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On fixed-time stability of a class of nonlinear time-varying systems ⋆

Youness Braidiz [*,**], **Andrey Polyakov** [**,***], **Denis Efimov** [**,***], and **Wilfrid Perruquetti** [*]

[*] *Ecole Centrale, Univ. Lille, CNRS, UMR 9189-CRIStAL, F-59000 Lille, France (email: youness.braidiz@centralelille.fr, wilfrid.perruquetti@centralelille.fr).*
[**] *Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France (email: denis.efimov@inria.fr, andrey.polyakov@inria.fr).*
[***] *ITMO University, 49 av. Kronverkskiy, 197101 Saint Petersburg, Russia.*

Abstract: Fixed-time stability of a class of nonlinear time-varying systems is investigated using the introduced notions of sub- and sup-homogeneity. These concepts allow the systems (even if they do not admit homogeneous approximations) to be analyzed using the homogeneity of their extensions. Then, finite-time and fixed-time stability properties can be recovered. The proposed stability conditions are not based on Lyapunov arguments. In the last section, an example illustrates the obtained results.

Keywords: Fixed-time stability, Homogeneity, Nonlinear systems

1. INTRODUCTION

In the last decades, analysis of asymptotic and exponential stability of dynamical systems were widely developed in the control community, e.g., see Hahn et al. (1963); Yoshizawa (1966); Hahn (1967); Bacciotti and Rosier (2006) and references therein. However, in many applications the time is a critical constraint which must be taken into account. Therefore, there are lot of researchers who work on the so called finite-time stability (FTS) and fixed-time stability (FxTS) as Zubov (1964); Bhat and Bernstein (2000b, 2005a); Hong et al. (2008); Braidiz et al. (2019). These concepts require that the system's trajectories converge to an equilibrium state in finite-time, (which does not depend on initial states in the FxTS case) and they are kept there then after.

FxTS plays an important role in many engineering problems, such as robot control, fixed-time attitude tracking of spacecraft, networked control issues, and so on. The concept of FxTS was proposed by Polyakov (2011); Polyakov et al. (2016) and he showed that this property can be investigated by using Lyapunov approach (necessary and sufficient Lyapunov characterizations of FxTS are obtained in Lopez-Ramirez et al. (2019)). However, in many applications, to find a Lyapunov function to study FTS and FxTS is not obvious. Therefore, in this paper, we propose a new method, which is based on construction of set-valued extensions. These extensions will satisfy some kind of symmetry called sup- or sub-homogeneity. This new relaxed symmetry still preserves the scaling property of the solutions of a sup- and sub-homogeneous differential inclusion (DI).

Homogeneous extensions were used in Braidiz et al. (2019) to investigate FTS of some class of nonlinear time-varying systems, which do not have a homogeneous approximation. Due to the features of their construction, all results that have been obtained for homogeneous systems/DIs (see Bhat and Bernstein (2005b); Moulay and Perruquetti (2005); Bernuau et al. (2015)) can be used to analyze these homogeneous extensions. Then, a non homogeneous system should satisfy some conditions to guarantee the existence of their homogeneous extensions. Similarly, in this paper, we introduce the so-called sup- and sub-homogeneous extensions, which will be used to study stability properties of some class of nonlinear systems.

The main contribution of this work is a method for establishment of the FxTS of some class of nonlinear systems which are uniformly globally asymptotically stable (GAS). The notion of sub-homogeneity which will be proposed in this work can also be used to study the FTS of uniformly GAS systems. Hence, this allows us to enlarge a lot the class of considered systems. Sup-homogeneity will be defined for DIs which will be given here as an extension of nonlinear systems. Particularly, we will prove that if a uniformly GAS system has a sup-homogeneous extension with positive degree of homogeneity, then it is nearly FxTS.

The paper is organized as follows. After introducing notations and definitions about asymptotic, finite-time, fixed-time stability and homogeneity in Section 2. We investigate the FTS and FxTS by using the introduced sup- and sub-homogeneous extensions (in Section 3). In the last section we present an example of a nonlinear model,

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(a form of mechanical system), to illustrate the obtained results of this paper.

2. PRELIMINARIES

2.1 Notations

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers. $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\lceil x \rfloor^\alpha = |x|^\alpha \text{sign}(x)$ for $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$.
- $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ denotes the unit sphere in $\mathbb{R}^n$. $B(x_0, r) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$ denotes the open ball of radius $r > 0$ centered at a point x_0, and $\overline{B(x_0, r)}$ is the closure of $B(x_0, r)$. We define $B(K, r) = \left\{ y \in \mathbb{R}^n : \inf_{x \in K} \|y - x\| < r \right\}$ for all compact $K \subset \mathbb{R}^n$.
- $\mathcal{M}_{m,n}$ is the set of all $m \times n-$matrices over the field of real numbers, and it forms a vector space. When $m = n$ we write $\mathcal{M}_n$ instead of $\mathcal{M}_{n,n}$. $\|\cdot\|_{\mathcal{M}_{m,n}}$ denotes the matrix norm induced by $\|\cdot\|$, i.e. $\|A\|_{\mathcal{M}_{m,n}} = \sup_{x \in S} \|Ax\|$.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ (resp., $\mathcal{GKL}$) if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\lim_{t \to +\infty} \beta(s, t) = 0$ (resp., if for each fixed $t \in (0, T)$, β is a strictly decreasing function of its second argument $t \in \mathbb{R}_+$ for any fixed first argument $s \in \mathbb{R}_+ \setminus \{0\}$ and $\beta(s, T) = 0$ for some $0 \leq T < +\infty$) for each fixed $s \in \mathbb{R}_+$.
- $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$ denotes the space of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}^n$.
- For $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we denote by $\mathcal{D}(F)$ the domain of definition of F.
- $\overline{\text{co}} A$ denotes the closed convex hull of $A \subset \mathbb{R}^n$.

2.2 Stability properties and generalized homogeneity

In this subsection we give some definitions of homogeneity and different kinds of stability for a non-autonomous system:

$$\begin{cases} \dot{x}(t) &= f(t, x(t)),\ t \geq t_0 \geq 0 \\ x(t_0) &= x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

the vector field $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n, f(t, 0) = 0, \forall t \geq 0$ is considered such that (1) possesses a unique solution in forward time for all initial conditions $x_0 \in \mathbb{R}^n$ except the origin and $t_0 \in \mathbb{R}_+$, which we will denote by $x(t, t_0, x_0)$

Uniform asymptotic, finite/fixed-time stability: The following definitions are inspired from Zubov (1964); Clarke et al. (1998); Bhat and Bernstein (2000a) and Polyakov and Fridman (2014).

Definition 2.1. The origin of (1) is

- **unifromly stable** if $\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0$, independent of t_0, such that
$$x_0 \in \overline{B(0, \delta)} \Longrightarrow x(t, t_0, x_0) \in \overline{B(0, \varepsilon)}, \forall t \geq t_0 \geq 0;$$
- **uniformly attractive** if $\exists c > 0$, independent on $t_0 \geq 0$, such that $\forall \varepsilon > 0,\ \exists \tilde{T}(\varepsilon, c) > 0$ and
$$x_0 \in \overline{B(0, c)} \Longrightarrow x(t, t_0, x_0) \in \overline{B(0, \varepsilon)}, \tag{2}$$
$\forall t \geq t_0 + \tilde{T}(\varepsilon, c)$;
- **uniformly asymptotically stable** if it is uniformly stable and uniformly attractive.

In addition, we say that (1) is **uniformly GAS** if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim_{\varepsilon \to +\infty} \delta(\varepsilon) = +\infty$ and for each pair of positive numbers ε and c, there is $\tilde{T}(\varepsilon, c)$ such that the inequality (2) holds for all $t \geq t_0 + \tilde{T}(\varepsilon, c)$.

Definition 2.2. The origin of (1) is

- **uniformly FTS**, if it is uniformly stable and there exists a nonempty open neighborhood of the origin $\mathcal{V} \subset \mathbb{R}^n$ and a function $T : \mathcal{V} \to \mathbb{R}_+$ such that $\forall t_0 \geq 0, \forall x_0 \in \mathcal{V}$,
$$x(t, t_0, x_0) = 0,\ \forall t \geq t_0 + T(x_0).$$
T is called a uniform settling-time function of the system (1);
- **globally uniformly FTS**, if $\mathcal{V} = \mathbb{R}^n$;
- **uniformly FxTS**, if the settling-time function T is bounded for all $x \in \mathbb{R}^n$;
- **uniformly nearly FxTS** if for any $r > 0$, there exists $0 < T_r < +\infty$ such that $x(t, t_0, x_0) \in B(0, r),\ \forall t \geq t_0 + T_r,\ \forall x_0 \in \mathbb{R}^n, \forall t_0 \geq 0$.

Now we define the stability properties for DIs. Let us consider the following DI

$$\begin{cases} \dot{x}(t) &\in F(x(t)),\ \forall t \geq 0, \\ x(0) &= x_0 \in \mathbb{R}^n, \end{cases} \tag{3}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. The next definition concerns the strong and weak stability properties of (3). We assume that the set-valued map F satisfies the classical conditions to guarantee the existence of solutions of (3) (see Filippov (1988)). For each $\Omega \subseteq \mathbb{R}^n$ we let $\mathbf{S}(\Omega) \subset \mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$ denote the set of solutions of (3), which satisfy $x(0) \in \Omega$. If Ω is a singleton $\{\zeta\}$, we will use the shorthand $\mathbf{S}(\zeta)$. We denote $\mathbf{S} = \mathbf{S}(\mathbb{R}^n)$ as the set of all solutions of (3).

Definition 2.3. The origin of the system (3) is **weakly uniformly GAS** (resp., weakly FTS/FxTS) if there exists a subset $\hat{\mathbf{S}}$ of the set $\mathbf{S}$ of solutions of (3), such that all $x(\cdot) \in \hat{\mathbf{S}}$ are uniformly GAS (resp., FTS/FxTS).

Definition 2.4. A set-valued map F is called upper semi-continuous at $\bar{x} \in \mathcal{D}(F)$ if $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $\forall x \in B(\bar{x}, \delta),\ F(x) \subset B(F(\bar{x}), \varepsilon)$.

Lemma 2.1. Filippov (1988) *Let a function F be upper semicontinuous on a compactum K and let for each $x \in K$ the set $F(x)$ be bounded. Then, the function F is bounded on K.*

By the definition of weak GAS (resp., FTS) $\exists \beta \in \mathcal{KL}$ (resp., $\beta \in \mathcal{GKL}$) such that $\|x(t)\| \leq \beta(\|x_0\|, t)$, for all $x_0 \in \mathbb{R}^n, x(\cdot) \in \hat{\mathbf{S}}^\beta(x_0)$. Notice that, in the case of the weak uniform GAS or FTS, the set $\hat{\mathbf{S}}$ may depend on the function β:

$$\hat{\mathbf{S}} = \hat{\mathbf{S}}^\beta.$$

In this case, for $\sigma \in \mathcal{K}_\infty$ and $R > 0$ we denote

$$\overline{\mathcal{C}}^{\beta}_{\sigma}(R) = \left\{ y \in \mathcal{C}([0,1],\mathbb{R}^n);\ \begin{array}{c} y(1)=0,\ y\left(\dfrac{\sigma(t)}{1+\sigma(t)}\right)=x(t), \\ t \geq 0, x(\cdot) \in \hat{\mathbf{S}}^{\beta}(B(0,R)) \end{array} \right\}.$$

The following lemma proves that the set $\hat{\mathbf{S}}^{\beta}$ of weakly uniformly stable solutions of (3) is homeomorphic to a compact in $\mathcal{C}([0,1],\mathbb{R}^n)$.

Lemma 2.2. If F is upper semicontinuous, $\hat{\mathbf{S}}^{\beta}$ is the set of all GAS solutions of (3) *and $R > 0$. Then, $\overline{\mathcal{C}}^{\beta}_{\sigma}(R)$ is a compactum in $(\mathcal{C}([0,1],\mathbb{R}^n), \|\cdot\|_{\mathcal{C}})$ with $\|f\|_{\mathcal{C}} = \sup_{x\in[0,1]} \|f(x)\|$.*

Homogeneity: In control theory, homogeneity simplifies qualitative analysis of nonlinear dynamic systems. So that, it allows a local properties (e.g. local stability) to be extended globally using a symmetry of the solutions (homogeneity is a Lie symmetry). In order to define this concept, let us introduce the notion of dilation group.

Definition 2.5. A map $\mathbf{d} : \mathbb{R} \to \mathcal{M}_n(\mathbb{R})$ is called a linear dilation in $\mathbb{R}^n$ if it satisfies

- **Group property:** $\mathbf{d}(0) = I_n$, $\mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s)$, $t, s \in \mathbb{R}$;
- **Limit property:** $\lim_{s\to-\infty} \|\mathbf{d}(s)x\| = 0$ and $\lim_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty$ uniformly on the unit sphere.

Definition 2.6. The dilation $\mathbf{d}$ is monotone in $\mathbb{R}^n$, if $\|\mathbf{d}(s)\|_{\mathcal{M}_n} \leq 1$, $\forall s \leq 0$.

Now we will define the canonical homogeneous norm.

Definition 2.7. If the dilation $\mathbf{d}$ is monotone, the continuous function $\|\cdot\|_{\mathbf{d}} : x \in \mathbb{R}^n \to \|x\|_{\mathbf{d}} = e^{s_x} \in \mathbb{R}_+$, where $s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1$, $\|x\|_{\mathbf{d}} \to 0$ as $x \to 0$ and $\|\mathbf{d}(s)x\|_{\mathbf{d}} = e^s\|x\|_{\mathbf{d}} > 0$, is called the **canonical homogeneous norm**.

Definition 2.8. A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ (resp., a function $h : \mathbb{R}^n \to \mathbb{R}$) is said to be $\mathbf{d}$-homogeneous of degree $\nu \in \mathbb{R}$ (resp., $\mu \in \mathbb{R}$) iff for all $s \in \mathbb{R}$ and all $x \in \mathbb{R}^n$ we have $e^{-\nu s}\mathbf{d}(-s)f(\mathbf{d}(s)x) = f(x)$, (resp., $e^{-\mu s}h(\mathbf{d}(s)x) = h(x)$).

In the next section we will show the main results of this paper. A uniform GAS system will be considered and we will study its global fixed-time stability.

3. FINITE/FIXED-TIME STABILITY RESULTS

In this section, we consider the following system

$$\begin{cases} \dot{x}(t) = H_1(x(t))b_1(t,x(t)) + H_2(x(t))b_2(t,x(t)); \\ x(0) = x_0;\ x_0, x(t) \in \mathbb{R}^n,\ \forall t \geq t_0 \geq 0. \end{cases} \quad (4)$$

with

- $\mathcal{H}_1$: $H_1 : \mathbb{R}^n \to \mathcal{M}_{n,m}$ (resp., $H_2 : \mathbb{R}^n \to \mathcal{M}_{n,m}$) are continuous and $\mathbf{d}$-homogeneous of degree $\nu_1 < 0$, (resp., $\nu_2 > 0$) (i.e., $H_1(\mathbf{d}(s)x) = e^{\nu_1 s}\mathbf{d}(s)H_1(x)$ and $H_2(\mathbf{d}(s)x) = e^{\nu_2 s}\mathbf{d}(s)H_2(x)$);
- $\mathcal{H}_2$: $b_1 : \mathbb{R}_+ \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m$ and $b_2 : \mathbb{R}_+ \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m$ are continuous, $\exists M_1, M_2 > 0$ such that $0 < \|b_1(t,x)\| \leq M_1$ and $0 < \|b_2(t,x)\| \leq M_2$, $\forall x \in \mathbb{R}^n, \forall t \geq 0$;
- $\mathcal{H}_3$: The origin of the system (4) is uniformly GAS.

Our goal is to prove that (4), under the assumptions $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, is FxTS. Therefore, we will study this system in two steps. The first step will analyze the FTS of (4) by using the notion of sub-homogeneous extension (Section 3.1). In the next step, we will show that (4) is nearly FxTS by using the notion of sup-homogeneous extensions (Section 3.2).

3.1 Sub-homogeneous extension

In this section we will investigate the FTS of the uniformly GAS system (4). To do so, we will introduce the notion of sub-homogeneous extensions. This concept will be used to prove that the system (4) is FTS if $\nu_1 < 0$.

Definition 3.1. A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is $\mathbf{d}$-sub-homogeneous with degree of homogeneity $\nu \in \mathbb{R}^n$, if for all $x \in \mathbb{R}^n$, $F(\mathbf{d}(\tau)x) \subseteq e^{\nu\tau}\mathbf{d}(\tau)F(x)$, $\forall \tau \leq 0$, where $\mathbf{d}$ is a generalized dilation and $\nu \in \mathbb{R}$.

We will consider the following example of sub-homogeneous extension:

$$F(x) = \overline{\text{co}} \bigcup_{s\leq 0} \bigcup_{t\geq 0} \left\{ e^{-\nu_1 s}\mathbf{d}(-s)f(t,\mathbf{d}(s)x) \right\}, \quad (5)$$

where f is defined in (1). Such a set-valued map F is closed, convex and $\mathbf{d}$-sub-homogeneous by construction, and for $f(t,x) = H_1(x)b_1(t,x) + H_2(x)b_2(t,x)$, if $\nu_2 > 0 > \nu_1$, F is bounded for every fixed $x \in \mathbb{R}^n$. Indeed,

$$\begin{aligned} &e^{-\nu_1 s}\mathbf{d}(-s)f(t,\mathbf{d}(s)x) \\ &= H_1(x)b_1(t,\mathbf{d}(s)x) + e^{(\nu_2-\nu_1)s}H_2(x)b_2(t,\mathbf{d}(s)x), \end{aligned}$$

because $(\nu_2 - \nu_1) > 0$ and $\lim_{s\to-\infty} e^{(\nu_2-\nu_1)s} = 0$, we get

$$\begin{aligned} \lim_{s\to-\infty} \sup_{x\in K} &\|H_1(x)b_1(t,\mathbf{d}(s)x) + e^{(\nu_2-\nu_1)s}H_2(x)b_2(t,\mathbf{d}(s)x)\| \\ &\leq M_1 \sup_{x\in K} \|H_1(x)\|_{\mathcal{M}_{m,n}}, \end{aligned}$$

where K is a compact set in $\mathbb{R}^n \setminus \{0\}$. This implies that for every fixed $x \in \mathbb{R}^n$ the set $F(x)$ is compact. Therefore, the system (4) has a sub-homogeneous extension. Hence, the solutions of the system (4) are in the set of solutions of the DI (3) with F is sub-homogeneous. The following proposition ensures symmetry of solutions of the sub-homogeneous differential inclusion (3).

Proposition 3.1. Braidiz et al. (2020) *Let F be a $\boldsymbol{d}$-sub-homogeneous set-valued map of degree $\nu_1 \in \mathbb{R}$. If for $x_0 \in \mathbb{R}^n$, the trajectory $x(\cdot, x_0)$ is a solution of* (3) *starting at x_0, then for all $s \geq 0$, the absolutely continuous curve $t \longmapsto \boldsymbol{d}(s)x(e^{\nu_1 s}t, x_0)$ is a trajectory of* (3) *starting at $\boldsymbol{d}(s)x_0$, (i.e., if $x(\cdot,x_0) \in \mathbf{S}(x_0)$, then $\boldsymbol{d}(s)x(e^{\nu_1 s}\cdot, x_0) \in \mathbf{S}(\boldsymbol{d}(s)x_0)$ for all $s \geq 0$.*

The proof of Proposition 3.1 is based on $\mathbf{d}$-sub-homogeneity of the set valued map F (see Proposition 3.1 in Braidiz et al. (2020)).

Remark 3.1. Due to the continuity of the right side of (4), the solutions $x(\cdot, t_0, x_0)$ exist, then the set $\mathbf{S}(x_0)$ is nonempty. All the solutions of the system

$$\begin{cases} \dot{x}(t) = e^{-\nu_1 s}\mathbf{d}(-s)f(t,\mathbf{d}(s)x(t)), \\ x(0) = x_0 \end{cases}$$

$s \leq 0$ and $t \geq 0$, with $f(t,x) = H_1(x)b_1(t,x) + H_2(x)b_2(t,x)$ are in $\mathbf{S}(x_0)$, $\forall x_0 \in \mathbb{R}^n$.

In the sequel of this paper, we will use Lemma 2.2. Therefore, to do so, the set-valued map F should be upper semicontinuous:

Lemma 3.1. If the vector field $f(t,x) = H_1(x)b_1(t,x) + H_2(x)b_2(t,x)$ *is continuous and satisfies the assumptions* $\mathcal{H}_1, \mathcal{H}_2$*, then the set-valued map* F *given by* (5) *is upper semicontinuous.*

Proof. The proof of this lemma will be based on proving that: $\exists \bar{h}(\bar{x}) = \sum_{i=1}^{q(\bar{x})} \lambda_i(\bar{x}) g_i(\bar{x})$ such that

$$h(x) \in F(x) \Longrightarrow \inf_{\bar{h}(\bar{x}) \in F(\bar{x})} \|h(x) - \bar{h}(\bar{x})\| \leq \varepsilon. \quad (6)$$

with

$$h(x) = \sum_{i=1}^{q(x)} \lambda_i(x) g_i(x),$$

$$g_i(x) = e^{-\nu_1 s_i(x)} \mathbf{d}(-s_i(x)) f(t_i(x), \mathbf{d}(s_i(x))x),$$

$\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^n, \mathbb{R}_+)$, $q : \mathbb{R}^n \to \mathbb{N}$, $\{t_i\}_{i\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^n, \mathbb{R}_+)$, $\{s_i\}_{i\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^n, (-\infty, 0])$ and $\sum_{i=1}^{q(x)} \lambda_i(x) = 1$, $\forall x \in \mathbb{R}^n$. Due to the continuity of f, one gets

$$h(x) = \sum_{i=1}^{q(x)} \lambda_i(x) g_i(x) \in B(F(\bar{x}), \varepsilon),$$

($\mathcal{H}_1, \mathcal{H}_2$ imply that the set valued $F(\bar{x})$ is compact by construction) which means that $F(x) \subset B(F(\bar{x}), \varepsilon)$. Then, we deduce that F is upper semicontinuous. ■

Now, after showing that the set valued map F is upper semicontinuous. The results of Lemma 2.2 will be used to prove the FTS of the system (4).

Theorem 3.1. If the system (4) *satisfies the assumptions* $\mathcal{H}_1, \mathcal{H}_2$ *and* $\mathcal{H}_3$*, then it is FTS.*

Proof. Let $\hat{\mathbf{S}}^\beta$ be the set of solutions of the DIs (3), which are uniformly GAS. Let $R > 0$, $x_0 \in B_{2R}$, $x(\cdot, x_0) \in \hat{\mathbf{S}}^\beta(x_0)$, $B_R = \{x \in \mathbb{R}^n : \ \|\mathbf{d}(-\ln(R))x\| \leq 1\}$ and let us define the time $\tau(x_0, B_R)$ for uniformly GAS solutions of (3) starting from x_0, to reach and stay in B_R by:

$$\tau(x_0, B_R) = \sup_{x(\cdot,x_0)\in\hat{\mathbf{S}}^\beta(x_0)} \inf\{T > 0 : \ x(t, x_0) \in B_R, \forall t \geq T\}.$$

Using Lemma 2.2, one can prove that τ is finite for every $x_0 \in B_R$. The fact that $\nu_1 < 0$ implies the weak FTS of the sub-homogeneous extension (3). Since the solutions of (4) are in the set of solutions of (3). (4) is finite-time stable.■

Theorem 3.1 generalizes to time-varying systems the results that have been proven in Bernuau et al. (2015).

3.2 Sup-homogeneous extension

In this subsection we consider (3) with F is given by

$$F(x) = \overline{\text{co}} \bigcup_{s\geq 0} \bigcup_{t\geq 0} \left\{ e^{-\nu_2 s} \mathbf{d}(-s) f(t, \mathbf{d}(s)x) \right\}, \quad (7)$$

with $f(t,x) = H_1(x)b_1(t,x) + H_2(x)b_2(t,x)$ and from the construction of F, we can see that for $s = 0$ and for $t \geq 0$ we have $f(t,x) \in F(x)$, for every fixed $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$. Let us now start by the following definition of sup-homogeneous set-valued map.

Definition 3.2. We say that F is $\mathbf{d}$-sup-homogeneous with degree of sup-homogeneity ν_2, if for all $x \in \mathbb{R}^n$, $F(\mathbf{d}(\tau)x) \subseteq e^{\nu_2 \tau} \mathbf{d}(\tau) F(x)$, $\forall \tau \geq 0$, where $\mathbf{d}$ is a generalized dilation and $\nu_2 \in \mathbb{R}$.

The extension (7), with $\nu_2 > 0 > \nu_1$, is compact. Indeed, this happens because $\lim_{s\to+\infty} e^{(\nu_1 - \nu_2)s} = 0$. The solutions to the DI (3), when F is $\mathbf{d}$-sup-homogeneous, satisfy some sort of symmetry. The following proposition show that.

Proposition 3.2. If F *is* $\boldsymbol{d}$*-sup-homogeneous set-valued map of degree* $\nu_2 \in \mathbb{R}$*, then for all* $x(\cdot, x_0) \in \mathbf{S}(x_0)$*,* $x_0 \in \mathbb{R}^n$*, the absolutely continuous curve* $t \longmapsto \boldsymbol{d}(s)x(e^{\nu_2 s}t, x_0)$*,* $s \leq 0$ *is a trajectory of the system* (3),(7) *starting at* $\boldsymbol{d}(s)x_0$ *(i.e., if* $x(\cdot, x_0) \in \mathbf{S}(x_0)$*, then* $\boldsymbol{d}(s)x(e^{\nu_2 s}\cdot, x_0) \in \mathbf{S}(\boldsymbol{d}(s)x_0)$ *for all* $s \leq 0$*.*

Proof of Proposition 3.2: The proof is similar to the one of Proposition 3.1.■

Lemma 3.2. If $f(t,x) = H_1(x)b_1(t,x) + H_2(x)b_2(t,x)$ *is continuous and satisfies the assumptions* $\mathcal{H}_1, \mathcal{H}_2$*, then the set-valued map* F *given by* (7) *is upper semicontinuous.*

Proof. To prove Lemma 3.2 we use the same method as in the proof of Lemma 3.1.■

Let us now give the following theorem which shows the nearly FxTS of (4).

Theorem 3.2. If the system (4) *satisfies the assumptions* $\mathcal{H}_1, \mathcal{H}_2$ *and* $\mathcal{H}_3$*, then the system* (4) *is nearly FxTS.*

Proof. Let $\hat{\mathbf{S}}^\beta(x_0)$ be the set of solutions of the differential inclusion (3), starting from $x_0 \in \mathbb{R}^n$, which are uniformly GAS. Let $x(\cdot, x_0) \in \hat{\mathbf{S}}^\beta(x_0)$, $B_R = \{x \in \mathbb{R}^n : \ \|\mathbf{d}(-\ln(R))x\| \leq 1\}$ for a given $R > 0$ and let us define $\tau(x_0, B_R) = \sup_{x(\cdot)\in\hat{\mathbf{S}}^\beta(x_0)} \inf\{T > 0 : \ x(t, x_0) \in B_R, \ \forall t \geq T\}$, as in the proof of Theorem 3.1. By the definition of $\tau(x_0, B_R)$ and the fact $\nu_2 > 0$, one deduces that the ball B_R is FxTS. Which means that all the solutions in $\hat{\mathbf{S}}^\beta(x_0)$ get in B_R in fixed-time, and stay there. In particular, the solutions of (4) are converging in fixed-time to the ball B_R. ■

Combining Theorems 3.2 and 3.1, we deduce the FxTS of (4):

Theorem 3.3. The system (4), under the assumptions $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, is FxTS.

Proof. The system (4) satisfies the assumptions $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, then

1) Theorem 3.2 implies that the system (4) is nearly FxTS.
2) Theorem 3.1 shows that the system (4) is FTS for all $x_0 \in B_R$.

These two facts imply that the state of the system (4) converge to the origin at fixed-time (as we see in Fig. 1) $T \leq \tau\left(x\left(\frac{2^{\nu_2}\tilde{\tau}(R)}{2^{\nu_2}-1}\right), B_R\right) + \frac{2^{\nu_1}\tilde{\tau}(R)}{1-2^{\nu_1}}$. ■

Remark 3.2. The results of this paper can be used to stabilize in fixed-time nonlinear systems, which have the form $\dot{x} = H(x,u)b(t,x,u)$, with u is a control (e.g., the

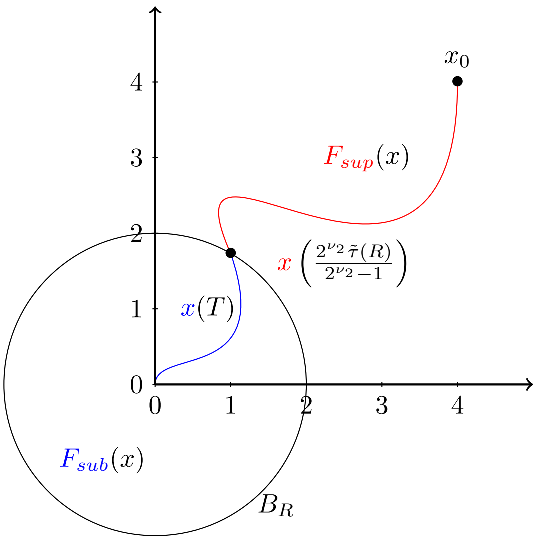

Fig. 1. Fixed-time stable solution.

dynamic model of quadrotor ($\dot{x} = A(x)x + Bu$)) with H is homogneoeus and b is bounded.

4. ACADEMIC EXAMPLE

Consider the following nonlinear system

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -\alpha x^{\frac{1}{3}} - \beta_1 \left(2 - \cos\left(\frac{t}{\|z\|}\right)\right) \lceil y \rfloor^{\frac{1}{2}} \\ & -\beta_2 \left(2 - \sin\left(\frac{t}{\|z\|}\right)\right) \lceil y \rfloor^{2}, \end{cases} \tag{8}$$

where $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ and $\beta_1, \beta_2 > 0$. We can rewrite the system (8) in the following form

$$\dot{z} = f(t,z) = H(z)b(t,z) + \bar{H}(z)\bar{b}(t,z),$$

with

$$b(t,z) = \left(1, 2 - \cos\left(\frac{t}{\|z\|}\right)\right)^T,$$

$$\bar{b}(t,z) = \left(0, 2 - \sin\left(\frac{t}{\|z\|}\right)\right)^T,$$

$H(x,y) = \begin{pmatrix} y & 0 \\ -\alpha x^{\frac{1}{3}} & -\beta_1 \lceil y \rfloor^{\frac{1}{2}} \end{pmatrix}$ and $\bar{H}(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & -\beta_2 \lceil y \rfloor^{2} \end{pmatrix}$.

The functions b and $\bar{b}$ are bounded and continuous on every $z \in \mathbb{R}^2 \setminus \{0\}$, for all $t \geq 0$. The functions H and $\bar{H}$ are continuous and **d**-homogeneous of the degree of homogeneity respectively $\nu = -\frac{1}{3}$ and $\bar{\nu} = \frac{2}{3}$ provided that the dilation **d** is defined as $\mathbf{d}(s) = \text{diag}\{e^s, e^{\frac{2}{3}s}\}$, which corresponds to a weighted homogeneity. We use the following Lyapunov function $U(x,y) = \frac{3}{4}\alpha|x|^{\frac{4}{3}} + \frac{1}{2}y^2$, we obtain

$$\langle DU(z), f(t,z)\rangle = -\beta_1 \left(2 - \cos\left(\frac{t}{\|z\|}\right)\right) |y|^{\frac{3}{2}} -\beta_2 \left(2 - \sin\left(\frac{t}{\|z\|}\right)\right) |y|^{3}.$$

From this we have $\langle DU(z), f(t,z)\rangle \leq 0, \forall z \in \mathbb{R}^2 \setminus \{0\}$ and all $t \geq 0$. Using LaSalle's theorem, one gets that the origin of the system (8) is uniformly GAS. The degree of homogeneity of H is negative and the degree of homogeneity of $\bar{H}$ is positive, then by using Theorem 3.3 the system (8) is FxTS. The Figure 2 shows the solution of the system (8) with initial condition $(2, 4)$ which reaches the origin in fixed-time (less than 50).

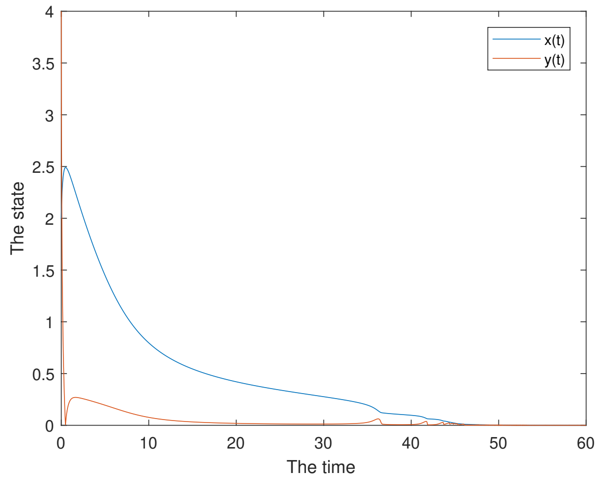

Fig. 2. The solutions of the system (4) with $\alpha = 1, \beta_1 = 2, \beta_2 = 1.5$ and the initial condition $(x_0, y_0) = (2, 4)$.

5. CONCLUSION

FxTS of some class of nonlinear time-varying systems is investigated. The notions of sup- and sub-homogeneity were introduced. These concepts are used to reach the FxTS results of this paper. Specifically, we presented some sufficient conditions on the vector fields to guarantee the FxTS of nonlinear dynamical systems. As we showed, these properties allow systems which do not admit homogeneous approximations to be analyzed. Example from Section 4 is considered to illustrate the obtained results.

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