate method in the context of nonlinear MPC has been proposed. However, the disturbance entering the system dynamics takes a finite set of values, while here we are concerned with disturbances having a continuous support.

The rest of the paper is organized as follows. After introducing some basic notions and notations, we precisely formulate the addressed problem in Section 2. In Section 3, we explain how the eMPC can be designed by rewriting the constrained optimization problem for the finite-horizon control computation as a quadratic convex problem and solving it parametrically in the initial state. Section 4 provides a numerical example. Some remarks conclude the paper in Section 5.

Basic notions and notations

Given two positive integers m and n, $\mathbb{R}^{m,n}$ denotes the space of the $m \times n$ real matrices, and $\mathbb{R}^m$ stands for $\mathbb{R}^{m,1}$. I_m and $O_{m,n}$ denote respectively the identity matrix of order m and the $m \times n$ zero matrix, while $\mathbf{0}_m$ and $\mathbf{1}_m$ are the elements of $\mathbb{R}^m$ with all zero and unitary entries, respectively.

Given a matrix M, M^s, $\mathrm{tr}(M)$ and $\mathrm{vec}(M)$ indicate the symmetric part, the trace and the vectorization of M, respectively. More precisely, the symmetric part of M is the matrix $0.5(M + M^T)$ and the vectorization of M is the column vector obtained by stacking the columns of M. Given two matrices $M_1 \in \mathbb{R}^{m,n}$ and $M_2 \in \mathbb{R}^{p,q}$, $M_1 \otimes M_2$ and $\mathrm{diag}(M_1, M_2)$ denote respectively the Kronecker product of M_1 and M_2 and the block-diagonal matrix formed by M_1 and M_2.

A (convex) polyhedron $\mathcal{P} \subseteq \mathbb{R}^h$ is defined as the intersection of q half-spaces (H-representation, Ziegler (2012)), and can be expressed through $P_A \in \mathbb{R}^{q,h}$ and $p_B \in \mathbb{R}^q$ as $\mathcal{P} = \{z \in \mathbb{R}^h | P_A z \le p_B\}$ or $\mathcal{P} = (P_A, p_B)$ for ease of notation. A polytope is a (convex) bounded polyhedron.

Given a random vector $v \in \mathbb{R}^h$, we denote by $\mathbb{C}_v$ its covariance matrix. $\mathbb{E}_v[g(v)]$ denotes the expectation of $g(v)$, with $g : \mathbb{R}^h \to \mathbb{R}^m$ measurable, with respect to the probability distribution of the random vector v.

2. PROBLEM FORMULATION

Consider a linear system governed by the equation:

$$x_{t+1} = Ax_t + B_u u_t + B_w w_t, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is a control input, and $w \in \mathbb{R}^{n_w}$ is a weakly stationary stochastic disturbance with known (constant) first and second order moments and compact support within a polytope $\mathcal{W}$. In particular, we assume without loss of generality that w has zero mean.[1] Matrices A, B_u and B_w have appropriate dimensions.

The aim is to design an eMPC of the form $u = \kappa_{\mathrm{mpc}}(x)$ by minimizing the average cost

$$J_{\mathrm{av}} = \mathbb{E}_w \left[\sum_{t=0}^{N-1} (x_t^T Q x_t + u_t^T R u_t) + x_N^T P x_N \right], \tag{2}$$

subject to the robust constraints

$$\begin{array}{l} C_{ax} x_t + C_{au} u_t \le c_b \\ x_N \in \mathcal{X}_f \\ \forall w_t \in \mathcal{W},\, t = 0, \dots, N-1. \end{array} \tag{3}$$

In the average cost J_{av}, matrices Q, R and P are symmetric and positive semidefinite. In the state-input constraints

$$C_{ax} x_t + C_{au} u_t \le c_b,\ t = 0, \dots, N-1, \tag{4}$$

$C_{ax} \in \mathbb{R}^{q_c, n_x}$, $C_{au} \in \mathbb{R}^{q_c, n_u}$ and $c_b \in \mathbb{R}^{q_c}$. Note that (4) can represent also constraints that depend only on the state or the input. The terminal set $\mathcal{X}_f$ is convex and polyhedral.

Matrix P and the terminal set $\mathcal{X}_f$ can be suitably chosen so as to guarantee stability and recursive feasibility (see Mayne et al. (2000), Bemporad and Morari (1999), Maciejowski (2002)).

Note that due to the weak stationarity of w, the cost J_{av} is time-invariant, which is essential in the eMPC design. Also, assuming a polytopic support $\mathcal{W}$ allows to rewrite the infinite number of constraints in (3) as a finite number of linear equalities and inequalities in the decision variables. This is detailed in the next section.

3. DERIVATION OF THE EXPLICIT MPC

In this section, we describe a method to solve the robust control problem introduced in Section 2 that rests on a suitable parametrization of the finite horizon law and on multiparametric programming.

Inspired by Goulart et al. (2006), we parametrize u_t, $t = 0, \dots, N-1$, as follows[2]

$$u_t = v_t + \sum_{j=0}^{t-1} M_{t,j} w_j, \tag{5}$$

where $v_t \in \mathbb{R}^{n_u}$ and $M_t = [M_{t,0} \ \dots \ M_{t,t-1}] \in \mathbb{R}^{n_u, t n_w}$ are design parameters that have to be optimally tuned. The control law parametrization (5) is affine in the past disturbance values, except for the control input $u_0 = v_0$ at time $t = 0$. If the finite horizon control problem is solved parametrically in the initial condition x_0, then, the optimal parametric expression $v_0^*(x_0)$ for v_0 provides the static state-feedback eMPC, i.e., $u = \kappa_{\mathrm{mpc}}(x) = v_0^*(x)$.

We next show how, by adopting parametrization (5), the cost (2) and the infinite number of constraints in (3) can be respectively expressed as a convex quadratic function and a finite number of linear constraints in the controller parameters. We shall then exploit the obtained properties for cost and constraints to design the eMPC by applying multiparametric Quadratic Programming (mp-QP) to the resulting convex quadratic optimization program.

[1] If the disturbance w has a nonzero mean, a suitable change of coordinates can be adopted to get a zero mean disturbance $w - \mathbb{E}_w[w]$.

[2] A summation where the index ranges between 0 and a negative integer is meant to be empty and, hence, provides a zero contribution.

3.1 Average cost reformulation

We introduce the variables:

$$X_N = \left[x_1^T \ \dots \ x_N^T\right]^T, \quad U_N = \left[u_0^T \ \dots \ u_{N-1}^T\right]^T,$$
$$V_N = \left[v_0^T \ \dots \ v_{N-1}^T\right]^T, \quad W_N = \left[w_0^T \ \dots \ w_{N-1}^T\right]^T,$$

and rewrite cost (2) as:

$$\begin{aligned} J_{\text{av}} &= x_0^T Q x_0 + \mathbb{E}_w \left[\sum_{t=1}^{N-1} x_t^T Q x_t + x_N^T P x_N\right] \\ &+ \mathbb{E}_w \left[\sum_{t=0}^{N-1} u_t^T R u_t\right] = \\ &= x_0^T Q x_0 + \mathbb{E}_w \left[X_N^T \operatorname{diag}(I_{N-1} \otimes Q, P) X_N\right] \\ &+ \mathbb{E}_w \left[U_N^T (I_N \otimes R) U_N\right]. \end{aligned}$$

Since the state of system (1) at time t, $t \geq 1$, can be expressed as

$$x_t = A^t x_0 + \sum_{j=0}^{t-1} A^{t-1-j}(B_u u_j + B_w w_j),$$

we can write X_N as

$$X_N = \mathbf{A}_N x_0 + \mathbf{B}_{u,N} U_N + \mathbf{B}_{w,N} W_N,$$

where

$$\mathbf{A}_N = \left[A^T \ \dots \ (A^N)^T\right]^T,$$

$$\mathbf{B}_{u/w,N} = \begin{bmatrix} B_{u/w} & O_{n_x,n_{u/w}} & \dots & O_{n_x,n_{u/w}} \\ AB_{u/w} & B_{u/w} & \dots & O_{n_x,n_{u/w}} \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B_{u/w} & A^{N-2}B_{u/w} & \dots & B_{u/w} \end{bmatrix}.$$

By exploiting parametrization (5), U_N is given by

$$U_N = V_N + \mathbf{M}_N W_{N-1}$$

where W_{N-1} is defined similarly to W_N and

$$\mathbf{M}_N = \begin{bmatrix} O_{n_u,n_w} & O_{n_u,n_w} & \dots & O_{n_u,n_w} \\ M_{1,0} & O_{n_u,n_w} & \dots & O_{n_u,n_w} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N-1,0} & M_{N-1,1} & \dots & M_{N-1,N-2} \end{bmatrix}.$$

We can then express X_N as

$$X_N = \mathbf{A}_N x_0 + \mathbf{B}_{u,N} V_N + (\mathbf{B}_{u,N}\mathbf{M}_{N,0} + \mathbf{B}_{w,N}) W_N,$$

with $\mathbf{M}_{N,0} = [\mathbf{M}_N \ O_{Nn_u,n_w}]$.

By recalling that, given a random vector v and a matrix P, we have that

$$\mathbb{E}_v\left[v^T P v\right] = \mathbb{E}_v\left[v\right]^T P \mathbb{E}_v\left[v\right] + \operatorname{tr}(P\mathbb{C}_v),$$

and by setting $D_{x,N} = \operatorname{diag}(I_{N-1} \otimes Q, P)$ and $D_{u,N} = I_N \otimes R$, the two terms of cost (2) can be rewritten as

$$\begin{aligned} &\mathbb{E}_w\left[X_N^T D_{x,N} X_N\right] = x_0^T \mathbf{A}_N^T D_{x,N} \mathbf{A}_N x_0 \\ &+ V_N^T \mathbf{B}_{u,N}^T D_{x,N} \mathbf{B}_{u,N} V_N + 2 x_0^T \mathbf{A}_N^T D_{x,N} \mathbf{B}_{u,N} V_N \\ &+ \operatorname{tr}[D_{x,N}(\mathbf{B}_{u,N}\mathbf{M}_{N,0} + \mathbf{B}_{w,N})\mathbb{C}_{W_N}(\mathbf{M}_{N,0}^T \mathbf{B}_{u,N}^T + \mathbf{B}_{w,N}^T)] \\ &\mathbb{E}_w\left[U_N^T D_{u,N} U_N\right] = V_N^T D_{u,N} V_N \\ &+ \operatorname{tr}[D_{u,N}\mathbf{M}_N \mathbb{C}_{W_{N-1}} \mathbf{M}_N^T], \end{aligned}$$

where we used the linearity of the expectation and the expression of the covariance matrix of the image of a random vector v through an affine map defined by a matrix P and a vector q, i.e.:

$$\mathbb{C}_{Pv+q} = P\mathbb{C}_v P^T.$$

Note that, being w stationary, the covariance matrices $\mathbb{C}_{W_N}$ and $\mathbb{C}_{W_{N-1}}$ are constant, which implies that cost J_{av} is time-invariant.

The next step is to express the trace terms as quadratic functions of the control law parameters. To this aim, we introduce a vector $m_N \in \mathbb{R}^{n_m}$ defined as

$$m_N = \left[\operatorname{vec}(M_1)^T \ \operatorname{vec}(M_2)^T \ \dots \ \operatorname{vec}(M_{N-1})^T\right]^T,$$

where $n_m = 0.5\, n_u n_w N(N-1)$, and exploit the following properties related to the trace, the vectorization operator and the Kronecker product:

$$\begin{aligned} &\operatorname{tr}(P^T) = \operatorname{tr}(P), \\ &\operatorname{tr}(P_1 + P_2) = \operatorname{tr}(P_1) + \operatorname{tr}(P_2), \\ &\operatorname{tr}(P_1P_2P_3P_4) = \operatorname{tr}(P_3P_4P_1P_2), \\ &\operatorname{tr}(P_1P_2) = \operatorname{vec}(P_1^T)^T \operatorname{vec}(P_2), \\ &\operatorname{tr}(P_1P_2^TP_3P_2) = \operatorname{vec}(P_2)^T(P_1 \otimes P_3)\operatorname{vec}(P_2), \\ &\operatorname{vec}(P_1P_2) = (I \otimes P_1)\operatorname{vec}(P_2). \end{aligned}$$

Also, we use the fact that the non-zero components of $\operatorname{vec}(\mathbf{M}_N)$ and the components of m_N are related with each other by means of a linear map defined by an invertible matrix $L \in \mathbb{R}^{n_m,n_m}$.

We then reformulate the trace terms as follows:

$$\begin{aligned} &\operatorname{tr}[D_{x,N}(\mathbf{B}_{u,N}\mathbf{M}_{N,0} + \mathbf{B}_{w,N})\mathbb{C}_{W_N}(\mathbf{M}_{N,0}^T\mathbf{B}_{u,N}^T + \mathbf{B}_{w,N}^T)] \\ &= m_N^T L^T C_{q,x}(D_{x,N}, \mathbf{B}_{u,N}, \mathbb{C}_{W_N}) L m_N \\ &+ C_{l,x}(D_{x,N}, \mathbf{B}_{u,N}, \mathbf{B}_{w,N}, \mathbb{C}_{W_N})^T L m_N \\ &+ \operatorname{tr}[D_{x,N}\mathbf{B}_{w,N}\mathbb{C}_{W_N}\mathbf{B}_{w,N}^T], \\ &\operatorname{tr}[D_{u,N}\mathbf{M}_N\mathbb{C}_{W_{N-1}}\mathbf{M}_N^T] \\ &= m_N^T L^T C_{q,u}(D_{u,N}, \mathbb{C}_{W_{N-1}}) L m_N, \end{aligned}$$

where $C_{q,x}(\cdot)$, $C_{q,u}(\cdot) \in \mathbb{R}^{n_m,n_m}$ are symmetric and positive semidefinite and $C_{l,x}(\cdot) \in \mathbb{R}^{n_m}$.

Finally, we can reformulate cost (2) as

$$\begin{aligned} J_{\text{av}} &= \frac{1}{2} x_0^T H_{xx} x_0 + x_0^T H_{xu} \begin{bmatrix} V_N \\ m_N \end{bmatrix} \\ &+ \frac{1}{2}\left[V_N^T \ m_N^T\right]^T H_{uu} \begin{bmatrix} V_N \\ m_N \end{bmatrix} + v_u^T \begin{bmatrix} V_N \\ m_N \end{bmatrix} + d, \end{aligned} \tag{6}$$

where

$$\begin{aligned} H_{xx} &= 2(Q + \mathbf{A}_N^T D_{x,N} \mathbf{A}_N) \\ H_{xu} &= 2\mathbf{A}_N^T D_{x,N}\left[\mathbf{B}_{u,N} \ O_{(N+1)n_x,n_m}\right] \\ H_{uu} &= 2\operatorname{diag}\Big(\mathbf{B}_{u,N}^T D_{x,N}\mathbf{B}_{u,N} + D_{u,N} \\ &\quad L^T C_{q,x}(D_{x,N}, \mathbf{B}_{u,N}, \mathbb{C}_{W_N}) L \\ &\quad + L^T C_{q,u}(D_{u,N}, \mathbb{C}_{W_{N-1}}) L\Big) \\ v_u^T &= [\mathbf{0}_{Nn_u}^T \ C_{l,x}(D_{x,N}, \mathbf{B}_{u,N}, \mathbf{B}_{w,N}, \mathbb{C}_{W_N})^T L] \\ d &= \operatorname{tr}[D_{x,N}\mathbf{B}_{w,N}\mathbb{C}_{W_N}\mathbf{B}_{w,N}^T]. \end{aligned}$$

Note that, since $D_{x,N}$, $D_{u,N}$, $C_{q,x}(\cdot)$ and $C_{q,u}(\cdot)$ are symmetric and positive semidefinite, matrix H_{uu} inherits the same properties, so that J_{av} is a convex quadratic function in the control law parameters V_N and m_N. We exploit this property in Subsection 3.3, when deriving the eMPC.

3.2 Constraints reformulation

In this subsection we show that, by introducing suitable auxiliary variables, constraints (3) can be formulated as a finite set of linear equalities and inequalities in the parameters V_N, m_N, x_0 and in the introduced auxiliary variables.

We start by considering the state-input inequality constraints (4). For $t = 0$ they are:

$$C_{ax}x_0 + \left[C_{au}\ O_{q_c,(N-1)n_u}\right] V_N \leq c_b,$$

while for $t \geq 1$ we express them as:

$$\begin{aligned}
&(I_{N-1} \otimes C_{ax})X_{N-1} + \left[O_{(N-1)q_c,n_u}\ I_{N-1} \otimes C_{au}\right] U_N \\
&= (I_{N-1} \otimes C_{ax})\mathbf{A}_{N-1}x_0 \\
&+ \left((I_{N-1} \otimes C_{ax}) \left[\mathbf{B}_{u,N-1}\ O_{(N-1)n_x,n_u}\right]\right. \\
&+ \left.\left[O_{(N-1)q_c,n_u}\ I_{N-1} \otimes C_{au}\right]\right) V_N \\
&+ \left((I_{N-1} \otimes C_{ax})\mathbf{B}_{u,N-1}\mathbf{M}_{N-1,0} + (I_{N-1} \otimes C_{ax})\mathbf{B}_{w,N-1}\right. \\
&+ \left.\left[O_{(N-1)q_c,n_u}\ I_{N-1} \otimes C_{au}\right] \mathbf{M}_N\right) W_{N-1} \\
&\leq \mathbf{1}_{N-1} \otimes c_b,\ W_{N-1} \in \mathcal{W}^{N-1}.
\end{aligned}$$

Such constraints are robustly satisfied only if they are satisfied in the worst case, i.e., when the left-hand side, affected by the disturbance, assumes its maximum value. Since the disturbance support is a polytope, given a H-representation (W_a, w_b) of $\mathcal{W}$ with $W_a \in \mathbb{R}^{q_w,n_w}$ and $w_b \in \mathbb{R}^{q_w}$, we can exploit LP duality as in Goulart et al. (2006) so as to equivalently reformulate the constraints by introducing a matrix $\mathbf{Z}_c \in \mathbb{R}^{(N-1)q_w,(N-1)q_c}$ of nonnegative auxiliary variables such that:

$$\begin{aligned}
&\mathbf{Z}_c^T(I_{N-1} \otimes W_a) = (I_{N-1} \otimes C_{ax})\mathbf{B}_{u,N-1}\mathbf{M}_{N-1,0} \\
&+ (I_{N-1} \otimes C_{ax})\mathbf{B}_{w,N-1} + \left[O_{(N-1)q_c,n_u}\ I_{N-1} \otimes C_{au}\right] \mathbf{M}_N \\
&\mathbf{Z}_c^T(\mathbf{1}_{N-1} \otimes w_b) \leq \mathbf{1}_{N-1} \otimes c_b - (I_{N-1} \otimes C_{ax})\mathbf{A}_{N-1}x_0 \\
&- \left((I_{N-1} \otimes C_{ax}) \left[\mathbf{B}_{u,N-1}\ O_{(N-1)n_x,n_u}\right]\right. \\
&+ \left.\left[O_{(N-1)q_c,n_u}\ I_{N-1} \otimes C_{au}\right]\right) V_N,
\end{aligned} \tag{7}$$

As for the terminal constraint $x_N \in \mathcal{X}_f$, it can be written as

$$\begin{aligned}
&X_{fa}[A^N x_0 + \mathbf{B}_{u,N}^{[N+1]} V_N + (\mathbf{B}_{u,N}^{[N+1]}\mathbf{M}_{N,0} \\
&+ \mathbf{B}_{w,N}^{[N+1]})W_N] \leq x_{fb},\ W_N \in \mathcal{W}^N,
\end{aligned}$$

where (X_{fa}, x_{fb}) with $X_{fa} \in \mathbb{R}^{q_f,n_x}$ and $x_{fb} \in \mathbb{R}^{q_f}$ is a H-representation of $\mathcal{X}_f$, and we set

$$\mathbf{B}_{u/w,N}^{[N+1]} = \left[A^{N-1}B_{u/w}\ \ldots\ AB_{u/w}\ B_{u/w}\right].$$

Similarly to (7), we can introduce a matrix $\mathbf{Z}_f \in \mathbb{R}^{Nq_w,q_f}$ of nonnegative auxiliary variables so as to satisfy the following equivalent constraints:

$$\begin{aligned}
&\mathbf{Z}_f^T(I_N \otimes W_a) = X_{fa}\mathbf{B}_{u,N}^{[N+1]}\mathbf{M}_{N,0} + X_{fa}\mathbf{B}_{w,N}^{[N+1]} \\
&\mathbf{Z}_f^T(\mathbf{1}_N \otimes w_b) \leq x_{fb} - X_{fa}A^N x_0 - X_{fa}\mathbf{B}_{u,N}^{[N+1]}V_N
\end{aligned} \tag{8}$$

Now, by considering columnwise the matrix equality constraints in (7) and (8), we are finally able to formulate constraints (3) as a finite set of equalities and inequalities that are linear in the overall decision variables $\vartheta = \left[V_N^T\ m_N^T\ \text{vec}(\mathbf{Z}_c^T)^T\ \text{vec}(\mathbf{Z}_f^T)^T\right]^T$:

$$\begin{aligned}
G_{eq}\vartheta &= g_{eq} + F_{eq}x_0 \\
G_{in}\vartheta &\leq g_{in} + F_{in}x_0,
\end{aligned}$$

where the expressions of $G_{eq} \in \mathbb{R}^{q_{eq},n_\vartheta}$, $G_{in} \in \mathbb{R}^{q_{in},n_\vartheta}$, $g_{eq} \in \mathbb{R}^{q_{eq}}$, $g_{in} \in \mathbb{R}^{q_{in}}$, $F_{eq} \in \mathbb{R}^{q_{eq},n_x}$ and $F_{in} \in \mathbb{R}^{q_{in},n_x}$ can be obtained by suitably stacking and rearranging constraints (7) and (8), together with the nonnegativity conditions on the introduced auxiliary variables $\mathbf{Z}_c$ and $\mathbf{Z}_f$.

3.3 Explicit MPC design

According to the derivations from the previous sections, the optimization problem to be solved offline in order to compute the eMPC takes the following form:

$$\begin{aligned}
&\min_{\vartheta} J(\vartheta, x_0) \\
&\text{subject to:} \\
&G_{eq}\vartheta = g_{eq} + F_{eq}x_0 \\
&G_{in}\vartheta \leq g_{in} + F_{in}x_0
\end{aligned} \tag{9}$$

with

$$\begin{aligned}
J(\vartheta, x_0) &= J_{\text{av}}(\vartheta) - \frac{1}{2}x_0^T H_{xx} x_0 - d \\
&= \frac{1}{2}\vartheta^T H_{\vartheta\vartheta}\vartheta + x_0^T H_{x\vartheta}\vartheta + v_\vartheta^T\vartheta,
\end{aligned}$$

where we set

$$\begin{aligned}
&H_{\vartheta\vartheta} = \text{diag}(H_{uu}, O_{n_z}), \\
&H_{x\vartheta} = \left[H_{xu}\ O_{n_x,n_z}\right],\ v_\vartheta = \begin{bmatrix} v_u \\ \mathbf{0}_{n_z} \end{bmatrix}, \\
&n_z = q_w((N-1)^2 q_c + Nq_f).
\end{aligned}$$

To solve problem (9) we resort to multiparametric quadratic programming (mp-QP) so as to determine an optimizer $\vartheta^*(x_0)$ that is a PieceWise Affine (PWA) function of x_0. Specifically, we exploit the fact that mp-QPs are special instances of Parametric Linear-Complementarity Problems (PLCPs), i.e., problems of the form:

$$\begin{aligned}
&\min_{w,z}\ 0 \\
&\text{subject to:} \\
&w - Mz = q + Qx_0 \\
&w^T z = 0 \\
&w, z \geq 0
\end{aligned} \tag{10}$$

for which numerically robust algorithms have been developed, Herceg et al. (2013). The idea is then to convert problem (9) into the form (10), solve it, and finally retrieve a solution to problem (9). Both the conversion of problem (9) into a PLCP and the computation of its solution can be performed through the Multi-Parametric Toolbox as described in Herceg et al. (2013).

By extracting from $\vartheta^*(x_0)$ the component $v_0^*(x_0)$, we finally obtain the eMPC:

$$u = \kappa_{\text{mpc}}(x) = v_0^*(x).$$

Note that, since $\vartheta^*(x_0)$ is a PWA function of x_0, then the eMPC is a PWA function of the state x.

4. NUMERICAL EXAMPLE

In this section we illustrate the simulation results obtained by applying the control methodology presented in Section 3 to a numerical example from Munoz de la Pena et al. (2005). We performed computations with a portable PC equipped with a 2.8 GHz quad-core Intel Core i7 processor and 16 GB of RAM. Multi-parametric quadratic programs have been solved through the Multi-Parametric Toolbox (version 3.21) as described in Herceg et al. (2013).

The system dynamics is given by equation (1) with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}. \tag{11}$$

We assume that the disturbance w is a white noise uniformly distributed in $[-1, 1]$, from which we readily obtain its mean and variance: $\mathbb{E}_w[w] = 0$ and $\mathbb{C}_w = \text{var}[w] = \frac{1}{3}$.

We address the problem of designing an eMPC to regulate system (11) around the origin despite of the additive uncertainty w.

The finite horizon problem to be solved offline has the following form:

$$\min_{u_t,\, t=0,\ldots,N-1} J \tag{12}$$

subject to:

$$\|x_t\|_\infty \leq 10, \quad t = 0, \ldots, N$$
$$|u_t| \leq 1, \quad t = 0, \ldots, N-1$$
$$\forall w_t \in [-1, 1], \quad t = 0, \ldots, N-1.$$

We consider two approaches to address problem (12), both resting on suitably parametrized control laws. The first one is the approach proposed in this paper, while the second one is presented in Munoz de la Pena et al. (2005).

In our approach we minimize $J = J_{\text{av}}$ in (2) using the control parametrization (5), while in the other approach the goal is to minimize $J = J_{\text{wc}}$, with

$$J_{\text{wc}} = \max_{w_t,\, t=0,\ldots,N-1} \left[\sum_{t=0}^{N-1} (x_t^T Q x_t + u_t^T R u_t) + x_N^T P x_N \right],$$

subject to the control parametrization

$$u_t = v_t + K x_t$$

with K fixed. We choose for both costs J_{av} and J_{wc} the weights $Q = P = I_2$ and $R = 10$. The gain K was set equal to the unconstrained LQR gain for system (A, B_u), which is $K = [-0.2054 \; -0.7835]$ for the considered dynamics and weights.

We now compare the results obtained with the two methods in terms of complexity of the eMPC structure, quantified by the number of regions with a different expression for the affine function $\kappa_{\text{mpc}}(x_0)$. Table 1 reports the number of regions for the two methods corresponding to different values of the time horizon length N.

Table 1. Number of regions generated with our approach and the one in Munoz de la Pena et al. (2005) for different values of N.

N	1	3	5	8
average cost	1	9	19	43
min-max cost	3	45	71	147

Our approach results in an eMPC structure with a smaller complexity for each value of N, which facilitates its online implementation since one has to identify the region of the partition to which the current state belongs. Indeed, the complexity of the partition is affected by the number of constraints of the mp-QP problem, which in our method grows polynomially with N, while in Munoz de la Pena et al. (2005) grows exponentially with N. The state-space partitions obtained for $N = 3, 5, 8$ are depicted in Figure 1, while the partitions for the approach in Munoz de la Pena et al. (2005) are not reported here and can be found in that paper.

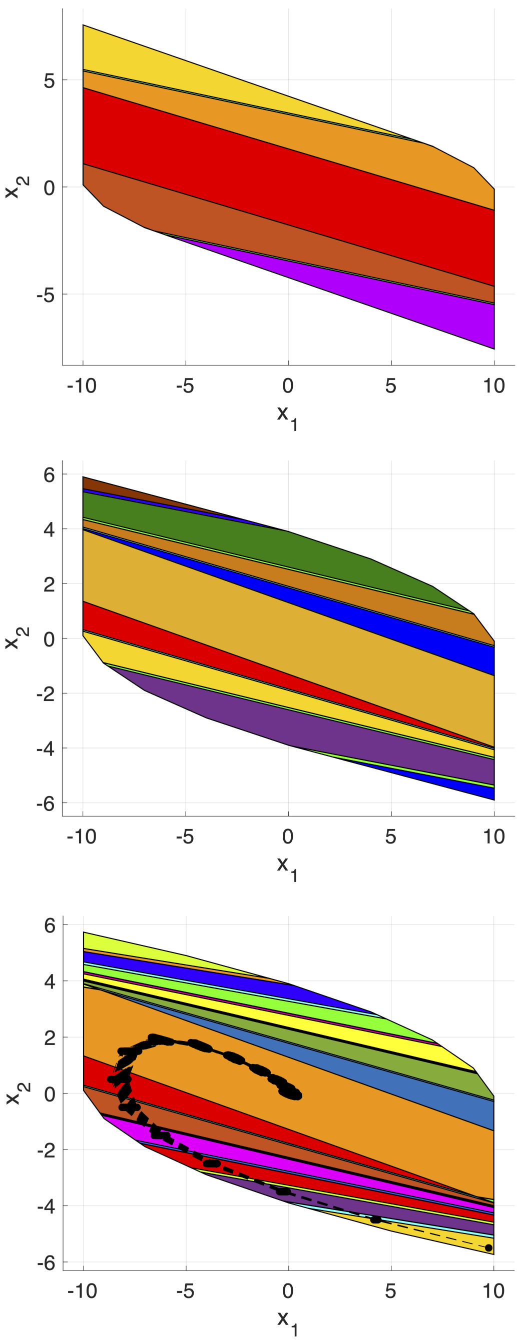

Fig. 1. State-space partitions obtained by applying the approach of Section 3 for $N = 3$ (top), $N = 5$ (middle) and $N = 8$ (bottom). The bottom plot reports also the trajectories of the closed-loop system when the eMPC is applied in the time horizon $[0, 40]$ starting from the same initial condition, for 1000 different disturbance realizations.

We finally consider the closed-loop behaviour of the system in the time horizon $[0, 40]$ when the eMPC determined with our approach for $N = 8$ is applied. Specifically,

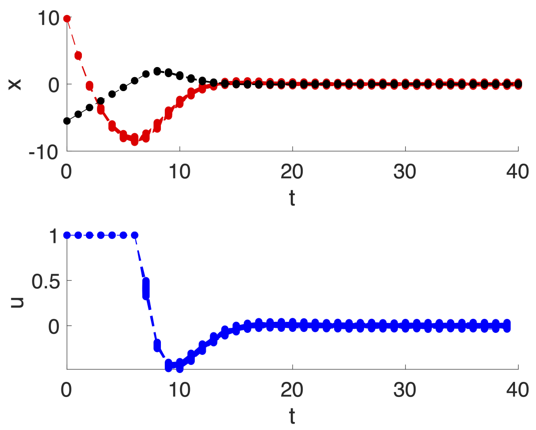

Fig. 2. Closed-loop behavior in the time horizon $[0, 40]$ obtained by applying the approach of Section 3 with prediction horizon of length $N = 8$, subject to 1000 disturbance realizations, starting from the same initial state x_0: in red the first state component, in black the second state component and in blue the applied control actions.

the closed-loop system is initialized at $x_0 = [9.75\ -5.5]^T$ and we analyze its evolution subject to 1000 different disturbance realizations. Figure 2 shows the behavior in time of each state component and of the control input when the prediction horizon length is $N = 8$ and Figure 1 reports the corresponding 2D state trajectories. Also, the closed-loop performance of our approach is similar to that reported in Munoz de la Pena et al. (2005), where the controller has been tested on a finite set of constant disturbances.

5. CONCLUSIONS AND FUTURE WORK

In this paper, we derived an eMPC for linear systems subject to a stochastic additive disturbance with bounded support. The controller is obtained by solving parametrically in the initial state a convex quadratic program and its structure is defined by a piecewise affine (PWA) function of the state. This result is obtained by integrating in the design a suitable parameterization of the finite horizon control law. Admittedly, such a parametrization is restricting the class of control policies with respect to the more powerful dynamic programming approach adopted in Bemporad et al. (2003). However, in our approach a single mp-QP problem has to be solved whose number of constraints is polynomial in the finite horizon length N, while in Bemporad et al. (2003) a sequence of N mp-LP problems has to be solved whose number of constraints may grow exponentially. We compared our approach with an alternative one still resorting to a control law parametrization but adopting a min-max approach as in Bemporad et al. (2003). The two approaches show the same performance in terms of closed-loop behavior of the controlled system but our approach generates a controller with a simpler structure, which makes its online implementation easier.

Admittedly, in our work we considered a single kind of uncertainty, which enters the system only through an additive disturbance. Other kinds of uncertainties should be taken into account as well. To this purpose, we are investigating a suitable extension of this work to the class of uncertain linear systems where both an additive disturbance and some parametric uncertainty on the dynamic matrices are present, like, e.g., in Bemporad et al. (2003).

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